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Relationships between a Central Quadrilateral and its Reference Quadrilateral

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Abstract. Let P be a point inside a convex quadrilateral $ABCD$. The lines from P to the vertices of the quadrilateral divide the quadrilateral into four triangles. If we locate a triangle center in each of these triangles, the four triangle centers form another quadrilateral called a central quadrilateral. For each of various shaped quadrilaterals, and each of 1000 different triangle centers, we compare the reference quadrilateral to the central quadrilateral. Using a computer, we determine how the two quadrilaterals are related. For example, we test to see if the two quadrilaterals are congruent, similar, have the same area, or have the same perimeter. We also look for such relationships when P is a special point associated with the reference quadrilateral, such as being the diagonal point, Steiner point, or Poncelet point.

Keywords. triangle centers, quadrilaterals, computer-discovered mathematics, Euclidean geometry. GeometricExplorer.

Mathematics Subject Classification (2020). 51M04, 51-08.

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1. Introduction

In this study, $ABCD$ always represents a convex quadrilateral known as the *reference quadrilateral*. A point E in the plane of the quadrilateral is chosen and will be called the *radiator*. The radiator can be an arbitrary point or it can be a notable point associated with the quadrilateral. Lines are drawn from the radiator to the vertices of the reference quadrilateral forming four triangles with the sides of the quadrilateral as shown in Figure 1. These triangles will be called the *radial triangles*.

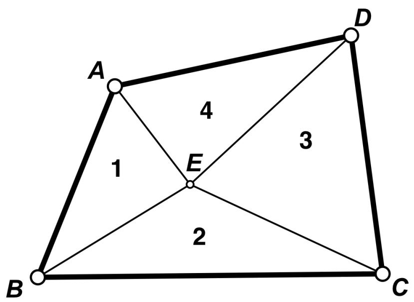

Figure 1. Radial Triangles

In the figure, the radial triangles have been numbered in a counterclockwise order starting with side AB: $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, $\triangle DAE$. Triangle centers (such as the incenter, centroid, or circumcenter) are selected in each triangle. The same type of triangle center is used with each radial triangle. In order, the names of these points are F, G, H, and I as shown in Figure 2. These four centers form a quadrilateral $FGHI$ that will be called the *central quadrilateral* (of quadrilateral $ABCD$ with respect to E). Quadrilateral $FGHI$ need not be convex.

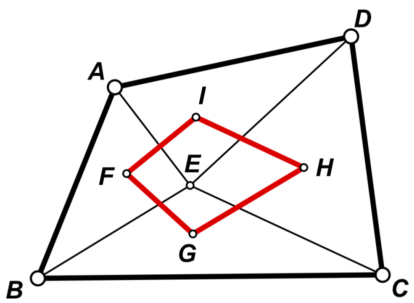

Figure 2. Central Quadrilateral

The purpose of this paper is to determine interesting relationships between a reference quadrilateral and its central quadrilateral.

2. Types of Quadrilaterals Studied

We are only interested in reference quadrilaterals that have a certain amount of symmetry. For example, we excluded bilateral quadrilaterals (those with two equal sides), bisect-diagonal quadrilaterals (where one diagonal bisects another), right kites, right trapezoids, and golden rectangles. The types of quadrilaterals we studied are shown in Table 1. The sides of the quadrilateral, in order, have lengths a, b, c, and d. The diagonals have lengths p and q. The measures of the angles of the quadrilateral, in order, are A, B, C, and D.

TABLE 1.

Types of Quadrilaterals Considered		
Quadrilateral Type	**Geometric Definition**	**Algebraic Condition**
general	convex	none
cyclic	has a circumcircle	$A+C=B+D$
tangential	has an incircle	$a+c=b+d$
extangential	has an excircle	$a+b=c+d$
parallelogram	opposite sides parallel	$a=c$, $b=d$
equalProdOpp	product of opposite sides equal	$ac=bd$
equalProdAdj	product of adjacent sides equal	$ab=cd$
orthodiagonal	diagonals are perpendicular	$a^2+c^2=b^2+d^2$
equidiagonal	diagonals have the same length	$p=q$
Pythagorean	equal sum of squares, adjacent sides	$a^2+b^2=c^2+d^2$
kite	two pair adjacent equal sides	$a=b$, $c=d$
trapezoid	one pair of opposite sides parallel	$A+B=C+D$
rhombus	equilateral	$a=b=c=d$
rectangle	equiangular	$A=B=C=D$
Hjelmslev	two opposite right angles	$A=C=90°$
isosceles trapezoid	trapezoid with two equal sides	$A=B$, $C=D$
APquad	sides in arithmetic progression	$d-c=c-b=b-a$

The following combinations of entries in the above list were also considered: bicentric quadrilaterals (cyclic and tangential), exbicentric quadrilaterals (cyclic and extangential), bicentric trapezoids, cyclic orthodiagonal quadrilaterals, equidiagonal kites, equidiagonal orthodiagonal quadrilaterals, equidiagonal orthodiagonal trapezoids, harmonic quadrilaterals (cyclic and equalProdOpp), orthodiagonal trapezoids, tangential trapezoids, and squares (equiangular rhombi).

So, in addition to the general convex quadrilateral, a total of 27 other types of quadrilaterals were considered in this study.

A graph of the types of quadrilaterals considered is shown in Figure 3. An arrow from A to B means that any quadrilateral of type B is also of type A. For example: all squares are rectangles and all kites are orthodiagonal. If a directed path leads from a quadrilateral of type A to a quadrilateral of type B, then we will say that A is an *ancestor* of B. For example, an equidiagonal quadrilateral is an ancestor of a rectangle. In other words, all rectangles are equidiagonal.

Unless otherwise specified, when we give a theorem or table of properties of a quadrilateral, we will omit an entry for a particular shape quadrilateral if the property is known to be true for an ancestor of that quadrilateral.

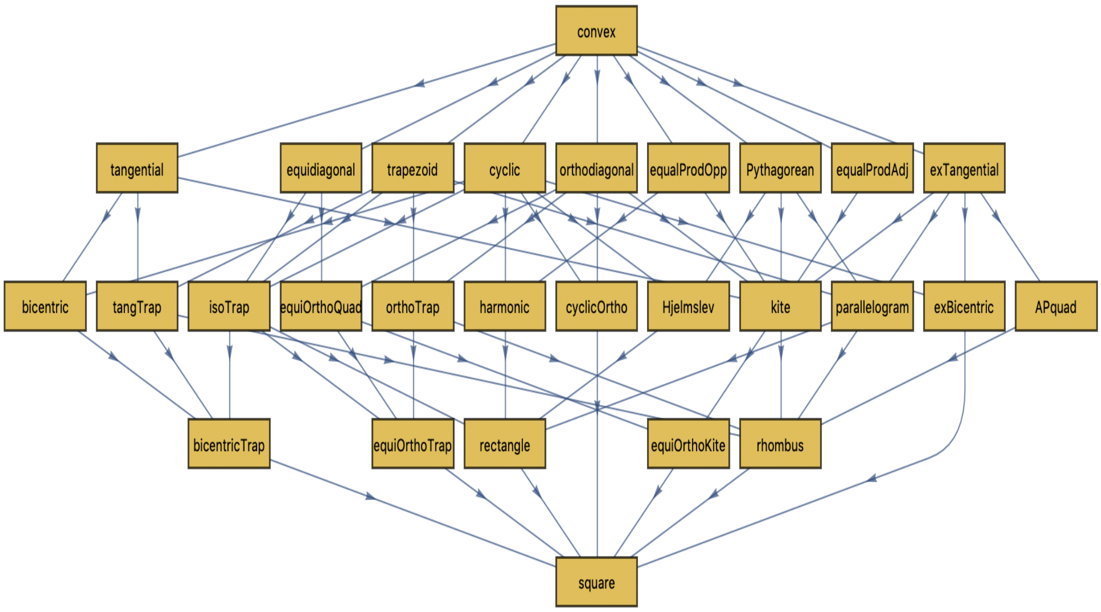

Figure 3. Quadrilateral Shapes

3. Centers

In this study, we will place triangle centers in the four radial triangles. We use Clark Kimberling's definition of a triangle center [4].

