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Generalization of the Gergonne Point

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1. Introduction

The Gergonne point [1] is the point of intersection of the vertices of a triangle with the points that are tangent with the incircle. It is denoted $X(7)$ in Clark Kimberling's Encyclopedia of Triangle Centers. Here, we give a result which gives the existence of this point as a special case.

2. Result

Theorem 2.1. *Let A, B, and C be the vertices of a given triangle.*
Let D be some point in the plane.
Let lines AB and CD intersect at point E.
Let lines AC and BD intersect at point F.
Let lines AD and BC intersect at point G.
Let the circle passing through E, F, and G intersect line AB at point H.
Let the circle passing through E, F, and G intersect line AC at point I.
Let the circle passing through E, F, and G intersect line BC at point J.
Then lines AJ, BI, and CH are concurrent.

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Proof. We use the first few coordinates calculated in [2].
A has coordinates $(0,0)$.
B has coordinates $(1,0)$.
C has coordinates (x_C, y_C).
D has coordinates (x_D, y_D).
E has coordinates $(\frac{x_C y_D - x_D y_C}{y_D - y_C}, 0)$.
F has coordinates $(\frac{x_C y_D}{x_C y_D - x_D y_C + y_C}, \frac{y_C y_D}{x_C y_D - x_D y_C + y_C})$.
G has coordinates $(\frac{x_D y_C}{-x_C y_D + x_D y_C + y_D}, \frac{y_C y_D}{-x_C y_D + x_D y_C + y_D})$.
Circle EFG is given by the equation
$(x - (-(((2x_C - 1)y_C^2 - 2x_C^5 + 3x_C^4 - x_C^2)y_D^5 - y_C((2x_D + 4x_C - 3)y_C^2 - (x_C - 1)x_C^2(2(5x_C - 1)x_D - 3x_C - 1))y_D^4 + y_C^2((4x_D + 2x_C - 3)y_C^2 - 2x_C x_D((10x_C^2 - 9x_C + 1)x_D - (2x_C - 1)(3x_C - 1)))y_D^3 - y_C^3((2x_D - 1)y_C^2 - 2x_C x_D(x_D(2(5x_C - 3)x_D - 9x_C + 5) + x_C - 1))y_D^2 - (x_D - 1)x_D^2((10x_C - 3)x_D - 2x_C - 1)y_C^4 y_D + (x_D - 1)^2 x_D^2(2x_D + 1)y_C^5)/(4(y_D - y_C)((x_C - 1)y_D - (x_D - 1)y_C)((x_C - 1)y_D - x_D y_C)(x_C y_D - (x_D - 1)y_C)(x_C y_D - x_D y_C))))^2 + (y - (-x_C^6 y_D^6 + 4x_C^5 x_D y_C y_D^5 - 2x_C^5 y_C y_D^5 + 3x_C^5 y_D^6 - 5x_C^4 x_D^2 y_C^2 y_D^4 + 5x_C^4 x_D y_C^2 y_D^4 - 10x_C^4 x_D y_C y_D^5 + 4x_C^4 y_C y_D^5 - 3x_C^4 y_D^6 + 10x_C^3 x_D^2 y_C^2 y_D^4 - 10x_C^3 x_D y_C^2 y_D^4 + 8x_C^3 x_D y_C y_D^5 - 2x_C^3 y_C y_D^5 + x_C^3 y_D^6 + 5x_C^2 x_D^4 y_C^4 y_D^2 - 10x_C^2 x_D^3 y_C^4 y_D^2 + 5x_C^2 x_D^2 y_C^4 y_D^2 - 5x_C^2 x_D^2 y_C^2 y_D^4 + 5x_C^2 x_D y_C^2 y_D^4 - 2x_C^2 x_D y_C y_D^5 + 3x_C^2 y_C^4 y_D^4 - 2x_C^2 y_C^3 y_D^5 - x_C^2 y_C^2 y_D^6 - 4x_C x_D^5 y_C^5 y_D + 10x_C x_D^4 y_C^5 y_D - 5x_C x_D^4 y_C^4 y_D^2 - 8x_C x_D^3 y_C^5 y_D + 10x_C x_D^3 y_C^4 y_D^2 + 2x_C x_D^2 y_C^5 y_D - 5x_C x_D^2 y_C^4 y_D^2 - 4x_C x_D y_C^5 y_D^3 + 4x_C x_D y_C^3 y_D^5 + 2x_C y_C^5 y_D^3 - 3x_C y_C^4 y_D^4 + x_C y_C^2 y_D^6 + x_D^6 y_C^6 - 3x_D^5 y_C^6 + 2x_D^5 y_C^5 y_D + 3x_D^4 y_C^6 - 4x_D^4 y_C^5 y_D - x_D^3 y_C^6 + 2x_D^3 y_C^5 y_D + x_D^2 y_C^6 y_D^2 + 2x_D^2 y_C^5 y_D^3 - 3x_D^2 y_C^4 y_D^4 - x_D y_C^6 y_D^2 + 3x_D y_C^4 y_D^4 - 2x_D y_C^3 y_D^5)/(4y_C y_D(y_C - y_D)(y_C + x_C y_D - x_D y_C)(y_D - x_C y_D + x_D y_C)(x_C^2 y_D^2 - 2x_C x_D y_C y_D + x_C y_C y_D - x_C y_C^2 + x_D^2 y_C^2 - x_D y_C^2 + x_D y_C y_D)))^2 - (((x_C y_D - x_D y_C)/(y_D - y_C) - (-(((2x_C - 1)y_C^2 - 2x_C^5 + 3x_C^4 - x_C^2)y_D^5 - y_C((2x_D + 4x_C - 3)y_C^2 - (x_C - 1)x_C^2(2(5x_C - 1)x_D - 3x_C - 1))y_D^4 + y_C^2((4x_D + 2x_C - 3)y_C^2 - 2x_C x_D((10x_C^2 - 9x_C + 1)x_D - (2x_C - 1)(3x_C - 1)))y_D^3 - y_C^3((2x_D - 1)y_C^2 - 2x_C x_D(x_D(2(5x_C - 3)x_D - 9x_C + 5) + x_C - 1))y_D^2 - (x_D - 1)x_D^2((10x_C - 3)x_D - 2x_C - 1)y_C^4 y_D + (x_D - 1)^2 x_D^2(2x_D + 1)y_C^5)/(4(y_D - y_C)((x_C - 1)y_D - (x_D - 1)y_C)((x_C - 1)y_D - x_D y_C)(x_C y_D - (x_D - 1)y_C)(x_C y_D - x_D y_C))))^2 + ((-x_C^6 y_D^6 + 4x_C^5 x_D y_C y_D^5 - 2x_C^5 y_C y_D^5 + 3x_C^5 y_D^6 - 5x_C^4 x_D^2 y_C^2 y_D^4 + 5x_C^4 x_D y_C^2 y_D^4 - 10x_C^4 x_D y_C y_D^5 + 4x_C^4 y_C y_D^5 - 3x_C^4 y_D^6 + 10x_C^3 x_D^2 y_C^2 y_D^4 - 10x_C^3 x_D y_C^2 y_D^4 + 8x_C^3 x_D y_C y_D^5 - 2x_C^3 y_C y_D^5 + x_C^3 y_D^6 + 5x_C^2 x_D^4 y_C^4 y_D^2 - 10x_C^2 x_D^3 y_C^4 y_D^2 + 5x_C^2 x_D^2 y_C^4 y_D^2 - 5x_C^2 x_D^2 y_C^2 y_D^4 + 5x_C^2 x_D y_C^2 y_D^4 - 2x_C^2 x_D y_C y_D^5 + 3x_C^2 y_C^4 y_D^4 - 2x_C^2 y_C^3 y_D^5 - x_C^2 y_C^2 y_D^6 - 4x_C x_D^5 y_C^5 y_D + 10x_C x_D^4 y_C^5 y_D - 5x_C x_D^4 y_C^4 y_D^2 - 8x_C x_D^3 y_C^5 y_D + 10x_C x_D^3 y_C^4 y_D^2 + 2x_C x_D^2 y_C^5 y_D - 5x_C x_D^2 y_C^4 y_D^2 - 4x_C x_D y_C^5 y_D^3 + 4x_C x_D y_C^3 y_D^5 + 2x_C y_C^5 y_D^3 - 3x_C y_C^4 y_D^4 + x_C y_C^2 y_D^6 + x_D^6 y_C^6 - 3x_D^5 y_C^6 + 2x_D^5 y_C^5 y_D + 3x_D^4 y_C^6 - 4x_D^4 y_C^5 y_D - x_D^3 y_C^6 + 2x_D^3 y_C^5 y_D + x_D^2 y_C^6 y_D^2 + 2x_D^2 y_C^5 y_D^3 - 3x_D^2 y_C^4 y_D^4 - x_D y_C^6 y_D^2 + 3x_D y_C^4 y_D^4 - 2x_D y_C^3 y_D^5)/(4y_C y_D(y_C - y_D)(y_C + x_C y_D - x_D y_C)(y_D - x_C y_D + x_D y_C)(x_C^2 y_D^2 - 2x_C x_D y_C y_D + x_C y_C y_D - x_C y_C^2 + x_D^2 y_C^2 - x_D y_C^2 + x_D y_C y_D)))^2) = 0.$
H has coordinates
$(-(((2x_C - 1)y_C^2 - x_C^4 + 2x_C^3 - x_C^2)y_D^4 - 2y_C((x_D + x_C - 1)y_C^2 - (x_C - 1)x_C^2(2x_D - 1))y_D^3 + y_C^2((2x_D - 1)y_C^2 - 2x_C x_D((3x_C - 2)x_D - 2x_C + 1))y_D^2 + 2(2x_C - 1)(x_D - 1)x_D^2 y_C^3 y_D - (x_D - 1)^2 x_D^2 y_C^4)/(2((x_C - 1)y_D - (x_D - 1)y_C)((x_C - 1)y_D - x_D y_C)(x_C y_D - (x_D - 1)y_C)(x_C y_D - x_D y_C)), 0)$.
I has coordinates
$(-(x_C(((2x_C - 1)y_C^2 - x_C^4 + 2x_C^3 - x_C^2)y_D^4 - 2y_C((x_D + x_C - 1)y_C^2 - (x_C - 1)x_C^2(2x_D - 1))y_D^3 + y_C^2((2x_D - 1)y_C^2 - 2x_C x_D((3x_C - 2)x_D - 2x_C + 1))y_D^2 + 2(2x_C - 1)(x_D - 1)x_D^2 y_C^3 y_D - (x_D - 1)^2 x_D^2 y_C^4))/(2(y_C^2 + x_C^2)y_D(y_D - y_C)((x_C - 1)y_D - (x_D - 1)y_C)((x_C - 1)y_D - x_D y_C)), -(y_C(((2x_C - 1)y_C^2 - x_C^4 + 2x_C^3 - x_C^2)y_D^4 - 2y_C((x_D + x_C - 1)y_C^2 - (x_C - 1)x_C^2(2x_D - 1))y_D^3 + y_C^2((2x_D - 1)y_C^2 - 2x_C x_D((3x_C - 2)x_D - 2x_C + 1))y_D^2 +$

