International Journal of Computer Discovered Mathematics (IJCDM)
ISSN 2367-7775 ©IJCDM
Volume 9, 2024, pp. 41-43.
Received March 23, 2024. Published on-line June 28, 2024.
web: http://www.journal-1.eu/
©The Author(s) This article is published with open access[1].

Generalization of a Result about Concurrent Newton Lines Formed from Regular Polygons

Benjamin Lee Warren[a] [2]
University of Binghamton, Mathematics,
4400 Vestel Pkwy E, Binghamton, NY 13902
e-mail: warrenbenjamin52@yahoo.com
web: https://orcid.org/0000-0002-2829-6953

Mathematics Subject Classification (2010). 51N20.

1. Introduction

In [1], we took a regular n-gon ($n \geq 5$) and dissected it into quadrilaterals by a vertex and the two adjacent vertex midpoints. In particular, we showed that the Newton lines of these quadrilaterals are concurrent. Here, we provide a generalization of this result.

2. Results

Theorem 2.1. *Fix the integer $n \geq 5$.*
Suppose we are given a regular n-gon in the plane.
Let G be any point in the plane.
For each vertex v_k, label points a_k on the left adjacent edge, and point b_k on the right adjacent edge, such that the distances a_k to v_k and b_k to v_k are equal, and respective distances are the same for all the vertices.
For each k, form the Newton line of the quadrilateral $\square v_k a_k G b_k$.
Then the Newton lines of all the quadrilaterals are concurrent.
Moreoever, if we denote this point as α, denote by $m(A, B)$ the midpoint of points A and B, and denote $|AB|$ as the distance between A and B, then we have that $\frac{|m(a_k,b_k)\alpha|}{|m(v_k,G)\alpha|} = 2(t\cos(\frac{2\pi}{n}) - t + 1)$ where $t = \frac{|b_1 v_1|}{|v_2 v_1|}$.

[1]This article is distributed under the terms of the Creative Commons Attribution License which permits any use, distribution, and reproduction in any medium, provided the original author(s) and the source are credited.
[2]Corresponding author