A *center function* is a nonzero function $f(a, b, c)$ homogeneous in a, b, and c and symmetric in b and c. *Homogeneous* in a, b, and c means that

$$f(ta, tb, tc) = t^n f(a, b, c)$$

for some nonnegative integer n, all $t > 0$, and all positive real numbers (a, b, c) satisfying $a < b + c$, $b < c + a$, and $c < a + b$. *Symmetric* in b and c means that

$$f(a, c, b) = f(a, b, c)$$

for all a, b, and c.

A *triangle center* is an equivalence class $x : y : z$ of ordered triples (x, y, z) given by

$$x = f(a, b, c), \quad y = f(b, c, a), \quad z = f(c, a, b).$$

Tens of thousands of interesting triangle centers have been cataloged in the Encyclopedia of Triangle Centers [5]. We use X_n to denote the nth named center in this encyclopedia.

Note that if the center function of a certain center is $f(a, b, c)$, then the trilinear coordinates of that point with respect to a triangle with sides a, b, and c are

$$\Big(f(a, b, c) : f(b, c, a) : f(c, a, b)\Big).$$

The barycentric coordinates for that point would then be

$$\Big(af(a, b, c) : bf(b, c, a) : cf(c, a, b)\Big).$$

4. Methodology

We used a computer program called GeometricExplorer to compare quadrilaterals with their central quadrilateral. Starting with each type of quadrilateral listed in Figure 3 for the reference quadrilateral, we picked various choices for point E, the radiator. The types of radiators studied are shown in Table 2.

TABLE 2.

Points Used as Radiators	
name	**description**
arbitrary point	any point in the plane of $ABCD$
diagonal point	intersection of the diagonals (QG–P1)
Poncelet point	(QA–P2)
Steiner point	(QA–P3)
circumcenter	center of circumscribed circle
incenter	center of inscribed circle
anticenter	(QA–P2 in a cyclic quadrilateral)
vertex centroid	(QA-P1)
midpoint of 3rd diagonal	(QG-P2)

Some notable points only exist for certain shape quadrilaterals. For example, the circumcenter only applies to cyclic quadrilaterals. A code in parentheses represents the name for the point as listed in the Encyclopedia of Quadri-Figures [22]. These will be defined in the section that reports relationships using these points.

For each n from 1 to 1000, we placed center X_n in each of the radial triangles of the reference quadrilateral. The program then analyzes the central quadrilateral formed by these four centers and reports if the central quadrilateral is related to the reference quadrilateral. Points at infinity were omitted. The types of relationships checked for are shown in Table 3.

TABLE 3.

Relationships Checked For	
notation	**description**
$[ABCD] = [FGHI]$	the quadrilaterals have the same area
$[ABCD] = k[FGHI]$	the area of $ABCD$ is k times the area of $FGHI$ †
$ABCD \cong FGHI$	the quadrilaterals are congruent
$ABCD \sim FGHI$	the quadrilaterals are similar
$\partial ABCD = \partial FGHI$	the quadrilaterals have the same perimeter
$\odot ABCD \cong \odot FGHI$	the quadrilaterals have congruent circumcircles
$\odot ABCD \equiv \odot FGHI$	the quadrilaterals have the same circumcircle
† Only rational values of k were checked for with denominators less than 6.	

5. Barycentric Coordinates and Quadrilaterals

The program we used to find results about central quadrilaterals (GeometricExplorer) is a useful tool for discovering results, but it does not prove that these results are true. GeometricExplorer uses numerical coordinates (to 15 digits of precision) for locating all the points. Thus, a relationship found by this program does not constitute a proof that the result is correct, but gives us compelling evidence for the validity of the result.

If a theorem in this paper is accompanied by a figure, this means that the figure was drawn using either Geometer's Sketchpad or GeoGebra. In either case, we used the drawing program to dynamically vary the points in the figure. Noticing that the result remains true as the points vary offers further evidence that the theorem is true. But again, this does not constitute a proof.

To prove the results that we have discovered, we use geometric methods, when possible. If we could not find a purely geometrical proof, we turned to analytic methods using barycentric coordinates and performing exact symbolic computation using Mathematica.

We assume the reader is familiar with barycentric coordinates. We give below some useful results that will be used when providing proofs of some of the theorems that we found.

The following result about the centroid of a triangle is well known [1, p. 65].

Lemma 5.1. *The centroid of a triangle divides a median in the ratio* $2:1$.

The following result about the area of a quadrilateral is well known [1, p. 124].

Lemma 5.2 (Varignon Parallelogram)**.** *The midpoints of the sides of a convex quadrilateral form a parallelogram. The area of the parallelogram is half the area of the quadrilateral. The sides of the parallelogram are parallel to the diagonals of the quadrilateral.*

This parallelogram is called the *Varignon paralellogram* of the given quadrilateral.

If $(u:v:w)$ are barycentric coordinates with the property that $u+v+w=1$, then we call the coordinates *normalized*.

The formula for the distance between two points in terms of their normalized barycentric coordinates is well known [3, Section 2].

Lemma 5.3 (Distance Formula)**.** *Let ABC be a triangle with sides of lengths a, b, and c. Relative to $\triangle ABC$, let the normalized barycentric coordinates for points P and Q be $(u_1:v_1:w_1)$ and $(u_2:v_2:w_2)$, respectively. Let $x=u_1-u_2$, $y=v_1-v_2$, and $z=w_1-w_2$. Then the distance from P to Q is*

$$PQ=\sqrt{-a^2yz-b^2zx-c^2xy}.$$

The formula for the area of a triangle is also well known [3, Equation 2].

Lemma 5.4 (Area Formula)**.** *Let ABC be a triangle with area K. Relative to $\triangle ABC$, let the normalized barycentric coordinates for points P, Q, and R be $(u_1 : v_1 : w_1)$, $(u_2 : v_2 : w_2)$, and $(u_3 : v_3 : w_3)$ respectively. Then the area of $\triangle PQR$ is*

$$[PQR] = \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} K.$$

Note that the area is signed. It is positive if the triangle has the same orientation as $\triangle ABC$ and negative if it has the opposite orientation.

The signed area of a quadrilateral $PQRS$ is defined as

$$[PQRS] = [PQR] + [RSP].$$

This definition is used even when the quadrilateral is self-intersecting.

Given the barycentric coordinates for a point P with respect to $\triangle ABC$, we sometimes want to find the corresponding point in some other triangle, $\triangle DEF$. This is accomplished using the well-known Change of Coordinates Formula [3, Section 3].

Lemma 5.5 (Change of Coordinates Formula)**.** *Relative to $\triangle ABC$, let the normalized barycentric coordinates for points D, E, and F be $(u_1 : v_1 : w_1)$, $(u_2 : v_2 : w_2)$, and $(u_3 : v_3 : w_3)$ respectively. Let the normalized barycentric coordinates for point P with respect to $\triangle DEF$ be $(p : q : r)$. Then the barycentric coordinates for P with respect to $\triangle ABC$ are $(u : v : w)$ where*

$$\begin{aligned} u &= u_1 p + u_2 q + u_3 r \\ v &= v_1 p + v_2 q + v_3 r \\ w &= w_1 p + w_2 q + w_3 r. \end{aligned}$$

Lemma 5.6. *Let ABC be a right triangle with right angle at A. Let f be a center function with the properties*

$$f(a, b, c) = 0 \qquad \text{and} \qquad f(b, c, a) = f(c, a, b)$$

when $c^2 = a^2 + b^2$. Then the center of $\triangle ABC$ corresponding to this center function coincides with the midpoint of the hypotenuse.