$2(2x_C-1)(x_D-1)x_D^2y_C^3y_D-(x_D-1)^2x_D^2y_C^4))/(2(y_C^2+x_C^2)y_D(y_D-y_C)((x_C-1)y_D-(x_D-1)y_C)((x_C-1)y_D-x_Dy_C)))$.

J has coordinates

$((((4x_C^2-3x_C+1)y_C^2+x_C^5-x_C^4-x_C^3+x_C^2)y_D^4-2y_C(((3x_C-1)x_D+(x_C-1)(2x_C-1))y_C^2+2(x_C-1)^2x_C^2x_D)y_D^3+y_C^2((2x_D(x_D+3x_C-2)-3x_C+1)y_C^2+2(x_C-1)^2x_Cx_D(3x_D-2))y_D^2-2(x_D-1)x_Dy_C^3(y_C^2+(x_C-1)((2x_C-1)x_D-x_C))y_D+(x_C-1)(x_D-1)^2x_D^2y_C^4)/(2(y_C^2+(x_C-1)^2)y_D(y_D-y_C)(x_Cy_D-(x_D-1)y_C)(x_Cy_D-x_Dy_C)),(y_C(((2x_C-1)y_C^2+x_C^4-2x_C^3+x_C^2)y_D^4-2y_C((x_D+x_C-1)y_C^2+(x_C-1)^2x_C(2x_D-1))y_D^3+y_C^2((2x_D-1)y_C^2+2(x_C-1)x_D((3x_C-1)x_D-4x_C+1)+2(x_C-1)x_C)y_D^2-2(2x_C-1)(x_D-1)^2x_Dy_C^3y_D+(x_D-1)^2x_D^2y_C^4))/(2(y_C^2+(x_C-1)^2)y_D(y_D-y_C)(x_Cy_D-(x_D-1)y_C)(x_Cy_D-x_Dy_C)))$.