Proof. Fix the integer $n \geq 3$ with $n = 4$.
On the Cartesian grid, we have the following.
G has coordinates given by (x_G, y_G).
v_k has coordinates given by $(\cos(\frac{2\pi k}{n}), \sin(\frac{2\pi k}{n}))$.
Fix $t \in \mathbb{R}$.
b_k has coordinates given by
$(\cos(\frac{2\pi k}{n}) + t(\cos(\frac{2\pi(k+1)}{n}) - \cos(\frac{2\pi k}{n})), \sin(\frac{2\pi k}{n}) + t(\sin(\frac{2\pi(k+1)}{n}) - \sin(\frac{2\pi k}{n}))).$
$\iff$
$(\cos(\frac{2\pi k}{n}) - 2t\sin(\frac{2\pi k+\pi}{n})\sin(\frac{\pi}{n}), \sin(\frac{2\pi k}{n}) + 2t\cos(\frac{2\pi k+\pi}{n})\sin(\frac{\pi}{n})).$
a_k has coordinates given by
$(\cos(\frac{2\pi(k-1)}{n}) - 2(1-t)\sin(\frac{2\pi(k-1)+\pi}{n})\sin(\frac{\pi}{n}), \sin(\frac{2\pi(k-1)}{n}) + 2(1-t)\cos(\frac{2\pi(k-1)+\pi}{n})\sin(\frac{\pi}{n})).$
$m(a_k, b_k)$ has coordinates given by
$(\frac{1}{2}(2\cos(\frac{2\pi k}{n}) + t(\cos(\frac{2\pi(k+1)}{n}) + \cos(\frac{2\pi(k-1)}{n}) - 2\cos(\frac{2\pi k}{n}))), \frac{1}{2}(2\sin(\frac{2\pi k}{n}) + t(\sin(\frac{2\pi(k+1)}{n}) +$
$\sin(\frac{2\pi(k-1)}{n}) - 2\sin(\frac{2\pi k}{n})))).$
$\iff$
$(\cos(\frac{2\pi k}{n}) + t\cos(\frac{2\pi k}{n})(\cos(\frac{2\pi}{n}) - 1), \sin(\frac{2\pi k}{n}) + t\sin(\frac{2\pi k}{n})(\cos(\frac{2\pi}{n}) - 1)).$
$m(v_k, G)$ has coordinates given by $(\frac{1}{2}(x_G + \cos(\frac{2\pi k}{n})), \frac{1}{2}(y_G + \sin(\frac{2\pi k}{n}))).$
The Newton line of the quadrilateral $\square v_k a_k G b_k$ is given by the equation
$y = \frac{1}{2}(y_G + \sin(\frac{2\pi k}{n})) + (x - \frac{1}{2}(x_G + \cos(\frac{2\pi k}{n})))(\frac{\sin(\frac{2\pi k}{n}) + t\sin(\frac{2\pi k}{n})(\cos(\frac{2\pi}{n}) - 1) - (\frac{1}{2}(y_G + \sin(\frac{2\pi k}{n})))}{\cos(\frac{2\pi k}{n}) + t\cos(\frac{2\pi k}{n})(\cos(\frac{2\pi}{n}) - 1) - \frac{1}{2}(x_G + \cos(\frac{2\pi k}{n}))}).$
Consider the point α whose coordinates are given by
$(\frac{x_G(t\cos(\frac{2\pi}{n}) - t + 1)}{2t\cos(\frac{2\pi}{n}) - 2t + 1}, \frac{y_G(t\cos(\frac{2\pi}{n}) - t + 1)}{2t\cos(\frac{2\pi}{n}) - 2t + 1}).$
We verify that α lies on all the Newton lines since
$\frac{1}{2}(y_G + \sin(\frac{2\pi k}{n})) + (\frac{x_G(t\cos(\frac{2\pi}{n}) - t + 1)}{2t\cos(\frac{2\pi}{n}) - 2t + 1} - \frac{1}{2}(x_G + \cos(\frac{2\pi k}{n})))(\frac{\sin(\frac{2\pi k}{n}) + t\sin(\frac{2\pi k}{n})(\cos(\frac{2\pi}{n}) - 1) - (\frac{1}{2}(y_G + \sin(\frac{2\pi k}{n})))}{\cos(\frac{2\pi k}{n}) + t\cos(\frac{2\pi k}{n})(\cos(\frac{2\pi}{n}) - 1) - \frac{1}{2}(x_G + \cos(\frac{2\pi k}{n}))})$
$- (\frac{y_G(t\cos(\frac{2\pi}{n}) - t + 1)}{2t\cos(\frac{2\pi}{n}) - 2t + 1}) = 0.$
Moreover, we calculate that
$\frac{|m(a_k,b_k)\alpha|}{|m(v_k,G)\alpha|} =$
$$\sqrt{\frac{(\cos(\frac{2\pi k}{n}) + t\cos(\frac{2\pi k}{n})(\cos(\frac{2\pi}{n}) - 1) - (\frac{x_G(t\cos(\frac{2\pi}{n}) - t + 1)}{2t\cos(\frac{2\pi}{n}) - 2t + 1}))^2 + (\sin(\frac{2\pi k}{n}) + t\sin(\frac{2\pi k}{n})(\cos(\frac{2\pi}{n}) - 1) - (\frac{y_G(t\cos(\frac{2\pi}{n}) - t + 1)}{2t\cos(\frac{2\pi}{n}) - 2t + 1}))^2}{(\frac{1}{2}(x_G + \cos(\frac{2\pi k}{n})) - (\frac{x_G(t\cos(\frac{2\pi}{n}) - t + 1)}{2t\cos(\frac{2\pi}{n}) - 2t + 1}))^2 + (\frac{1}{2}(y_G + \sin(\frac{2\pi k}{n})) - (\frac{y_G(t\cos(\frac{2\pi}{n}) - t + 1)}{2t\cos(\frac{2\pi}{n}) - 2t + 1}))^2}}$$
$= \sqrt{4(t\cos(\frac{2\pi}{n}) - t + 1)^2} = 2(t\cos(\frac{2\pi}{n}) - t + 1)$ which is independent of G and k as desired.

□

An analogous result holds for triangles in general [2].