Proof. Since $\triangle ABC$ is a right triangle with hypotenuse BC, we must have $c^2 = a^2 + b^2$. The trilinear coordinates for the center then are

$$\begin{aligned} \Big(f(a, b, c) : f(b, c, a) : f(c, a, b)\Big) &= \Big(0 : f(b, c, a) : f(b, c, a)\Big) \\ &= (0 : 1 : 1) \end{aligned}$$

which corresponds to the trilinear coordinates for the midpoint of the hypotenuse. $\square$

Lemma 5.7. *The nine-point center of a right triangle coincides with the midpoint of the median to the hypotenuse.*

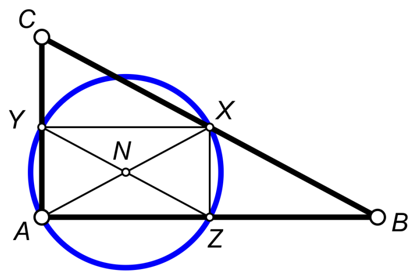

Figure 4. Nine point center of a right triangle

Proof. Let X, Y, and Z be the midpoints of the sides of right triangle ABC as shown in Figure 4. Since X, Y, and Z are midpoints, $XZ \parallel CA$ and $XY \parallel BA$. Since $\angle BAC$ is a right angle, $AYXZ$ must be a rectangle. The nine-point circle of $\triangle ABC$ passes through X, Y, and Z and is therefore the circumcircle of this rectangle. The nine-point center is the center of this rectangle and is therefore the midpoint of AX. □

Lemma 5.8. *Let M be midpoint of the hypotenuse BC of right triangle ABC. Let X be any point on AM. Let Y and Z be the feet of the perpendiculars dropped from X to AC and AB, respectively (Figure 5). Then*

$$\frac{[ABC]}{[AZXY]} = 2\left(\frac{AM}{AX}\right)^2.$$

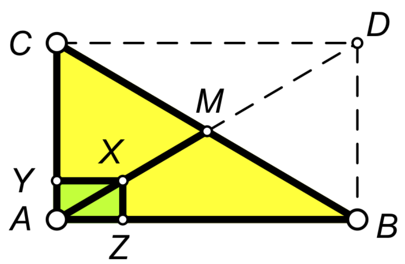

Figure 5. right triangle, $\frac{[ABC]}{[AZXY]} = 2\left(\frac{AM}{AX}\right)^2$.

Proof. Reflect A about M to get point D, making $ACDB$ a rectangle. Rectangles $AYXZ$ and $ACDB$ are similar. The ratio of the area of two similar figures is the square of the ratio of similarity. So

$$\frac{[ABDC]}{[AZXY]} = \left(\frac{AD}{AX}\right)^2.$$

Thus,

$$\frac{[ABC]}{[AZXY]} = 2\left(\frac{AM}{AX}\right)^2$$

since $AD = 2AM$ and $[ABCD] = 2[ABC]$. □

The following result comes from [20].

Lemma 5.9. *The condition for a triangle center with center function $f(x,y,z)$ to lie on the angle bisector at vertex A in right triangle ABC (with right angle at A) is*

$$f(x,y,z) = f(y,x,z)$$

for all x, y, and z satisfying $x^2 + y^2 = z^2$.

Lemma 5.10. *Let ABC be an isosceles triangle with $AB = AC = b$ and $BC = a$. Let M be the midpoint of BC. Let X be any triangle center of $\triangle ABC$. Suppose the barycentric coordinates for X are $(u : v : w)$ with respect to $\triangle ABC$ (Figure 6). Then X lies on AM, the median to side BC, $v = w$, and*

$$\frac{XM}{AM} = \frac{u}{u+2v}.$$

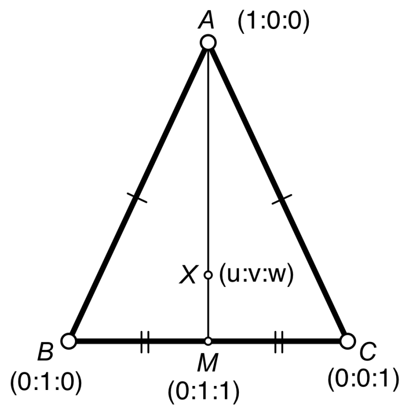

FIGURE 6. Center X of an isosceles triangle

Proof. Since the barycentric coordinates of a point are proportional to the areas of the triangles formed by that point and the sides of a triangle, we must have

$$\frac{[AXB]}{[AXC]} = \frac{v}{w}.$$

But triangles AXB and AXC are congruent. Therefore $v = w$.

The equation of line BC is $x = 0$ and the equation of line AM is $y = z$, so the barycentric coordinates for M are $(0 : 1 : 1)$.

Again, by the area property,

$$\frac{[BXC]}{[ABC]} = \frac{u}{u+v+w}.$$

But the area of a triangle is half the base times the height, so

$$\frac{[BXC]}{[ABC]} = \frac{XM}{AM}.$$

Thus,

$$\frac{XM}{AM} = \frac{u}{u+v+w} = \frac{u}{u+2v}.$$

□

Lemma 5.11. *Let ABC be an isosceles triangle with $AB = AC$. Then the center X_n coincides with A for the following values of n:*

59, 99, 100, 101, 107, 108, 109, 110, 112, 162, 163, 190, 249, 250, 476, 643, 644, 645, 646, 648, 651, 653, 655, 658, 660, 662, 664, 666, 668, 670, 677, 681, 685, 687, 689, 691, 692, 765, 769, 771, 773, 777, 779, 781, 783, 785, 787, 789, 791, 793, 795, 797, 799, 803, 805, 807, 809, 811, 813, 815, 817, 819, 823, 825, 827, 831, 833, 835, 839, 874, 877, 880, 883, 886, 889, 892, 898, 901, 906, 907, 919, 925, 927, 929, 930, 931, 932, 933, 934, 935.

Lemma 5.12. *Let ABC be an isosceles triangle with $AB = AC$. Let M be the midpoint of BC. Then the center X_n coincides with M for the following values of n:*

11, 115, 116, 122, 123, 124, 125, 127, 130, 134, 135, 136, 137, 139, 244, 245, 246, 247, 338, 339, 865, 866, 867, 868.

Lemma 5.13. *The center X_n lies on the line at infinity for all isosceles triangles, but not for all triangles, for the following values of n:*

351, 647, 649, 650, 652, 654, 656, 657, 659, 661, 663, 665, 667, 669, 676, 684, 686, 693, 764, 770, 798, 810, 822, 850, 875, 876, 878, 879, 881, 882, 884, 885, 887, 890, 905.

For reference, X_n lies on the line at infinity for all triangles for the following values of n: 30, 511, 512, 513, 514, 515, 516, 517, 518, 519, 520, 521, 522, 523, 524, 525, 526, 527, 528, 529, 530, 531, 532, 533, 534, 535, 536, 537, 538, 539, 540, 541, 542, 543, 544, 545, 674, 680, 688, 690, 696, 698, 700, 702, 704, 706, 708, 710, 712, 714, 716, 718, 720, 722, 724, 726, 730, 732, 734, 736, 740, 742, 744, 746, 752, 754, 758, 760, 766, 768, 772, 776, 778, 780, 782, 784, 786, 788, 790, 792, 794, 796, 802, 804, 806, 808, 812, 814, 816, 818, 824, 826, 830, 832, 834, 838, 888, 891, 900, 912, 916, 918, 924, 926, 928, 952, 971.

6. Results Using an Arbitrary Point

In this configuration, the radiator, E, is any point in the plane of the reference quadrilateral $ABCD$.

Our computer analysis found only two relationships that hold for all quadrilaterals when E is an arbitrary point in the plane. We examined all the types of quadrilaterals listed in Table 1 and all triangle centers from X_1 to X_{1000}. The two relationships only occur when the chosen center is X_2, the centroid. The relationships are shown in Table 4.

Table 4.

Central Quadrilaterals formed by an Arbitrary Point		
Quadrilateral Type	**Relationship**	**centers**
general	$[ABCD] = \frac{9}{2}[FGHI]$	2
square	$ABCD \sim FGHI$	2

We were able to find geometric proofs for these relationships.