Line AJ is given by the equation

$y=x(((y_C(((2x_C-1)y_C^2+x_C^4-2x_C^3+x_C^2)y_D^4-2y_C((x_D+x_C-1)y_C^2+(x_C-1)^2x_C(2x_D-1))y_D^3+y_C^2((2x_D-1)y_C^2+2(x_C-1)x_D((3x_C-1)x_D-4x_C+1)+2(x_C-1)x_C)y_D^2-2(2x_C-1)(x_D-1)^2x_Dy_C^3y_D+(x_D-1)^2x_D^2y_C^4))/(2(y_C^2+(x_C-1)^2)y_D(y_D-y_C)(x_Cy_D-(x_D-1)y_C)(x_Cy_D-x_Dy_C)))/((((4x_C^2-3x_C+1)y_C^2+x_C^5-x_C^4-x_C^3+x_C^2)y_D^4-2y_C(((3x_C-1)x_D+(x_C-1)(2x_C-1))y_C^2+2(x_C-1)^2x_C^2x_D)y_D^3+y_C^2((2x_D(x_D+3x_C-2)-3x_C+1)y_C^2+2(x_C-1)^2x_Cx_D(3x_D-2))y_D^2-2(x_D-1)x_Dy_C^3(y_C^2+(x_C-1)((2x_C-1)x_D-x_C))y_D+(x_C-1)(x_D-1)^2x_D^2y_C^4)/(2(y_C^2+(x_C-1)^2)y_D(y_D-y_C)(x_Cy_D-(x_D-1)y_C)(x_Cy_D-x_Dy_C))))$.

Line BI is given by the equation

$y=-(y_C(((2x_C-1)y_C^2-x_C^4+2x_C^3-x_C^2)y_D^4-2y_C((x_D+x_C-1)y_C^2-(x_C-1)x_C^2(2x_D-1))y_D^3+y_C^2((2x_D-1)y_C^2-2x_Cx_D((3x_C-2)x_D-2x_C+1))y_D^2+2(2x_C-1)(x_D-1)x_D^2y_C^3y_D-(x_D-1)^2x_D^2y_C^4)(x-1))/(-x_C(((2x_C-1)y_C^2-x_C^4+2x_C^3-x_C^2)y_D^4-2y_C((x_D+x_C-1)y_C^2-(x_C-1)x_C^2(2x_D-1))y_D^3+y_C^2((2x_D-1)y_C^2-2x_Cx_D((3x_C-2)x_D-2x_C+1))y_D^2+2(2x_C-1)(x_D-1)x_D^2y_C^3y_D-(x_D-1)^2x_D^2y_C^4)-2(y_C^2+x_C^2)y_D(y_D-y_C)((x_C-1)y_D-(x_D-1)y_C)((x_C-1)y_D-x_Dy_C))$.

Line CH is given by the equation

$y=y_C-(y_C(x-x_C))/(-(((2x_C-1)y_C^2-x_C^4+2x_C^3-x_C^2)y_D^4-2y_C((x_D+x_C-1)y_C^2-(x_C-1)x_C^2(2x_D-1))y_D^3+y_C^2((2x_D-1)y_C^2-2x_Cx_D((3x_C-2)x_D-2x_C+1))y_D^2+2(2x_C-1)(x_D-1)x_D^2y_C^3y_D-(x_D-1)^2x_D^2y_C^4)/(2((x_C-1)y_D-(x_D-1)y_C)((x_C-1)y_D-x_Dy_C)(x_Cy_D-(x_D-1)y_C)(x_Cy_D-x_Dy_C))-x_C)$.

