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An Extension of the Van Lamoen Circle

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1. Introduction

In [1], a generalization of the Van Lamoen circle theorem was proven. Here, we give a generalization of this result.

2. Results

Theorem 2.1. *Let A, B, and C be the vertices of a given triangle.*
Let D be any point in the plane.
Let lines AB and CD intersect at E.
Let lines AC and BD intersect at F.
Let lines AD and BC intersect at G.
Let H be a point on line DE.
Let I be the point on line DF such that $DI : FI = DH : EH$.
Let J be the point on line DG such that $DJ : GJ = DH : EH$.
Let K be the circumcenter of A, D, and I.
Let L be the circumcenter of A, D, and H.
Let M be the circumcenter of C, D, and J.
Let N be the circumcenter of C, D, and I.
Let O be the circumcenter of B, D, and H.
Let P be the circumcenter of B, D, and J.
Then K, L, M, N, O, and P lie on the same conic section.

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Proof. We borrow some phrasing and results from [1] and [2].
A has coordinates $(0,0)$.
B has coordinates $(1,0)$.
C has coordinates (x_C, y_C).
D has coordinates (x_D, y_D).
E has coordinates $(\frac{x_C y_D - x_D y_C}{y_D - y_C}, 0)$.
F has coordinates $(\frac{x_C y_D}{x_C y_D - x_D y_C + y_C}, \frac{y_C y_D}{x_C y_D - x_D y_C + y_C})$.
G has coordinates $(\frac{x_D y_C}{-x_C y_D + x_D y_C + y_D}, \frac{y_C y_D}{-x_C y_D + x_D y_C + y_D})$.
H has coordinates $(x_H, y_C + (\frac{y_C - y_D}{x_C - x_D})(x_H - x_C))$.
I has coordinates
$(x_D - \frac{(x_D - x_H)(x_D - (\frac{x_C y_D}{x_C y_D - x_D y_C + y_C}))}{x_D - (\frac{x_C y_D - x_D y_C}{y_D - y_C})}, (\frac{y_D}{x_D - 1})(x_D - \frac{(x_D - x_H)(x_D - (\frac{x_C y_D}{x_C y_D - x_D y_C + y_C}))}{x_D - (\frac{x_C y_D - x_D y_C}{y_D - y_C})} - 1))$.
J has coordinates
$(x_D + \frac{(x_D - x_H)(\frac{x_D y_C}{-x_C y_D + x_D y_C + y_D} - x_D)}{x_D - (\frac{x_C y_D - x_D y_C}{y_D - y_C})}, (\frac{y_D}{x_D})(x_D + \frac{(x_D - x_H)(\frac{x_D y_C}{-x_C y_D + x_D y_C + y_D} - x_D)}{x_D - (\frac{x_C y_D - x_D y_C}{y_D - y_C})}))$.
K has coordinates
$x_K = -(x_D^2 y_C^2 - 2x_D^3 y_C^2 + x_C^2 y_D^4 + x_D^4 y_C^2 + x_C x_D y_D^2 - x_C x_H y_D^2 - x_D x_H y_C^2 - x_C x_H y_D^4 + x_C y_C y_D^3 + x_D^2 y_C y_D - x_D y_C y_D^3 - x_D^3 y_C y_D - 2x_C^2 x_D y_D^2 + 2x_D^2 x_H y_C^2 - x_D^3 x_H y_C^2 + x_C^2 x_D^2 y_D^2 + x_D^2 y_C^2 y_D^2 - x_C x_D^2 x_H y_D^2 - x_D x_H y_C^2 y_D^2 - 3x_C x_D y_C y_D + x_C x_H y_C y_D + x_D x_H y_C y_D + 2x_C x_D x_H y_D^2 + 5x_C x_D^2 y_C y_D - 2x_C x_D y_C y_D^3 - 2x_C x_D^3 y_C y_D + x_C x_H y_C y_D^3 - 2x_D^2 x_H y_C y_D + x_D x_H y_C y_D^3 + x_D^3 x_H y_C y_D + x_C x_D^2 x_H y_C y_D - 2x_C x_D x_H y_C y_D)/(2y_D(x_C - x_D)(y_C + x_C y_D - x_D y_C))$