Relationship $[ABCD] = \frac{9}{2}[FGHI]$

Theorem 6.1. *Let E be any point in the plane of convex quadrilateral $ABCD$ not on a sideline of the quadrilateral. Let F, G, H, and I be the centroids of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively (Figure 7). Then $FGHI$ is a parallelogram and*

$$[ABCD] = \frac{9}{2}[FGHI].$$

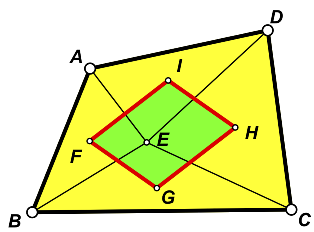

Figure 7. E arbitrary, centroids $\implies \dfrac{[ABCD]}{[FGHI]} = \dfrac{9}{2}$

Proof. Let P be the midpoint of BC and let Q be the midpoint of CD (Figure 8).

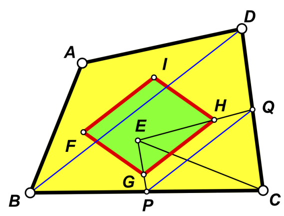

Figure 8.

Since G is the centroid of $\triangle BEC$, EP is a median of $\triangle BEC$ and $EG/GP = 2$ by Lemma 5.1. Similarly, $EH/HQ = 2$. Thus, $GH \parallel PQ$. But $PQ \parallel BD$, so $GH \parallel BD$. In the same manner, $FI \parallel BD$. Hence, $GH \parallel FI$. Likewise, $FG \parallel IH$. Thus, $FGHI$ is a parallelogram.

Now, let R be the midpoint of DA and let S be the midpoint of AB (Figure 9).

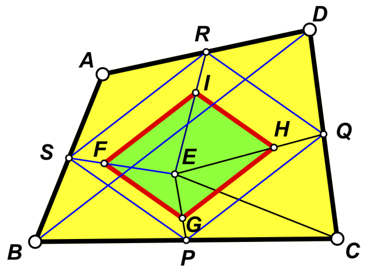

Figure 9.

Then $PQRS$ is a parallelogram similar to parallelogram $FGHI$ with ratio of similarity $3:2$ since $EQ/EH = 3/2$. Thus,

$$\frac{[PQRS]}{[FGHI]} = \frac{9}{4}. \tag{1}$$

Now

$$\frac{[ABCD]}{[PQRS]} = 2 \tag{2}$$

by Lemma 5.2. Combining equations (1) and (2) gives

$$\frac{[ABCD]}{[FGHI]} = \frac{9}{2}$$

and we are done. □

Relationship $ABCD \sim FGHI$

Theorem 6.2. *Let E be any point in the plane of square $ABCD$ not on a sideline of the square. Let F, G, H, and I be the centroids of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively (Figure 10). Then $FGHI$ is a square.*

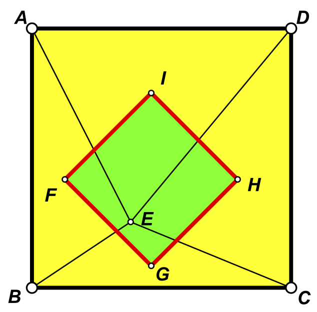

Figure 10. E arbitrary, square, centroids $\implies$ square

Proof. By Theorem 6.1, $FGHI$ is a parallelogram. From the proof of Theorem 6.1, we see that each side of this parallelogram has length equal to half the length of one of the diagonals of the square. Since the diagonals of a square are equal, the parallelogram must also be a square. $\square$

Conjecture 1. *Let E be any point inside square $ABCD$. Let F, G, H, and I be centers of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively, with the same center function. If $FGHI$ is a square independent of point E, then the four centers must be centroids.*

7. Results Using the Diagonal Point

In this configuration, the radiator, E, is the diagonal point of the reference quadrilateral $ABCD$ (the point of intersection of the diagonals). In this case, the radial triangles are also called *quarter triangles*.

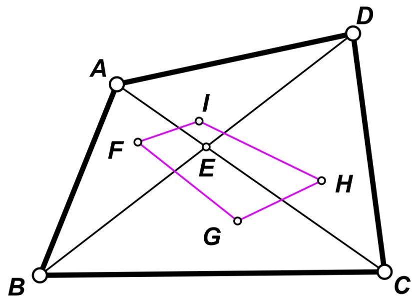

Figure 11. Central quadrilateral formed using the diagonal point

Our computer analysis found a number of relationships between the reference quadrilateral $ABCD$ and the central quadrilateral $FGHI$. Table 5 shows the relationship found for an arbitrary quadrilateral. Table 6 on page 230 shows the relationships found for specific shaped quadrilaterals, other than a square.

Table 5.

Central Quadrilaterals formed by the Diagonal Point		
Quadrilateral Type	**Relationship**	**centers**
general	$[ABCD]=\frac{1}{2}[FGHI]$	20

7.1. Proofs for General Quadrilaterals.

We now give a proof for the result listed in Table 5 for a general quadrilateral.

Relationship $[ABCD]=\frac{1}{2}[FGHI]$

Theorem 7.1. *Let E be the diagonal point of convex quadrilateral $ABCD$. Let F, G, H, and I be the X_{20} points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively (Figure 12). Then $FGHI$ is a parallelogram and*

$$[ABCD]=\frac{1}{2}[FGHI].$$

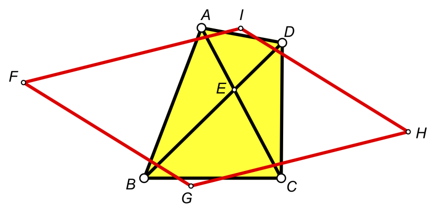

Figure 12. $X_{20} \implies [ABCD]=\frac{1}{2}[FGHI]$

Proof. We set up a barycentric coordinate system using $\triangle ABC$ as the reference triangle, so that

$$A = (1 : 0 : 0)$$
$$B = (0 : 1 : 0)$$
$$C = (0 : 0 : 1).$$

We let the barycentric coordinates of D be $(p : q : r)$ with $p + q + r = 1$ and without loss of generality, assume $p > 0$, $q < 0$, and $r > 0$ (Figure 13).

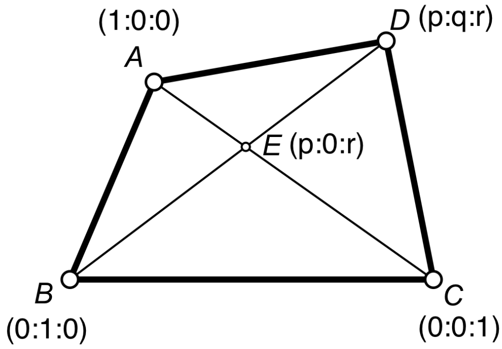

FIGURE 13. Set-up for Quadrilateral coordinate system

The equation for line AC is $y = 0$. The equation for line BD is $rx = pz$. Therefore, the barycentric coordinates for the diagonal point E are $(p : 0 : r)$. Note that the barycentric coordinates for A, B, C, and D are already normalized. The normalized barycentric coordinates for E are

$$E = \left(\frac{p}{p+r} : 0 : \frac{r}{p+r}\right).$$

From [7], we find that barycentric coordinates for the X_{20} point of a triangle ABC with sides a, b, and c are

$$\begin{aligned} X_{20} =\big(&3a^4 - 2a^2b^2 - 2a^2c^2 - b^4 + 2b^2c^2 - c^4 : \\ &- a^4 - 2a^2b^2 + 2a^2c^2 + 3b^4 - 2b^2c^2 - c^4 : \\ &- a^4 + 2a^2b^2 - 2a^2c^2 - b^4 - 2b^2c^2 + 3c^4\big). \end{aligned}$$