Lines AJ and BI intersect at the point K which has coordinates

$(-((x_C^4y_D^4-4x_C^3x_Dy_Cy_D^3+2x_C^3y_Cy_D^3-2x_C^3y_D^4+6x_C^2x_D^2y_C^2y_D^2-4x_C^2x_Dy_C^2y_D^2+4x_C^2x_Dy_Cy_D^3-2x_C^2y_Cy_D^3+x_C^2y_D^4-4x_Cx_D^3y_C^3y_D+4x_Cx_D^2y_C^3y_D-4x_Cx_D^2y_C^2y_D^2+2x_Cx_Dy_C^2y_D^2+2x_Cy_C^3y_D^3-2x_Cy_C^2y_D^4+x_D^4y_C^4-2x_D^3y_C^4+2x_D^3y_C^3y_D+x_D^2y_C^4-2x_D^2y_C^3y_D-2x_Dy_C^4y_D^2+2x_Dy_C^3y_D^3+y_C^4y_D^2-2y_C^3y_D^3+y_C^2y_D^4)(x_C^5y_D^4-4x_C^4x_Dy_Cy_D^3-x_C^4y_D^4+6x_C^3x_D^2y_C^2y_D^2-4x_C^3x_Dy_C^2y_D^2+8x_C^3x_Dy_Cy_D^3-x_C^3y_D^4-4x_C^2x_D^3y_C^3y_D+6x_C^2x_D^2y_C^3y_D-12x_C^2x_D^2y_C^2y_D^2-2x_C^2x_Dy_C^3y_D+8x_C^2x_Dy_C^2y_D^2-4x_C^2x_Dy_Cy_D^3-4x_C^2y_C^3y_D^3+4x_C^2y_C^2y_D^4+x_C^2y_D^4+x_Cx_D^4y_C^4-2x_Cx_D^3y_C^4+6x_Cx_D^3y_C^3y_D+x_Cx_D^2y_C^4-8x_Cx_D^2y_C^3y_D+6x_Cx_D^2y_C^2y_D^2+6x_Cx_Dy_C^4y_D^2-6x_Cx_Dy_C^3y_D^3+2x_Cx_Dy_C^3y_D-4x_Cx_Dy_C^2y_D^2-3x_Cy_C^4y_D^2+6x_Cy_C^3y_D^3-3x_Cy_C^2y_D^4-x_D^4y_C^4+2x_D^3y_C^4-2x_D^3y_C^3y_D-2x_D^2y_C^5y_D+2x_D^2y_C^4y_D^2-x_D^2y_C^4+2x_D^2y_C^3y_D+2x_Dy_C^5y_D-4x_Dy_C^4y_D^2+2x_Dy_C^3y_D^3+y_C^4y_D^2-2y_C^3y_D^3+y_C^2y_D^4))/(4x_C^8y_Cy_D^7-3x_C^8y_D^8-24x_C^7x_Dy_C^2y_D^6+16x_C^7x_Dy_Cy_D^7+12x_C^7y_C^2y_D^6-24x_C^7y_Cy_D^7+12x_C^7y_D^8+60x_C^6x_D^2y_C^3y_D^5-32x_C^6x_D^2y_C^2y_D^6-60x_C^6x_Dy_C^3y_D^5+116x_C^6x_Dy_C^2y_D^6-56x_C^6x_Dy_Cy_D^7+4x_C^6y_C^3y_D^7+12x_C^6y_C^3y_D^5-4x_C^6y_C^2y_D^8-48x_C^6y_C^2y_D^6+52x_C^6y_Cy_D^7-18x_C^6y_D^8-80x_C^5x_D^3y_C^4y_D^4+24x_C^5x_D^3y_C^3y_D^5+120x_C^5x_D^2y_C^4y_D^4-216x_C^5x_D^2y_C^3y_D^5+96x_C^5x_D^2y_C^2y_D^6-24x_C^5x_Dy_C^4y_D^6-48x_C^5x_Dy_C^4y_D^4+24x_C^5x_Dy_C^3y_D^7+192x_C^5x_Dy_C^3y_D^5-204x_C^5x_Dy_C^2y_D^6+72x_C^5x_Dy_Cy_D^7+12x_C^5y_C^4y_D^6+4x_C^5y_C^4y_D^4-24x_C^5y_C^3y_D^7-36x_C^5y_C^3y_D^5+12x_C^5y_C^2y_D^8+72x_C^5y_C^2y_D^6-52x_C^5y_Cy_D^7+12x_C^5y_D^8+60x_C^4x_D^4y_C^5y_D^3+$