and
$y_K = (x_D^3 y_C^2 + x_C^2 y_D^4 - 2x_D^4 y_C^2 + x_D^5 y_C^2 - x_C x_D y_D^4 + x_C y_C y_D^3 - x_D y_C y_D^3 + x_C x_D^2 y_D^2 - x_C x_D^3 y_D^2 + x_C^2 x_D y_D^4 - x_D^2 x_H y_C^2 + 2x_D^3 x_H y_C^2 - x_D^4 x_H y_C^2 - x_C^2 x_D^2 y_D^2 + x_C^2 x_D^3 y_D^2 + x_D^3 y_C^2 y_D^2 + 2x_C x_D^2 x_H y_D^2 - x_C x_D^3 x_H y_D^2 - 2x_C x_D^2 y_C y_D^3 + x_D^2 x_H y_C y_D^3 - x_D^2 x_H y_C^2 y_D^2 - x_C x_D x_H y_D^2 - x_C x_D x_H y_D^4 - 2x_C x_D^2 y_C y_D + 4x_C x_D^3 y_C y_D - 2x_C x_D^4 y_C y_D + x_D^2 x_H y_C y_D - 2x_D^3 x_H y_C y_D + x_D^4 x_H y_C y_D - 2x_C x_D^2 x_H y_C y_D + x_C x_D x_H y_C y_D^3 + x_C x_D^3 x_H y_C y_D + x_C x_D x_H y_C y_D)/(2y_D^2(x_C - x_D)(y_C + x_C y_D - x_D y_C))$.
L has coordinates
$x_L = (x_C^2 x_D y_D + x_H x_C^2 y_D - x_C x_D^2 y_C - x_C x_D^2 y_D - 2x_H x_C x_D y_D + x_C y_C y_D^2 - x_C y_D^3 + x_D^3 y_C + x_H x_D^2 y_D - x_D y_C^2 y_D + x_D y_C y_D^2 + x_H y_C^2 y_D - 2x_H y_C y_D^2 + x_H y_D^3)/(2(x_C y_D - x_D y_C)(x_C - x_D))$
and
$y_L = -(x_H x_C^2 x_D - x_C^2 y_D^2 - 2x_H x_C x_D^2 + 2x_C x_D y_C y_D + x_H x_D^3 - x_D^2 y_C^2 + x_H x_D y_C^2 - 2x_H x_D y_C y_D + x_H x_D y_D^2)/(2(x_C y_D - x_D y_C)(x_C - x_D))$.
M has coordinates
$x_M = (x_C^3 y_D^3 - x_D^3 y_C^3 - x_C^2 y_D^5 - x_C^4 y_D^3 + x_D^4 y_C^3 + x_C y_D^5 - x_H y_D^5 + x_C x_H y_D^5 - 2x_C y_C y_D^4 - x_D y_C y_D^4 + 3x_H y_C y_D^4 + x_C x_D^2 y_D^3 - x_C^2 x_D y_D^3 + x_C^3 x_D y_D^3 + x_D^2 x_H y_C^3 - x_D^3 x_H y_C^3 - x_C^2 x_H y_D^3 + x_C y_C^2 y_D^3 + 2x_C^2 y_C y_D^4 + 2x_D y_C^2 y_D^3 - x_D y_C^3 y_D^2 - x_C^3 y_C y_D^2 + 3x_D^3 y_C^2 y_D - 3x_H y_C^2 y_D^3 + x_H y_C^3 y_D^2 - x_C^2 x_D^2 y_D^3 - x_C^2 y_C^2 y_D^3 + x_C x_D^2 x_H y_D^3 - 2x_C x_D^2 y_C y_D^2 - 2x_C x_D y_C^2 y_D^3 + x_C x_D y_C^3 y_D^2 + x_C x_D^3 y_C y_D^2 - 3x_C x_D^3 y_C^2 y_D + x_C^3 x_D y_C y_D^2 + x_C x_H y_C^2 y_D^3 + 3x_D^2 x_H y_C y_D^2 - 3x_D^2 x_H y_C^2 y_D + 2x_D x_H y_C^2 y_D^3 - x_D x_H y_C^3 y_D^2 - x_D^3 x_H y_C y_D^2 + 2x_D^3 x_H y_C^2 y_D + x_C^2 x_D^2 y_C y_D^2 + x_C x_D y_C y_D^4 - 2x_C x_H y_C y_D^4 - x_D x_H y_C y_D^4 - 2x_C x_D^2 x_H y_C y_D^2 + x_C x_D^2 x_H y_C^2 y_D)/(2y_D(x_C y_D - x_D y_C)(x_C - x_D)(y_D - x_C y_D + x_D y_C))$
and
$y_M = -(2x_C^2 y_D^4 - x_D^3 y_C^2 + x_D^4 y_C^2 - 2x_C y_D^4 + x_H y_D^4 - x_C x_H y_D^4 + 2x_D y_C y_D^3 + 2x_D^3 y_C y_D - 2x_H y_C y_D^3 - 2x_C x_D^2 y_D^2 + x_C^2 x_D y_D^2 - x_C^3 x_D y_D^2 + x_D^2 x_H y_C^2 - x_D^3 x_H y_C^2 + x_D^2 x_H y_D^2 + x_D^2 y_C^3 y_D + x_H y_C^2 y_D^2 + 2x_C^2 x_D^2 y_D^2 + x_D^2 y_C^2 y_D^2 - x_C x_D^2 x_H y_D^2 - x_C x_D y_C^2 y_D^2 + x_C^2 x_D^2 y_C y_D -$