To convert these to normalized barycentric coordinates, divide each coordinate by their sum,

$$a^4 + b^4 + c^4 - 2a^2b^2 - 2b^2c^2 - 2c^2a^2.$$

Now we want to find the barycentric coordinates (with respect to $\triangle ABC$) for the X_{20} point of $\triangle ABE$. First, we compute the lengths of the sides of $\triangle ABE$ using the Distance Formula (Lemma 5.3). We find that

$$AB = c$$
$$BE = \frac{\sqrt{a^2r(p+r) - b^2pr + c^2p(p+r)}}{p+r}$$
$$CE = \frac{br}{p+r}$$

If we call these lengths a_1, b_1, and c_1, respectively, then we can use the Change of Coordinates Formula (Lemma 5.5) to find the coordinates for point F, the X_{20}

point of $\triangle ABE$, by substituting a_1, b_1, and c_1 for a, b, and c in the expression for the normalized barycentric coordinates for X_{20}. We find that the barycentric coordinates for F before normalization (with respect to $\triangle ABC$) are

$$F = \Big(a^4(2p+3r) - 2a^2\left(b^2(2p+r)+c^2r\right) + \left(b^2-c^2\right)\left(b^2(2p-r)+c^2(2p+r)\right) :$$
$$-a^4(p+r) + 2a^2\left(b^2(p-r)+c^2(p+r)\right) - b^4(p-3r) - 2b^2c^2(3p+r) - c^4(p+r) :$$
$$a^4(-r) - 2a^2\left(c^2(2p+r)-b^2r\right) - b^4r + 2b^2c^2(2p-r) + c^4(4p+3r)\Big).$$

Using the same procedure, similar expressions are found for the coordinates of points G, H, and I, the X_{20} points of triangles BCE, CDE, and DAE, respectively.

Next, we want to compare the area of quadrilaterals $ABCD$ and $FGHI$. Letting K be the area of $\triangle ABC$, we can use the Area Formula (Lemma 5.4) to find the area of $\triangle CDA$. We find that

$$[CDA] = -qK.$$

Note that this area is positive since we assumed $q < 0$. Thus,

$$[ABCD] = [ABC] + [CDA] = K - qK = K(1-q).$$

To compute the area of quadrilateral $FGHI$, we take a shortcut. By Theorem 5.1 of [20], quadrilateral $FGHI$ is a parallelogram. Thus,

$$[FGHI] = 2[FGH].$$

Computing the area of $\triangle FGH$ using the Area Formula, we find (after simplifying and using the fact that $p+q+r=1$):

$$[FGH] = K(1-q).$$

Consequently, $[ABCD] = [FGH] = \frac{1}{2}[FGHI]$. □

Open Question 1. *Is there a simpler or purely geometric proof for Theorem 7.1?*

7.2. Proofs for Orthodiagonal Quadrilaterals.

We now give proofs for the results listed in Table 6 for orthodiagonal quadrilaterals.

TABLE 6.

Central Quadrilaterals formed by the Diagonal Point		
Quadrilateral Type	**Relationship**	**centers**
orthodiagonal	$[ABCD] = 32[FGHI]$	546
	$[ABCD] = 18[FGHI]$	381
	$[ABCD] = 8[FGHI]$	5, 402
	$[ABCD] = 2[FGHI]$	3, 97, 122, 123, 127, 131, 216, 268, 339, 382, 408, 417, 418, 426, 440, 441, 454, 464–466, 577, 828, 852, 856
	$[ABCD] = \frac{1}{2}[FGHI]$	22, 23, 151, 175, 253, 280, 347, 401, 858, 925
equiorthodiagonal	$[ABCD] = 2[FGHI]$	124
	$[ABCD] = \frac{1}{2}[FGHI]$	102
rhombus	$[ABCD] = 4[FGHI]$	10
	$[ABCD] = [FGHI]$	40, 84
	$\partial ABCD = \partial FGHI$	40, 84
rectangle	$[ABCD] = 8[FGHI]$	402, 620
	$[ABCD] = 6[FGHI]$	395, 396
	$[ABCD] = 2[FGHI]$	11, 115, 116, 122–125, 127, 130, 134–137, 139, 244–247, 338, 339, 865–868
	$[ABCD] = \frac{3}{2}[FGHI]$	616, 617
	$[ABCD] = \frac{1}{2}[FGHI]$	146–153

Relationship $[ABCD] = 32[FGHI]$

Proposition 7.2 (X_{546} Property of a Right Triangle)**.** *Let P be the X_{546} point of right triangle ABC, with A being the vertex of the right angle. Then $BC = 8AP$.*

Proof. Let M be the midpoint of the hypotenuse of $\triangle ABC$. Let N be the nine-point center of $\triangle ABC$. Let P be the X_{546} point of $\triangle ABC$ (Figure 14). By Lemma 5.7, N is the midpoint of AM. Since $AM = \frac{1}{2}BC$, this means $AN = \frac{1}{4}BC$. According to [18], P is the midpoint of HN where H is the orthocenter of $\triangle ABC$. But the orthocenter of a right triangle is the vertex of the right angle, so P is the midpoint of AN and $AP = \frac{1}{2}AN$. Thus $AP = \frac{1}{8}BC$. $\square$

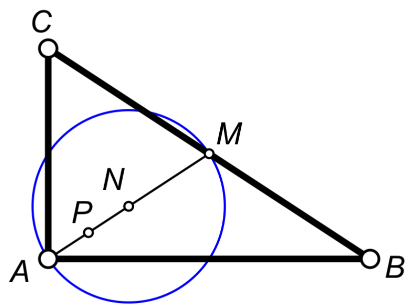

FIGURE 14. right triangle, $P = X_{546} \implies BC = 8AP$

Theorem 7.3. *Let E be the diagonal point of orthodiagonal quadrilateral $ABCD$. Let F, G, H, and I be the X_{546} point of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively. Then*

$$[ABCD] = 32[FGHI].$$

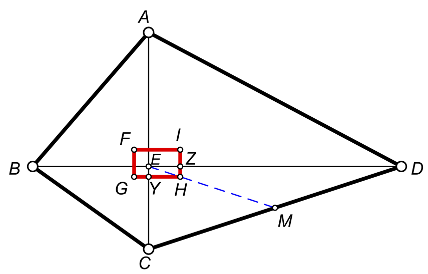

FIGURE 15. orthodiagonal, $X_{546} \implies [ABCD] = 32[FGHI]$

Proof. Since $ABCD$ is orthodiagonal, $\triangle ECD$ is a right triangle. By Proposition 7.2, $EH = \frac{1}{8}CD$. Let M be the midpoint of CD and let Y and Z be the feet of the perpendiculars from H to EC and ED, respectively. By Lemma 5.8,

$$\frac{[ECD]}{[EYHZ]} = 2\left(\frac{EM}{EH}\right)^2 = 2\left(\frac{CD/2}{CD/8}\right)^2 = 32.$$

By symmetry, the same is true for the other three radial triangles. Thus,

$$[ABCD] = 32[FGHI].$$

□

Relationship $[ABCD] = 18[FGHI]$

Theorem 7.4. *Let E be the diagonal point of orthodiagonal quadrilateral $ABCD$. Let F, G, H, and I be the X_{381} point of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively (Figure 16). Then*

$$[ABCD] = 18[FGHI].$$

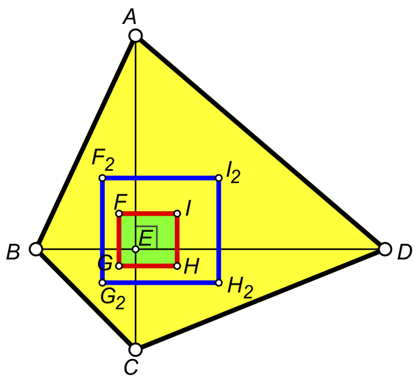

FIGURE 16. orthodiagonal, $X_{381} \implies [ABCD] = 18[FGHI]$

Proof. Let F_2, G_2, H_2, and I_2 be the X_2 point (centroid) of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively (Figure 16). By Theorem 6.1,

$$[FGHI] = \frac{2}{9}[ABCD].$$

From [13] we learn that the X_{381} point of a right triangle is the midpoint of the centroid and orthocenter of that triangle. The orthocenter of a right triangle coincides with the vertex of the right angle. Therefore, F is the midpoint of EF_2. The same reasoning applies to G, H, and I. Therefore, quadrilateral $FGHI$ is similar to quadrilateral $F_2G_2H_2I_2$, with similarity ratio 2. Thus,

$$[FGHI] = \frac{1}{4}[F_2G_2H_2I_2] = \frac{1}{4}\left(\frac{2}{9}[ABCD]\right) = \frac{1}{18}[ABCD]$$

or $[ABCD] = 18[FGHI]$. □

Relationship $[ABCD] = 8[FGHI]$

Lemma 7.5. *In a right triangle, the X_5 point coincides with the X_{402} point.*