$10x_C^4x_D^4y_C^4y_D^4 - 120x_C^4x_D^3y_C^5y_D^3 + 180x_C^4x_D^3y_C^4y_D^4 - 60x_C^4x_D^3y_C^3y_D^5 + 60x_C^4x_D^2y_C^5y_D^5 +$
$72x_C^4x_D^2y_C^5y_D^3 - 60x_C^4x_D^2y_C^4y_D^6 - 278x_C^4x_D^2y_C^4y_D^4 + 276x_C^4x_D^2y_C^3y_D^5 - 100x_C^4x_D^2y_C^2y_D^6 -$
$60x_C^4x_Dy_C^5y_D^5 - 12x_C^4x_Dy_C^5y_D^3 + 120x_C^4x_Dy_C^4y_D^6 + 108x_C^4x_Dy_C^4y_D^4 - 60x_C^4x_Dy_C^3y_D^7 -$
$216x_C^4x_Dy_C^3y_D^5 + 160x_C^4x_Dy_C^2y_D^6 - 40x_C^4x_Dy_Cy_D^7 + 12x_C^4y_C^5y_D^5 - 42x_C^4y_C^4y_D^6 - 8x_C^4y_C^4y_D^4 +$
$44x_C^4y_C^3y_D^7 + 36x_C^4y_C^3y_D^5 - 14x_C^4y_C^2y_D^8 - 48x_C^4y_C^2y_D^6 + 24x_C^4y_Cy_D^7 - 3x_C^4y_D^8 - 24x_C^3x_D^5y_C^6y_D^2 -$
$32x_C^3x_D^5y_C^5y_D^3 + 60x_C^3x_D^4y_C^6y_D^2 - 40x_C^3x_D^4y_C^5y_D^3 - 20x_C^3x_D^4y_C^4y_D^4 - 80x_C^3x_D^3y_C^6y_D^4 -$
$48x_C^3x_D^3y_C^6y_D^2 + 80x_C^3x_D^3y_C^5y_D^5 + 160x_C^3x_D^3y_C^5y_D^3 - 120x_C^3x_D^3y_C^4y_D^4 + 48x_C^3x_D^3y_C^3y_D^5 +$
$120x_C^3x_D^2y_C^6y_D^4 + 12x_C^3x_D^2y_C^6y_D^2 - 240x_C^3x_D^2y_C^5y_D^5 - 104x_C^3x_D^2y_C^5y_D^3 + 120x_C^3x_D^2y_C^4y_D^6 +$
$196x_C^3x_D^2y_C^4y_D^4 - 144x_C^3x_D^2y_C^3y_D^5 + 40x_C^3x_D^2y_C^2y_D^6 - 48x_C^3x_Dy_C^6y_D^4 + 176x_C^3x_Dy_C^5y_D^5 +$
$16x_C^3x_Dy_C^5y_D^3 - 192x_C^3x_Dy_C^4y_D^6 - 72x_C^3x_Dy_C^4y_D^4 + 64x_C^3x_Dy_C^3y_D^7 + 96x_C^3x_Dy_C^3y_D^5 -$
$52x_C^3x_Dy_C^2y_D^6 + 8x_C^3x_Dy_Cy_D^7 + 4x_C^3y_C^6y_D^4 - 32x_C^3y_C^5y_D^5 + 60x_C^3y_C^4y_D^6 + 4x_C^3y_C^4y_D^4 -$
$40x_C^3y_C^3y_D^7 - 12x_C^3y_C^3y_D^5 + 8x_C^3y_C^2y_D^8 + 12x_C^3y_C^2y_D^6 - 4x_C^3y_Cy_D^7 + 4x_C^2x_C^6y_Cy_D^7 + 24x_C^2x_D^6y_C^6y_D^2 -$
$12x_C^2x_D^5y_C^7y_D - 36x_C^2x_D^5y_C^6y_D^2 + 48x_C^2x_D^5y_C^5y_D^3 + 60x_C^2x_D^4y_C^7y_D^3 + 12x_C^2x_D^4y_C^7y_D -$
$60x_C^2x_D^4y_C^6y_D^4 - 8x_C^2x_D^4y_C^6y_D^2 - 44x_C^2x_D^4y_C^5y_D^3 + 10x_C^2x_D^4y_C^4y_D^4 - 120x_C^2x_D^3y_C^7y_D^3 -$
$4x_C^2x_D^3y_C^7y_D + 240x_C^2x_D^3y_C^6y_D^4 + 28x_C^2x_D^3y_C^6y_D^2 - 120x_C^2x_D^3y_C^5y_D^5 - 32x_C^2x_D^3y_C^5y_D^3 +$
$20x_C^2x_D^3y_C^4y_D^4 - 12x_C^2x_D^3y_C^3y_D^5 + 72x_C^2x_D^2y_C^7y_D^3 - 268x_C^2x_D^2y_C^6y_D^4 - 8x_C^2x_D^2y_C^6y_D^2 +$
$296x_C^2x_D^2y_C^5y_D^5 + 32x_C^2x_D^2y_C^5y_D^3 - 100x_C^2x_D^2y_C^4y_D^6 - 38x_C^2x_D^2y_C^4y_D^4 + 24x_C^2x_D^2y_C^3y_D^5 -$
$4x_C^2x_D^2y_C^2y_D^6 - 12x_C^2x_Dy_C^7y_D^3 + 100x_C^2x_Dy_C^6y_D^4 - 200x_C^2x_Dy_C^5y_D^5 - 4x_C^2x_Dy_C^5y_D^3 +$
$148x_C^2x_Dy_C^4y_D^6 + 12x_C^2x_Dy_C^4y_D^4 - 36x_C^2x_Dy_C^3y_D^7 - 12x_C^2x_Dy_C^3y_D^5 + 4x_C^2x_Dy_C^2y_D^6 +$
$4x_C^2y_C^6y_D^6 - 8x_C^2y_C^6y_D^4 - 8x_C^2y_C^5y_D^7 + 32x_C^2y_C^5y_D^5 + 4x_C^2y_C^4y_D^8 - 42x_C^2y_C^4y_D^6 + 20x_C^2y_C^3y_D^7 -$
$2x_C^2y_C^2y_D^8 - 8x_Cx_D^7y_C^7y_D + 24x_Cx_D^6y_C^7y_D - 24x_Cx_D^6y_C^6y_D^2 - 24x_Cx_D^5y_C^8y_D^2 + 24x_Cx_D^5y_C^7y_D^3 -$
$24x_Cx_D^5y_C^7y_D + 52x_Cx_D^5y_C^6y_D^2 - 16x_Cx_D^5y_C^5y_D^3 + 60x_Cx_D^4y_C^8y_D^2 - 120x_Cx_D^4y_C^7y_D^3 +$
$8x_Cx_D^4y_C^7y_D + 60x_Cx_D^4y_C^6y_D^4 - 32x_Cx_D^4y_C^6y_D^2 + 24x_Cx_D^4y_C^5y_D^3 - 48x_Cx_D^3y_C^8y_D^2 + 176x_Cx_D^3y_C^7y_D^3 -$
$192x_Cx_D^3y_C^6y_D^4 + 4x_Cx_D^3y_C^6y_D^2 + 64x_Cx_D^3y_C^5y_D^5 - 8x_Cx_D^3y_C^5y_D^3 + 12x_Cx_D^2y_C^8y_D^2 - 96x_Cx_D^2y_C^7y_D^3 +$
$196x_Cx_D^2y_C^6y_D^4 - 152x_Cx_D^2y_C^5y_D^5 + 40x_Cx_D^2y_C^4y_D^6 - 8x_Cx_Dy_C^7y_D^5 + 16x_Cx_Dy_C^7y_D^3 +$