$x_D x_H y_C^2 y_D^2 - 3x_C x_D y_C y_D^3 - 3x_C x_D^3 y_C y_D + x_C x_H y_C y_D^3 - 2x_D^2 x_H y_C y_D + x_D x_H y_C y_D^3 + x_D^3 x_H y_C y_D + x_C x_D^2 x_H y_C y_D)/(2y_D(x_C y_D - x_D y_C)(y_D - x_C y_D + x_D y_C))$.
N has coordinates
$x_N = -(x_D^2 y_C^3 - 2x_D^3 y_C^3 - x_C^2 y_D^5 - x_C^4 y_D^3 + x_D^4 y_C^3 - x_C x_D y_D^3 + x_C x_H y_D^3 - x_D x_H y_C^3 + x_C x_H y_D^5 - x_C y_C y_D^4 + x_D y_C y_D^4 + 2x_C^2 x_D y_D^3 + x_C^3 x_D y_D^3 + 2x_D^2 x_H y_C^3 - x_D^3 x_H y_C^3 + 2x_C y_C^2 y_D^3 - x_C y_C^3 y_D^2 - x_C^3 y_C y_D^2 + 2x_C^2 y_C y_D^4 - x_D^2 y_C y_D^2 - 2x_D y_C^2 y_D^3 + x_D y_C^3 y_D^2 + x_D^3 y_C y_D^2 - x_C^2 x_D^2 y_D^3 - x_C^2 y_C^2 y_D^3 + x_C x_D^2 x_H y_D^3 - 5x_C x_D^2 y_C y_D^2 + 6x_C x_D^2 y_C^2 y_D - x_C^2 x_D y_C y_D^2 - 2x_C x_D y_C^2 y_D^3 + x_C x_D y_C^3 y_D^2 + x_C x_D^3 y_C y_D^2 - 3x_C x_D^3 y_C^2 y_D + x_C^3 x_D y_C y_D^2 + x_C x_H y_C^2 y_D^3 + 2x_D^2 x_H y_C y_D^2 - 4x_D^2 x_H y_C^2 y_D + 2x_D x_H y_C^2 y_D^3 - x_D x_H y_C^3 y_D^2 - x_D^3 x_H y_C y_D^2 + 2x_D^3 x_H y_C^2 y_D + x_C^2 x_D^2 y_C y_D^2 - 2x_C x_D x_H y_D^3 + 4x_C x_D y_C y_D^2 - 3x_C x_D y_C^2 y_D + x_C x_D y_C y_D^4 - 2x_C x_H y_C y_D^2 + x_C x_H y_C^2 y_D - 2x_C x_H y_C y_D^4 - x_D x_H y_C y_D^2 + 2x_D x_H y_C^2 y_D - x_D x_H y_C y_D^4 + 4x_C x_D x_H y_C y_D^2 - 2x_C x_D x_H y_C^2 y_D - 2x_C x_D^2 x_H y_C y_D^2 + x_C x_D^2 x_H y_C^2 y_D)/(2y_D(x_C - x_D)(y_C + x_C y_D - x_D y_C)(y_C - y_D + x_C y_D - x_D y_C))$
and
$y_N = (x_D^2 y_C^2 + x_C^3 y_D^2 - 2x_D^3 y_C^2 + 2x_C^2 y_D^4 + x_D^4 y_C^2 - x_C y_D^4 - y_C y_D^3 + y_C^3 y_D + x_C x_D y_D^2 - x_C x_H y_D^2 - x_D x_H y_C^2 - x_C x_H y_D^4 + x_C^2 y_C y_D + 2x_C y_C y_D^3 - 2x_D y_C^3 y_D - x_C x_D^2 y_D^2 - 2x_C^2 x_D y_D^2 - x_C^3 x_D y_D^2 + 2x_D^2 x_H y_C^2 - x_D^3 x_H y_C^2 + x_C y_C^2 y_D^2 + x_D^2 y_C^3 y_D + 2x_C^2 x_D^2 y_D^2 + x_D^2 y_C^2 y_D^2 - x_C x_D^2 x_H y_D^2 - x_C x_D y_C^2 y_D^2 + x_C^2 x_D^2 y_C y_D - x_D x_H y_C^2 y_D^2 - 3x_C x_D y_C y_D + x_C x_H y_C y_D + x_D x_H y_C y_D + 2x_C x_D x_H y_D^2 + 6x_C x_D^2 y_C y_D - 2x_C^2 x_D y_C y_D - 3x_C x_D y_C y_D^3 - 3x_C x_D^3 y_C y_D + x_C x_H y_C y_D^3 - 2x_D^2 x_H y_C y_D + x_D x_H y_C y_D^3 + x_D^3 x_H y_C y_D + x_C x_D^2 x_H y_C y_D - 2x_C x_D x_H y_C y_D)/(2y_D(y_C + x_C y_D - x_D y_C)(y_C - y_D + x_C y_D - x_D y_C))$.
O has coordinates