Proof. From [16], we find that the barycentric coordinates for the X_{402} point are $(p : q : r)$ where
$p = (2a^4 - a^2b^2 - a^2c^2 - b^4 + 2b^2c^2 - c^4)(a^8 - a^6b^2 - a^6c^2 - 2a^4b^4 + 5a^4b^2c^2 - 2a^4c^4 + 3a^2b^6 - 3a^2b^4c^2 - 3a^2b^2c^4 + 3a^2c^6 - b^8 - b^6c^2 + 4b^4c^4 - b^2c^6 - c^8)$,
$q = (-a^4 - a^2b^2 + 2a^2c^2 + 2b^4 - b^2c^2 - c^4)(-a^8 + 3a^6b^2 - a^6c^2 - 2a^4b^4 - 3a^4b^2c^2 + 4a^4c^4 - a^2b^6 + 5a^2b^4c^2 - 3a^2b^2c^4 - a^2c^6 + b^8 - b^6c^2 - 2b^4c^4 + 3b^2c^6 - c^8)$,
and
$r = (-a^4 + 2a^2b^2 - a^2c^2 - b^4 - b^2c^2 + 2c^4)(-a^8 - a^6b^2 + 3a^6c^2 + 4a^4b^4 - 3a^4b^2c^2 - 2a^4c^4 - a^2b^6 - 3a^2b^4c^2 + 5a^2b^2c^4 - a^2c^6 - b^8 + 3b^6c^2 - 2b^4c^4 - b^2c^6 + c^8)$.
When $a^2 = b^2 + c^2$, these coordinates simplify to

$$X_{402} = \left(16b^6c^6 : 8b^6c^6 : 8b^6c^6\right) = (2 : 1 : 1).$$

The barycentric coordinates for the X_5 point are

$$\left(a^2\left(b^2+c^2\right) - \left(b^2-c^2\right)^2 : b^2\left(a^2+c^2\right) - \left(c^2-a^2\right)^2 : c^2\left(a^2+b^2\right) - \left(a^2-b^2\right)^2\right).$$

When $a^2 = b^2 + c^2$, these coordinates simplify to

$$X_5 = \left(4b^2c^2 : 2b^2c^2 : 2b^2c^2\right) = (2 : 1 : 1).$$

Thus, for right triangles, X_{402} and X_5 coincide. The common point is the midpoint of the hypotenuse by Lemma 5.7. $\square$

Theorem 7.6. *Let E be the diagonal point of orthodiagonal quadrilateral $ABCD$. Let F, G, H, and I be the X_5 or X_{402} point of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively (Figure 17). Then $FGHI$ is a rectangle and*

$$[ABCD] = 8[FGHI].$$

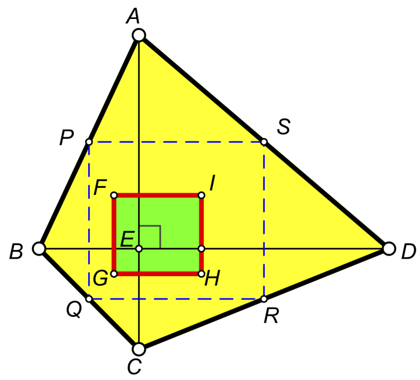

FIGURE 17. orthodiagonal, $X_5 \implies [ABCD] = 8[FGHI]$

Proof. By Lemma 7.5, we need only prove this theorem when the chosen center is the X_5 point (the nine-point center). Since the quadrilateral is orthodiagonal, each of the radial triangles is a right triangle. By Lemma 5.7, the X_5 point (nine-point center) of each of these triangles is the midpoint of the median to the hypotenuse. Thus, $FGHI$ is similar to the Varignon parallelogram $PQRS$ of the quadrilateral with ratio of similarity $\frac{1}{2}$ (Figure 17). So the ratio of their areas is $1 : 4$. By Lemma 5.2, the Varignon parallelogram has half the area of the quadrilateral. So

$$[FGHI] = \frac{1}{4}[PQRS] = \frac{1}{4}\left(\frac{1}{2}[ABCD]\right) = \frac{1}{8}[ABCD]$$

or $[ABCD] = 8[FGHI]$. $\square$

$\boxed{\text{Relationship } [ABCD] = 2[FGHI]}$

Theorem 7.7. *Let E be the diagonal point of orthodiagonal quadrilateral $ABCD$. Let F, G, H, and I be centers of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively (Figure 18). If the center function for the four centers have the properties $f(a, b, c) = 0$ and $f(b, c, a) = f(c, a, b)$ when $c^2 = a^2+b^2$, then $FGHI$ is a rectangle and $[ABCD] = 2[FGHI]$.*

Proof. By Lemma 5.6, F, G, H, and I are the midpoints of the sides of the quadrilateral (Figure 18). By Lemma 5.2, these midpoints form a parallelogram whose sides are parallel to the diagonals of the quadrilateral. Since the diagonals of the quadrilateral are perpendicular, this parallelogram must be a rectangle. Also by Lemma 5.2, the area of this rectangle is half the area of the quadrilateral. $\square$

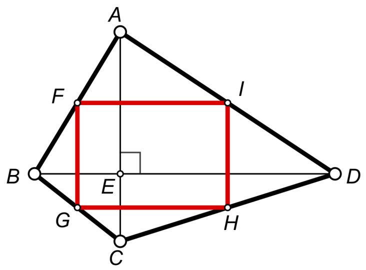

Figure 18. $[ABCD] = 2[FGHI]$

Examples. Some examples of centers described by Theorem 7.7 are X_3, X_{97}, X_{122}, X_{123}, X_{127}, X_{131}, X_{216}, X_{268}, X_{339}, X_{382}, X_{408}, X_{417}, X_{418}, X_{426}, X_{440}, X_{441}, X_{454}, X_{464}, X_{465}, X_{466}, X_{577}, X_{828}, X_{852}, and X_{856}. These agree with the centers found for orthodiagonal quadrilaterals listed in Table 6 with relationship $[ABCD] = 2[FGHI]$. For an orthodiagonal quadrilateral, these centers all coincide with the midpoints of the sides of the quadrilateral.

Theorem 7.8. *Let ABC be a right triangle with right angle at A. Let M be the midpoint of the hypotenuse. Then the center X_n coincides with M for the following values of n:*

3, 97, 122, 123, 127, 131, 216, 268, 339, 408, 417, 418, 426, 440, 441, 454, 464, 465, 466, 577, 828, 852, 856.

Theorem 7.9. *Let ABC be a right triangle with right angle at A. Let M be the midpoint of the hypotenuse. Then the center X_n lies on the line AM (but does not coincide with A or M) for the following values of n:*

2, 5, 20, 21, 22, 23, 95, 140, 199, 233, 237, 253, 280, 347, 376, 377, 379, 381, 382, 383, 401, 402, 404, 405, 409, 411, 413, 416, 439, 442, 443, 446, 448, 449, 452, 453, 474, 546, 547, 548, 549, 550, 631, 632, 851, 853, 854, 855, 857, 858, 859, 861, 863, 864, 865, 866, 867, 868, 925, 964.

We have excluded values of n for which X_n lies on the line at infinity.