$16x_Cx_Dy_C^6y_D^6 - 68x_Cx_Dy_C^6y_D^4 - 8x_Cx_Dy_C^5y_D^7 + 96x_Cx_Dy_C^5y_D^5 - 52x_Cx_Dy_C^4y_D^6 + 8x_Cx_Dy_C^3y_D^7 +$
$4x_Cy_C^7y_D^5 - 12x_Cy_C^6y_D^6 + 4x_Cy_C^6y_D^4 + 12x_Cy_C^5y_D^7 - 12x_Cy_C^5y_D^5 - 4x_Cy_C^4y_D^8 + 12x_Cy_C^4y_D^6 -$
$4x_Cy_C^3y_D^7 + x_D^8y_C^8 - 4x_D^7y_C^8 + 4x_D^7y_C^7y_D + 4x_D^6y_C^9y_D - 4x_D^6y_C^8y_D^2 + 6x_D^6y_C^8 - 12x_D^6y_C^7y_D +$
$4x_D^6y_C^6y_D^2 - 12x_D^5y_C^9y_D + 24x_D^5y_C^8y_D^2 - 4x_D^5y_C^8 - 12x_D^5y_C^7y_D^3 + 12x_D^5y_C^7y_D - 8x_D^5y_C^6y_D^2 +$
$12x_D^4y_C^9y_D - 42x_D^4y_C^8y_D^2 + x_D^4y_C^8 + 44x_D^4y_C^7y_D^3 - 4x_D^4y_C^7y_D - 14x_D^4y_C^6y_D^4 + 4x_D^4y_C^6y_D^2 -$
$4x_D^3y_C^9y_D + 28x_D^3y_C^8y_D^2 - 56x_D^3y_C^7y_D^3 + 44x_D^3y_C^6y_D^4 - 12x_D^3y_C^5y_D^5 + 4x_D^2y_C^8y_D^4 - 6x_D^2y_C^8y_D^2 -$
$8x_D^2y_C^7y_D^5 + 28x_D^2y_C^7y_D^3 + 4x_D^2y_C^6y_D^6 - 42x_D^2y_C^6y_D^4 + 24x_D^2y_C^5y_D^5 - 4x_D^2y_C^4y_D^6 - 4x_Dy_C^8y_D^4 +$
$12x_Dy_C^7y_D^5 - 4x_Dy_C^7y_D^3 - 12x_Dy_C^6y_D^6 + 12x_Dy_C^6y_D^4 + 4x_Dy_C^5y_D^7 - 12x_Dy_C^5y_D^5 + 4x_Dy_C^4y_D^6 +$
$y_C^8y_D^4 - 4y_C^7y_D^5 + 6y_C^6y_D^6 - 4y_C^5y_D^7 + y_C^4y_D^8), -(y_C(x_C^4y_D^4 - 4x_C^3x_Dy_Cy_D^3 + 2x_C^3y_Cy_D^3 -$
$2x_C^3y_D^4 + 6x_C^2x_D^2y_C^2y_D^2 - 4x_C^2x_Dy_C^2y_D^2 + 4x_C^2x_Dy_Cy_D^3 - 2x_C^2y_Cy_D^3 + x_C^2y_D^4 - 4x_Cx_D^3y_C^3y_D +$
$4x_Cx_D^2y_C^3y_D - 4x_Cx_D^2y_C^2y_D^2 + 2x_Cx_Dy_C^2y_D^2 + 2x_Cy_C^3y_D^3 - 2x_Cy_C^2y_D^4 + x_D^4y_C^4 - 2x_D^3y_C^4 +$
$2x_D^3y_C^3y_D + x_D^2y_C^4 - 2x_D^2y_C^3y_D - 2x_Dy_C^4y_D^2 + 2x_Dy_C^3y_D^3 + y_C^4y_D^2 - 2y_C^3y_D^3 + y_C^2y_D^4)(x_C^4y_D^4 -$
$4x_C^3x_Dy_Cy_D^3 + 2x_C^3y_Cy_D^3 - 2x_C^3y_D^4 + 6x_C^2x_D^2y_C^2y_D^2 - 8x_C^2x_Dy_C^2y_D^2 + 8x_C^2x_Dy_Cy_D^3 +$
$2x_C^2y_C^2y_D^2 - 4x_C^2y_Cy_D^3 + x_C^2y_D^4 - 4x_Cx_D^3y_C^3y_D + 8x_Cx_D^2y_C^3y_D - 8x_Cx_D^2y_C^2y_D^2 - 4x_Cx_Dy_C^3y_D +$
$10x_Cx_Dy_C^2y_D^2 - 4x_Cx_Dy_Cy_D^3 - 2x_Cy_C^3y_D^3 + 2x_Cy_C^2y_D^4 - 2x_Cy_C^2y_D^2 + 2x_Cy_Cy_D^3 + x_D^4y_C^4 -$
$2x_D^3y_C^4 + 2x_D^3y_C^3y_D + x_D^2y_C^4 - 4x_D^2y_C^3y_D + 2x_D^2y_C^2y_D^2 + 2x_Dy_C^4y_D^2 - 2x_Dy_C^3y_D^3 + 2x_Dy_C^3y_D -$
$2x_Dy_C^2y_D^2 - y_C^4y_D^2 + 2y_C^3y_D^3 - y_C^2y_D^4))/(4x_C^8y_Cy_D^7 - 3x_C^8y_D^8 - 24x_C^7x_Dy_C^2y_D^6 + 16x_C^7x_Dy_Cy_D^7 +$
$12x_C^7y_C^2y_D^6 - 24x_C^7y_Cy_D^7 + 12x_C^7y_D^8 + 60x_C^6x_D^2y_C^3y_D^5 - 32x_C^6x_D^2y_C^2y_D^6 - 60x_C^6x_Dy_C^3y_D^5 +$
$116x_C^6x_Dy_C^2y_D^6 - 56x_C^6x_Dy_Cy_D^7 + 4x_C^6y_C^3y_D^7 + 12x_C^6y_C^3y_D^5 - 4x_C^6y_C^2y_D^8 - 48x_C^6y_C^2y_D^6 +$
$52x_C^6y_Cy_D^7 - 18x_C^6y_D^8 - 80x_C^5x_D^3y_C^4y_D^4 + 24x_C^5x_D^3y_C^3y_D^5 + 120x_C^5x_D^2y_C^4y_D^4 - 216x_C^5x_D^2y_C^3y_D^5 +$
$96x_C^5x_D^2y_C^2y_D^6 - 24x_C^5x_Dy_C^4y_D^6 - 48x_C^5x_Dy_C^4y_D^4 + 24x_C^5x_Dy_C^3y_D^7 + 192x_C^5x_Dy_C^3y_D^5 -$
$204x_C^5x_Dy_C^2y_D^6 + 72x_C^5x_Dy_Cy_D^7 + 12x_C^5y_C^4y_D^6 + 4x_C^5y_C^4y_D^4 - 24x_C^5y_C^3y_D^7 - 36x_C^5y_C^3y_D^5 +$
$12x_C^5y_C^2y_D^8 + 72x_C^5y_C^2y_D^6 - 52x_C^5y_Cy_D^7 + 12x_C^5y_D^8 + 60x_C^4x_D^4y_C^5y_D^3 + 10x_C^4x_D^4y_C^4y_D^4 -$
$120x_C^4x_D^3y_C^5y_D^3 + 180x_C^4x_D^3y_C^4y_D^4 - 60x_C^4x_D^3y_C^3y_D^5 + 60x_C^4x_D^2y_C^5y_D^5 + 72x_C^4x_D^2y_C^5y_D^3 -$