$x_O = (x_C^2 x_D y_D + x_H x_C^2 y_D - x_C x_D^2 y_C - x_C x_D^2 y_D - 2x_H x_C x_D y_D + x_C y_C y_D^2 + x_C y_C - x_C y_D^3 - x_C y_D + x_D^3 y_C + x_H x_D^2 y_D - x_D y_C^2 y_D + x_D y_C y_D^2 - x_D y_C + x_D y_D + x_H y_C^2 y_D - 2x_H y_C y_D^2 + x_H y_D^3)/(2(x_C - x_D)(y_C - y_D + x_C y_D - x_D y_C))$
and
$y_O = (x_C^2 y_D^2 + x_D^2 y_C^2 + 2x_C x_D - 2x_C x_D^2 + x_C^2 x_D + x_C^2 x_H + x_D^2 x_H - x_D^3 x_H - 2x_C y_D^2 - x_D y_C^2 + x_D y_D^2 + x_H y_C^2 + x_H y_D^2 - x_C^2 - x_D^2 + x_D^3 + 2x_C x_D^2 x_H - x_C^2 x_D x_H - x_D x_H y_C^2 - x_D x_H y_D^2 - 2x_C x_D x_H + 2x_C y_C y_D - 2x_H y_C y_D - 2x_C x_D y_C y_D + 2x_D x_H y_C y_D)/(2(x_C - x_D)(y_C - y_D + x_C y_D - x_D y_C))$.
P has coordinates
$x_P = -(x_C^2 y_D^2 - x_D^3 y_C^2 + x_C^2 y_D^4 + x_D^4 y_C^2 - x_C y_D^2 - x_C y_D^4 + x_D y_D^2 + x_H y_D^4 - x_C x_D y_D^2 - x_C x_H y_D^4 + x_D^2 y_C y_D + 2x_D y_C y_D^3 + 2x_D^3 y_C y_D - 2x_H y_C y_D^3 - x_C x_D^2 y_D^2 + x_D^2 x_H y_C^2 - x_D^3 x_H y_C^2 + x_D^2 x_H y_D^2 - x_D y_C^2 y_D^2 + x_H y_C^2 y_D^2 + x_C^2 x_D^2 y_D^2 + x_D^2 y_C^2 y_D^2 - x_C x_D^2 x_H y_D^2 - x_D x_H y_C^2 y_D^2 - x_C x_D y_C y_D - 2x_C x_D y_C y_D^3 - 2x_C x_D^3 y_C y_D + x_C x_H y_C y_D^3 - 2x_D^2 x_H y_C y_D + x_D x_H y_C y_D^3 + x_D^3 x_H y_C y_D + x_C x_D^2 x_H y_C y_D)/(2y_D(x_C - x_D)(y_D - x_C y_D + x_D y_C))$
and
$y_P = (x_D^3 y_C^2 - 2x_C^2 y_D^4 - 2x_D^4 y_C^2 + x_D^5 y_C^2 + x_D^2 y_D^2 - x_D^3 y_D^2 + 2x_C y_D^4 - x_D y_D^4 - x_H y_D^4 - x_C x_D y_D^2 + x_C x_H y_D^4 + x_D x_H y_D^4 - 2x_D y_C y_D^3 - x_D^3 y_C y_D + x_D^4 y_C y_D + 2x_H y_C y_D^3 + x_C x_D^2 y_D^2 + x_C^2 x_D y_D^2 + x_C^2 x_D y_D^4 - x_D^2 x_H y_C^2 + 2x_D^3 x_H y_C^2 - x_D^4 x_H y_C^2 - x_D^2 x_H y_D^2 + x_D^3 x_H y_D^2 + x_D y_C^2 y_D^2 + x_D^2 y_C y_D^3 - x_H y_C^2 y_D^2 - 2x_C^2 x_D^2 y_D^2 + x_C^2 x_D^3 y_D^2 - 2x_D^2 y_C^2 y_D^2 + x_D^3 y_C^2 y_D^2 + x_C x_D^2 x_H y_D^2 - x_C x_D^3 x_H y_D^2 - 2x_C x_D^2 y_C y_D^3 + 2x_D x_H y_C^2 y_D^2 + x_D^2 x_H y_C y_D^3 - x_D^2 x_H y_C^2 y_D^2 - x_C x_D x_H y_D^4 - x_C x_D^2 y_C y_D + 3x_C x_D y_C y_D^3 + 3x_C x_D^3 y_C y_D - 2x_C x_D^4 y_C y_D - x_C x_H y_C y_D^3 + 2x_D^2 x_H y_C y_D - 3x_D x_H y_C y_D^3 - 3x_D^3 x_H y_C y_D + x_D^4 x_H y_C y_D - x_C x_D^2 x_H y_C y_D + x_C x_D x_H y_C y_D^3 + x_C x_D^3 x_H y_C y_D)/(2y_D^2(x_C - x_D)(y_D - x_C y_D + x_D y_C))$.