Proof. The normalized barycentric coordinates for M are $\left(0 : \frac{1}{2} : \frac{1}{2}\right)$, using Formula (12) from [3]. The barycentric equation for line AM is $y = z$, using Equation (3) from [3]. If P lies on line AM and has barycentric coordinates $(u : v : w)$, we must have $v = w$. Examining the first 1,000 centers in [5], those for which $v = w$ when $a^2 = b^2 + c^2$ are the ones given by Theorems 7.8 and 7.9. The ones for which $u = 0$ are the ones given by Theorem 7.8. □

Theorem 7.10. *Let ABC be a right triangle with right angle at A. Let M be the midpoint of the hypotenuse. Then the center X_n lies on the line AM and the ratio of AM to AX_n is a constant for the values shown in Table 7:*

Proof. The lengths of AM and AX_n are easily found (by computer) using the Distance Formula (Lemma 5.3). The resulting ratio is simplified using the constraint that $a^2 = b^2 + c^2$. Values of this ratio that are not constant are discarded. □

Combining the data in Theorem 7.10 with Lemma 5.8 gives cases where $\frac{[ABC]}{[AZXY]}$ is rational, which in turn gives cases where $[ABCD]/[FGHI]$ is rational where

Table 7. Values of n for which the ratio $AM : AX_n$ is a constant

n	**ratio**	n	**ratio**	n	**ratio**	n	**ratio**
2	$\frac{3}{2}$	140	$\frac{4}{3}$	402	2	547	$\frac{12}{7}$
3	1	216	1	408	1	548	$\frac{4}{5}$
5	2	233	$\frac{5}{3}$	417	1	549	$\frac{6}{5}$
20	$\frac{1}{2}$	253	$\frac{1}{2}$	418	1	550	$\frac{2}{3}$
22	$\frac{1}{2}$	268	1	426	1	577	1
23	$\frac{1}{2}$	280	$\frac{1}{2}$	440	1	631	$\frac{5}{4}$
95	$\frac{5}{4}$	339	1	441	1	632	$\frac{10}{7}$
97	1	347	$\frac{1}{2}$	454	1	828	1
122	1	376	$\frac{3}{4}$	464	1	852	1
123	1	381	3	465	1	856	1
127	1	382	1	466	1	858	$\frac{1}{2}$
131	1	401	$\frac{1}{2}$	546	4	925	$\frac{1}{2}$

$FGHI$ is the central quadrilateral formed by the diagonal point using center X_n in an orthodiagonal quadrilateral $ABCD$. This therefore proves the entries in Table 6 for the orthodiagonal quadrilateral entries.

It is interesting to note that the entries in Table 6 for the orthodiagonal quadrilateral with $[ABCD] = 2[FGHI]$ includes the point X_{382} which does not appear in Theorem 7.8. This is because the X_{382} of a right triangle coincides with the reflection of the hypotenuse midpoint M about the vertex of the right angle, A. This agrees with [14] which states that X_{382} is the reflection of the circumcenter about the orthocenter. (In a right triangle, the circumcenter is M and the orthocenter is A.)

Table 7 implies additional results about central quadrilaterals associated with orthodiagonal quadrilaterals using the diagonal point as the radiator that do not appear in Table 6. This is because the computer analysis that produced Table 6 only checked area ratios where the denominator was less than 6. These supplementary results are shown in Table 8. They are obtained by applying Lemma 5.8 to the entries in Table 7.

TABLE 8. Supplementary Results

Central Quadrilaterals formed by the Diagonal Point		
Quadrilateral Type	**Relationship**	**centers**
orthodiagonal	$[ABCD] = \frac{288}{49}[FGHI]$	547
	$[ABCD] = \frac{50}{9}[FGHI]$	233
	$[ABCD] = \frac{200}{49}[FGHI]$	632
	$[ABCD] = \frac{32}{9}[FGHI]$	140
	$[ABCD] = \frac{25}{8}[FGHI]$	95, 631
	$[ABCD] = \frac{72}{25}[FGHI]$	549
	$[ABCD] = \frac{32}{25}[FGHI]$	548
	$[ABCD] = \frac{9}{8}[FGHI]$	376
	$[ABCD] = \frac{8}{9}[FGHI]$	550

7.3. **Proofs for Equidiagonal Orthodiagonal Quadrilaterals.**

We now give proofs for the results listed in Table 6 for equidiagonal orthodiagonal quadrilaterals.

Lemma 7.11. *The area of an orthodiagonal quadrilateral with diagonals of length x and y is $\frac{1}{2}xy$.*

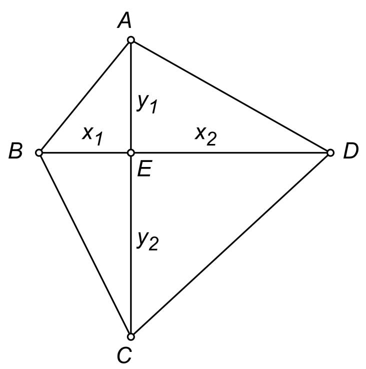

FIGURE 19. An orthodiagonal quadrilateral

Proof. Let the segments of the diagonal of length x be x_1 and x_2. Let the segments of the diagonal of length y be y_1 and y_2 (Figure 19). Then

$$\begin{aligned}[ABCD] &= [ABE] + [BCE] + [CDE] + [DAE] \\ &= \frac{1}{2}x_1y_1 + \frac{1}{2}x_1y_2 + \frac{1}{2}x_2y_2 + \frac{1}{2}x_2y_1 \\ &= \frac{1}{2}(x_1 + x_2)(y_1 + y_2) \\ &= \frac{1}{2}xy.\end{aligned}$$

□

Lemma 7.12. *The area of an equidiagonal orthodiagonal quadrilateral with diagonal of length d is $\frac{1}{2}d^2$.*

Proof. Let $x = y = d$ in Lemma 7.11. □

Relationship $[ABCD] = \frac{1}{2}[FGHI]$

Lemma 7.13. *Let ABC be a right triangle with right angle at A (Figure 20). Then the X_{102} point of $\triangle ABC$ lies on the angle bisector of $\angle BAC$ and also lies on the perpendicular bisector of BC.*

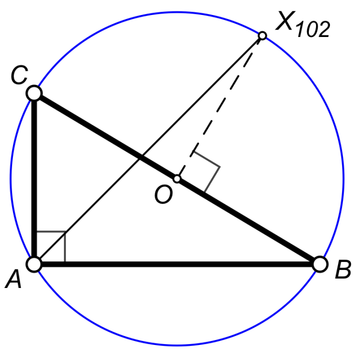

FIGURE 20. The X_{102} point of a right triangle

Proof. From [11], we find that the center function corresponding to X_{102} is

$$f(a,b,c) = a \times U \times V$$

where

$$U = a^4 - a^3c - 2a^2b^2 + a^2bc + a^2c^2 + ab^2c - 2abc^2 + ac^3 + b^4 - b^3c + b^2c^2 + bc^3 - 2c^4$$

and

$$V = a^4 - a^3b + a^2b^2 + a^2bc - 2a^2c^2 + ab^3 - 2ab^2c + abc^2 - 2b^4 + b^3c + b^2c^2 - bc^3 + c^4.$$

(Recall that the center function is the first component of the *trilinear* coordinates for X_{102}.) A little computation shows that $f(a,b,c) - f(b,a,c)$ factors as

$$(a-b)(a+b-c)(a+b+c)\left(a^2+b^2-c^2\right)W$$

where

$$W = a^4 - a^3c + a^2\left(-2b^2 + bc + c^2\right) + ac(b-c)^2 + b^4 - b^3c + b^2c^2 + bc^3 - 2c^4.$$

Thus, $f(a,b,c) = f(b,a,c)$ when $a^2 + b^2 = c^2$. Therefore, by Lemma 5.9, X_{102} lies on the angle bisector of vertex A.