$60x_C^4x_D^2y_C^4y_D^6 - 278x_C^4x_D^2y_C^4y_D^4 + 276x_C^4x_D^2y_C^3y_D^5 - 100x_C^4x_D^2y_C^2y_D^6 - 60x_C^4x_Dy_C^5y_D^5 -$
$12x_C^4x_Dy_C^5y_D^3 + 120x_C^4x_Dy_C^4y_D^6 + 108x_C^4x_Dy_C^4y_D^4 - 60x_C^4x_Dy_C^3y_D^7 - 216x_C^4x_Dy_C^3y_D^5 +$
$160x_C^4x_Dy_C^2y_D^6 - 40x_C^4x_Dy_Cy_D^7 + 12x_C^4y_C^5y_D^5 - 42x_C^4y_C^4y_D^6 - 8x_C^4y_C^4y_D^4 + 44x_C^4y_C^3y_D^7 +$
$36x_C^4y_C^3y_D^5 - 14x_C^4y_C^2y_D^8 - 48x_C^4y_C^2y_D^6 + 24x_C^4y_Cy_D^7 - 3x_C^4y_D^8 - 24x_C^3x_D^5y_C^6y_D^2 - 32x_C^3x_D^5y_C^5y_D^3 +$
$60x_C^3x_D^4y_C^6y_D^2 - 40x_C^3x_D^4y_C^5y_D^3 - 20x_C^3x_D^4y_C^4y_D^4 - 80x_C^3x_D^3y_C^6y_D^4 - 48x_C^3x_D^3y_C^6y_D^2 +$
$80x_C^3x_D^3y_C^5y_D^5 + 160x_C^3x_D^3y_C^5y_D^3 - 120x_C^3x_D^3y_C^4y_D^4 + 48x_C^3x_D^3y_C^3y_D^5 + 120x_C^3x_D^2y_C^6y_D^4 +$
$12x_C^3x_D^2y_C^6y_D^2 - 240x_C^3x_D^2y_C^5y_D^5 - 104x_C^3x_D^2y_C^5y_D^3 + 120x_C^3x_D^2y_C^4y_D^6 + 196x_C^3x_D^2y_C^4y_D^4 -$
$144x_C^3x_D^2y_C^3y_D^5 + 40x_C^3x_D^2y_C^2y_D^6 - 48x_C^3x_Dy_C^6y_D^4 + 176x_C^3x_Dy_C^5y_D^5 + 16x_C^3x_Dy_C^5y_D^3 -$
$192x_C^3x_Dy_C^4y_D^6 - 72x_C^3x_Dy_C^4y_D^4 + 64x_C^3x_Dy_C^3y_D^7 + 96x_C^3x_Dy_C^3y_D^5 - 52x_C^3x_Dy_C^2y_D^6 +$
$8x_C^3x_Dy_Cy_D^7 + 4x_C^3y_C^6y_D^4 - 32x_C^3y_C^5y_D^5 + 60x_C^3y_C^4y_D^6 + 4x_C^3y_C^4y_D^4 - 40x_C^3y_C^3y_D^7 - 12x_C^3y_C^3y_D^5 +$
$8x_C^3y_C^2y_D^8 + 12x_C^3y_C^2y_D^6 - 4x_C^3y_Cy_D^7 + 4x_C^2x_D^6y_C^7y_D + 24x_C^2x_D^6y_C^6y_D^2 - 12x_C^2x_D^5y_C^7y_D -$
$36x_C^2x_D^5y_C^6y_D^2 + 48x_C^2x_D^5y_C^5y_D^3 + 60x_C^2x_D^4y_C^7y_D^3 + 12x_C^2x_D^4y_C^7y_D - 60x_C^2x_D^4y_C^6y_D^4 -$
$8x_C^2x_D^4y_C^6y_D^2 - 44x_C^2x_D^4y_C^5y_D^3 + 10x_C^2x_D^4y_C^4y_D^4 - 120x_C^2x_D^3y_C^7y_D^3 - 4x_C^2x_D^3y_C^7y_D + 240x_C^2x_D^3y_C^6y_D^4 +$
$28x_C^2x_D^3y_C^6y_D^2 - 120x_C^2x_D^3y_C^5y_D^5 - 32x_C^2x_D^3y_C^5y_D^3 + 20x_C^2x_D^3y_C^4y_D^4 - 12x_C^2x_D^3y_C^3y_D^5 +$
$72x_C^2x_D^2y_C^7y_D^3 - 268x_C^2x_D^2y_C^6y_D^4 - 8x_C^2x_D^2y_C^6y_D^2 + 296x_C^2x_D^2y_C^5y_D^5 + 32x_C^2x_D^2y_C^5y_D^3 -$
$100x_C^2x_D^2y_C^4y_D^6 - 38x_C^2x_D^2y_C^4y_D^4 + 24x_C^2x_D^2y_C^3y_D^5 - 4x_C^2x_D^2y_C^2y_D^6 - 12x_C^2x_Dy_C^7y_D^3 +$
$100x_C^2x_Dy_C^6y_D^4 - 200x_C^2x_Dy_C^5y_D^5 - 4x_C^2x_Dy_C^5y_D^3 + 148x_C^2x_Dy_C^4y_D^6 + 12x_C^2x_Dy_C^4y_D^4 -$
$36x_C^2x_Dy_C^3y_D^7 - 12x_C^2x_Dy_C^3y_D^5 + 4x_C^2x_Dy_C^2y_D^6 + 4x_C^2y_C^6y_D^6 - 8x_C^2y_C^6y_D^4 - 8x_C^2y_C^5y_D^7 +$
$32x_C^2y_C^5y_D^5 + 4x_C^2y_C^4y_D^8 - 42x_C^2y_C^4y_D^6 + 20x_C^2y_C^3y_D^7 - 2x_C^2y_C^2y_D^8 - 8x_Cx_D^7y_C^7y_D + 24x_Cx_D^6y_C^7y_D -$
$24x_Cx_D^6y_C^6y_D^2 - 24x_Cx_D^5y_C^8y_D^2 + 24x_Cx_D^5y_C^7y_D^3 - 24x_Cx_D^5y_C^7y_D + 52x_Cx_D^5y_C^6y_D^2 -$
$16x_Cx_D^5y_C^5y_D^3 + 60x_Cx_D^4y_C^8y_D^2 - 120x_Cx_D^4y_C^7y_D^3 + 8x_Cx_D^4y_C^7y_D + 60x_Cx_D^4y_C^6y_D^4 -$
$32x_Cx_D^4y_C^6y_D^2 + 24x_Cx_D^4y_C^5y_D^3 - 48x_Cx_D^3y_C^8y_D^2 + 176x_Cx_D^3y_C^7y_D^3 - 192x_Cx_D^3y_C^6y_D^4 +$
$4x_Cx_D^3y_C^6y_D^2 + 64x_Cx_D^3y_C^5y_D^5 - 8x_Cx_D^3y_C^5y_D^3 + 12x_Cx_D^2y_C^8y_D^2 - 96x_Cx_D^2y_C^7y_D^3 + 196x_Cx_D^2y_C^6y_D^4 -$
$152x_Cx_D^2y_C^5y_D^5 + 40x_Cx_D^2y_C^4y_D^6 - 8x_Cx_Dy_C^7y_D^5 + 16x_Cx_Dy_C^7y_D^3 + 16x_Cx_Dy_C^6y_D^6 -$
$68x_Cx_Dy_C^6y_D^4 - 8x_Cx_Dy_C^5y_D^7 + 96x_Cx_Dy_C^5y_D^5 - 52x_Cx_Dy_C^4y_D^6 + 8x_Cx_Dy_C^3y_D^7 + 4x_Cy_C^7y_D^5 -$
$12x_Cy_C^6y_D^6 + 4x_Cy_C^6y_D^4 + 12x_Cy_C^5y_D^7 - 12x_Cy_C^5y_D^5 - 4x_Cy_C^4y_D^8 + 12x_Cy_C^4y_D^6 - 4x_Cy_C^3y_D^7 +$
$x_D^8y_C^8 - 4x_D^7y_C^8 + 4x_D^7y_C^7y_D + 4x_D^6y_C^9y_D - 4x_D^6y_C^8y_D^2 + 6x_D^6y_C^8 - 12x_D^6y_C^7y_D + 4x_D^6y_C^6y_D^2 -$
$12x_D^5y_C^9y_D + 24x_D^5y_C^8y_D^2 - 4x_D^5y_C^8 - 12x_D^5y_C^7y_D^3 + 12x_D^5y_C^7y_D - 8x_D^5y_C^6y_D^2 + 12x_D^4y_C^9y_D -$
$42x_D^4y_C^8y_D^2 + x_D^4y_C^8 + 44x_D^4y_C^7y_D^3 - 4x_D^4y_C^7y_D - 14x_D^4y_C^6y_D^4 + 4x_D^4y_C^6y_D^2 - 4x_D^3y_C^9y_D +$
$28x_D^3y_C^8y_D^2 - 56x_D^3y_C^7y_D^3 + 44x_D^3y_C^6y_D^4 - 12x_D^3y_C^5y_D^5 + 4x_D^2y_C^8y_D^4 - 6x_D^2y_C^8y_D^2 - 8x_D^2y_C^7y_D^5 +$
$28x_D^2y_C^7y_D^3 + 4x_D^2y_C^6y_D^6 - 42x_D^2y_C^6y_D^4 + 24x_D^2y_C^5y_D^5 - 4x_D^2y_C^4y_D^6 - 4x_Dy_C^8y_D^4 + 12x_Dy_C^7y_D^5 -$
$4x_Dy_C^7y_D^3 - 12x_Dy_C^6y_D^6 + 12x_Dy_C^6y_D^4 + 4x_Dy_C^5y_D^7 - 12x_Dy_C^5y_D^5 + 4x_Dy_C^4y_D^6 + y_C^8y_D^4 -$
$4y_C^7y_D^5 + 6y_C^6y_D^6 - 4y_C^5y_D^7 + y_C^4y_D^8))$.