References

[1] Warren, B.L., The Newton Lines of the Quadrilaterals which Dissect a Regular n-gon by the Adjacent Vertex Midpoints are Concurrent, *International Journal of Computer Discovered Mathematics*

[2] Warren, B.L., The Newton Lines of Three Certain Quadrilaterals Formed from a Triangle are Concurrent, *International Journal of Computer Discovered Mathematics*

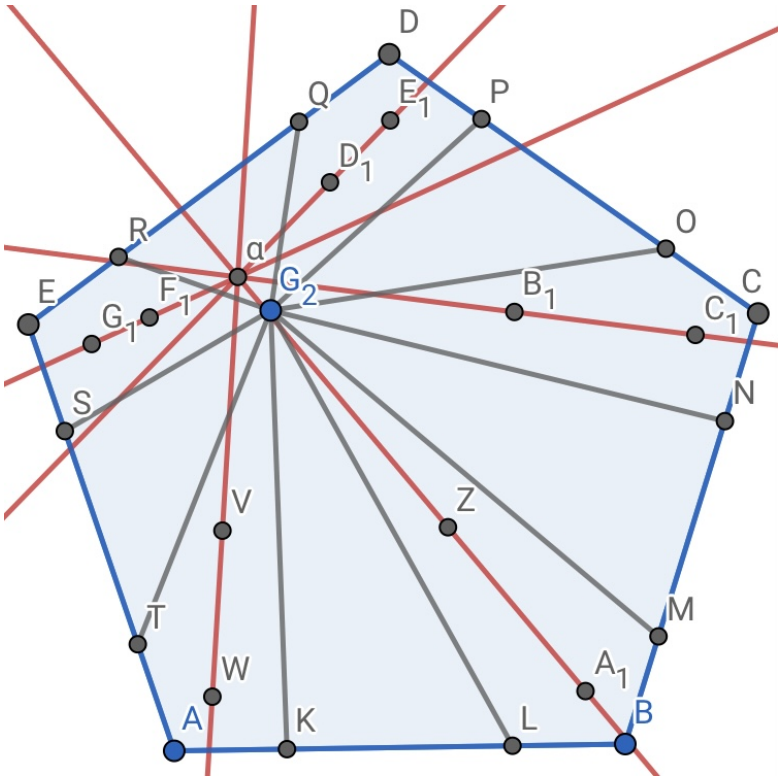

Figure 1.

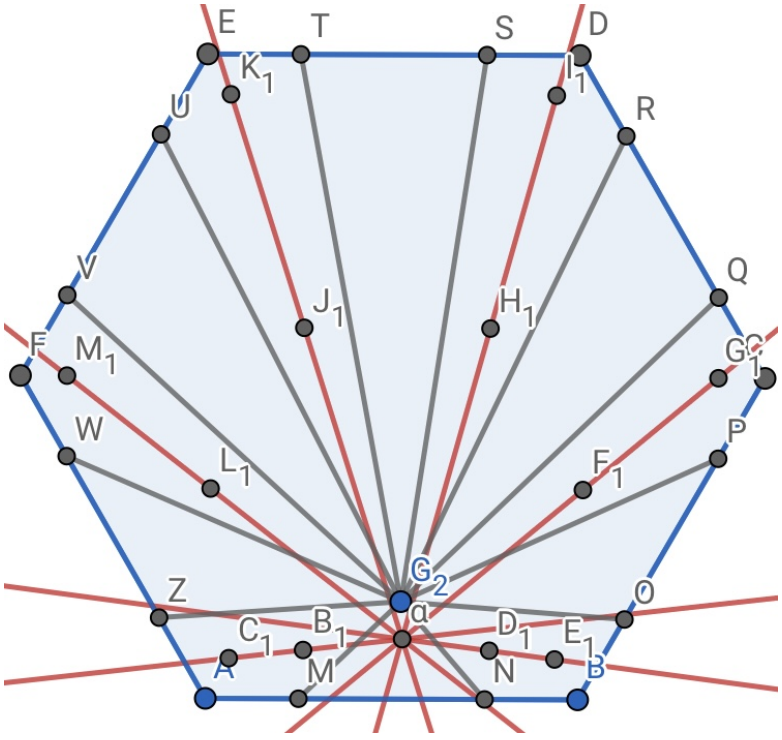

Figure 2. Result for the cases of regular pentagons and hexagons.