One can then make the finite algebraic calculation to verify that

$$det\begin{bmatrix} x_K^2 & y_K^2 & x_K y_K & x_K & y_K & 1 \\ x_L^2 & y_L^2 & x_L y_L & x_L & y_L & 1 \\ x_M^2 & y_M^2 & x_M y_M & x_M & y_M & 1 \\ x_N^2 & y_N^2 & x_N y_N & x_N & y_N & 1 \\ x_O^2 & y_O^2 & x_O y_O & x_O & y_O & 1 \\ x_P^2 & y_P^2 & x_P y_P & x_P & y_P & 1 \end{bmatrix} = 0$$

as desired. □

The theorem in [1] is the special case of this result where we take D to be the centroid.

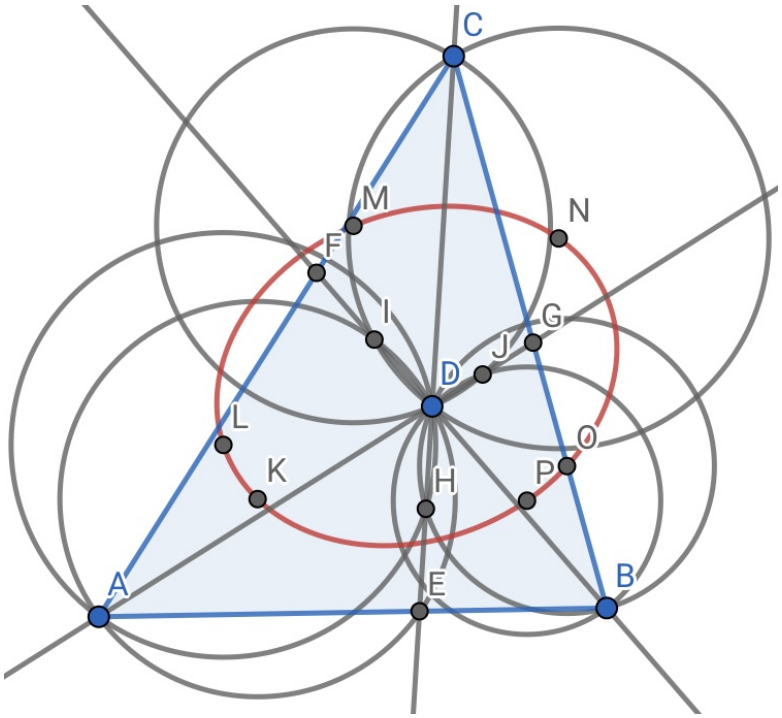

FIGURE 1. Illustration of the result.

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