We then verify that this point lies on line CH since we subsitute the above values and calculate that

$y_C - (y_C(x_K - x_C))/(-(((2x_C - 1)y_C^2 - x_C^4 + 2x_C^3 - x_C^2)y_D^4 - 2y_C((x_D + x_C - 1)y_C^2 -$
$(x_C - 1)x_C^2(2x_D - 1))y_D^3 + y_C^2((2x_D - 1)y_C^2 - 2x_Cx_D((3x_C - 2)x_D - 2x_C + 1))y_D^2 +$
$2(2x_C - 1)(x_D - 1)x_D^2y_C^3y_D - (x_D - 1)^2x_D^2y_C^4)/(2((x_C - 1)y_D - (x_D - 1)y_C)((x_C -$
$1)y_D - x_Dy_C)(x_Cy_D - (x_D - 1)y_C)(x_Cy_D - x_Dy_C)) - x_C) - y_K = 0$ as desired. □

The theorem of the Gergonne point follows when the circle in the construction is taken to be the incircle, for which D and the mentioned point of concurrence coincide.

References

[1] Weisstein, Eric W. "Gergonne Point." From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/GergonnePoint.html

[2] Warren, B.L., An Extension of the Van Lamoen Circle, under review *International Journal of Computer Discovered Mathematics.*

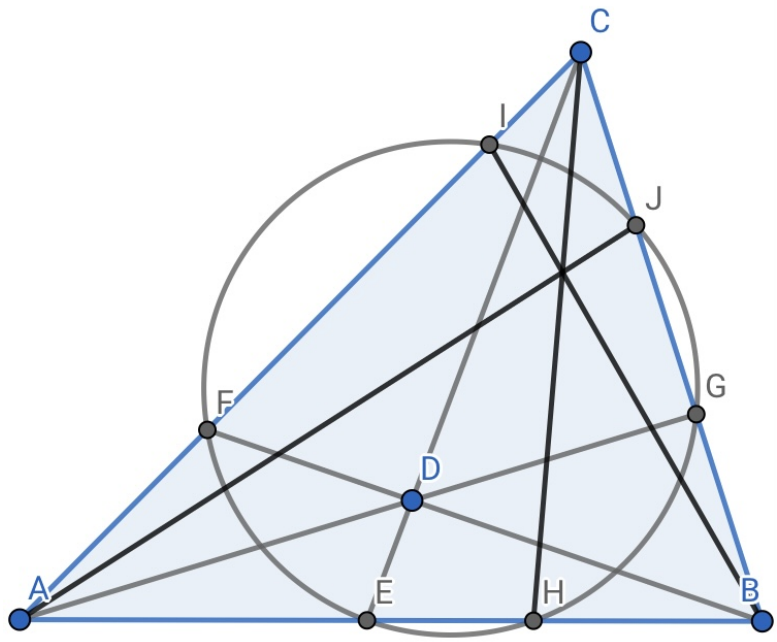

FIGURE 1. Illustration of theorem 2.1.