Also from [11], we learn that the X_{102} point of a triangle lies on its circumcircle. Since the angle bisector of $\angle BAC$ bisects the arc from B to C, X_{102} must lie on the perpendicular bisector of side BC. □

Theorem 7.14. *Let E be the diagonal point of equidiagonal orthodiagonal quadrilateral $ABCD$. Let F, G, H, and I be the X_{102} points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively (Figure 21). Then $FGHI$ is an equidiagonal orthodiagonal quadrilateral and*

$$\frac{[ABCD]}{[FGHI]} = \frac{1}{2}.$$

Furthermore, quadrilaterals $ABCD$ and $FGHI$ have the same diagonal point and centroid and the diagonals of $FGHI$ bisect the right angles formed by the diagonals of $ABCD$.

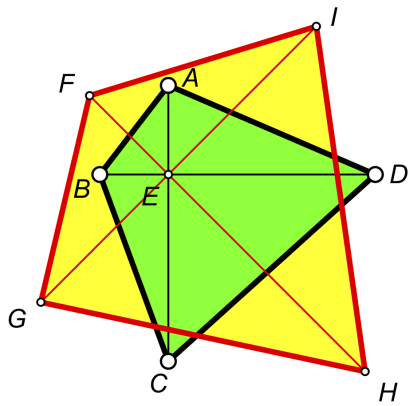

FIGURE 21. equiortho, $X_{102} \implies \frac{[ABCD]}{[FGHI]} = \frac{1}{2}$

The following proof is due to Ahmet Çetin [2].

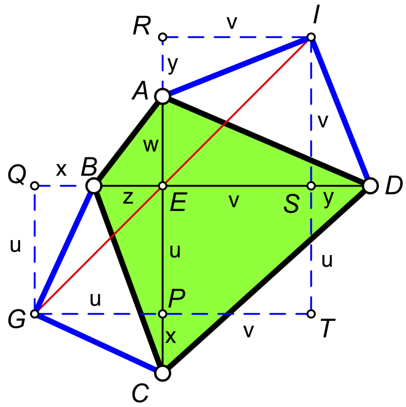

FIGURE 22.

Proof. Draw lines through G and I parallel to AC. Draw lines through G and I parallel to BD. These lines form intersection points, P, Q, R, S, and T as shown in Figure 22. Let $\angle ADE = \theta$. By Lemma 7.13, I lies on the perpendicular bisector of AD, so $\angle IDA = 45°$. Hence $\angle IDS = \theta + 45°$. Now

$$\angle IAR = 180° - \angle DAI - \angle EAD = 180° - 45° - (90° - \theta) = \theta + 45°.$$

Thus, $\angle IDS = \angle IAR$. Since angles $\angle DSI$ and $\angle ARI$ are right angles, and $ID = IA$, we can conclude that $\triangle IDS \cong \triangle IAR$. Hence, $DS = AR = y$. Similarly, $CP = BQ = x$.

Since $ERIS$ is a parallelogram and diagonal EI bisects $\angle SER$, $ERIS$ mist be a square. Similarly, $EPGQ$ is a square. Thus $v = w + y$ where $ES = v$. Similarly, $EPGQ$ is a square and $PG = PE = u = x + z$ where $BQ = x$ and $EB = z$. Quadrilateral $PEST$ is a rectangle, so $PE = TS = u$. Hence $GT = u + v = TI$. Thus $\triangle GTI$ is an isosceles right triangle and therefore $GI = GT\sqrt{2}$.

Since $ABCD$ is equidiagonal,

$$w + u + x = z + v + y.$$

Since $u = x + z$,

$$w + u + x = w + (x + z) + x.$$

Since $v = w + y$,

$$z + v + y = z + (w + y) + y.$$

Thus

$$w + (x + z) + x = z + (w + y) + y$$

which implies that $x = y$.

But $GT = u + v = u + w + y = u + w + x = AC$. Since $GI = GT\sqrt{2}$, we have $GI^2 = 2AC^2$. By Lemma 7.12, $[FGHI] = 2[ABCD]$. □

$\boxed{\text{Relationship } [ABCD] = 2[FGHI]}$

Theorem 7.15. *Let E be the diagonal point of equidiagonal orthodiagonal quadrilateral $ABCD$. Let F, G, H, and I be the X_{124} points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively (Figure 23). Then $FGHI$ is an equidiagonal orthodiagonal quadrilateral and*

$$[ABCD] = 2[FGHI].$$

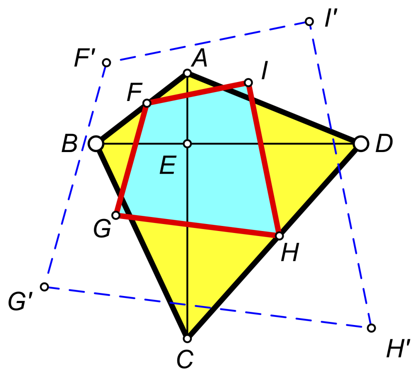

FIGURE 23. equiortho, $X_{124} \implies [ABCD] = 2[FGHI]$

Proof. Let F', G', H', and I' be the X_{102} points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively (Figure 23). According to [12], the X_{124} point of a triangle is the midpoint of its X_4 point and its X_{102} point. But the X_4 point of a right triangle is the vertex of the right angle, so the X_4 points of all four radial triangles is point E. Thus, F is the midpoint of EF' with similar results for G, H, and I. Hence quadrilateral $FGHI$ is homothetic to quadrilateral $F'G'H'I'$ with ratio of similitude 2. But $[F'G'H'I'] = 2[ABCD]$ by Theorem 7.14. Therefore, $[FGHI] = \frac{1}{2}[ABCD]$. Also $F'G'H'I'$ is equidiagonal and orthodiagonal. Since the quadrilaterals are homothetic, $FGHI$ must also be an equidiagonal orthodiagonal quadrilateral. $\square$

7.4. **Proofs for Rhombi.**

We now give proofs for the results listed in Table 6 for rhombi.

Relationship $[ABCD] = 4[FGHI]$

Lemma 7.16. *Let ABC be a right triangle with right angle at A. Let S be the Spieker center (X_{10} point) of $\triangle ABC$. Let J and K be the feet of perpendiculars from S to AC and AB, respectively, so that $AKSJ$ is a rectangle (Figure 24). Then $[ABC] = 4[AKSJ]$.*

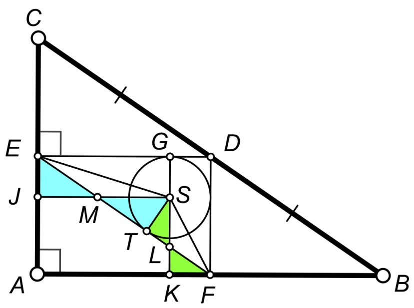

FIGURE 24. S is the Spieker center of $\triangle ABC$

Proof. Let D, E, and F be the midpoints of the sides of $\triangle ABC$ as shown in Figure 24. From [6], it is known that S is the incenter of the medial triangle DEF. Let G and T be the feet of the perpendiculars from S to DE and EF, respectively. Segments SG and ST are radii of the incircle of $\triangle DEF$, so $ST = SG$. Since $SG \perp DE$, $SGEJ$ is a rectangle and $SG = EJ$. Therefore, $EJ = ST$ and $\triangle EJM \cong \triangle STM$. Similarly, $\triangle FKL \cong \triangle STL$.

So the green triangles have the same area and the blue triangles have the same area. Therefore, $[AKSJ] = [AFE] = \frac{1}{4}[ABC]$. ☐

Theorem 7.17. *Let E be the diagonal point of rhombus $ABCD$. Let F, G, H, and I be the X_{10} points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively (Figure 25). Then $FGHI$ is a rectangle and*

$$[ABCD] = 4[FGHI].$$

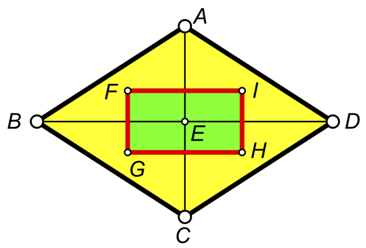

FIGURE 25. rhombus, $X_{10} \implies [ABCD] = 4[FGHI]$

Proof. Since $ABCD$ is a rhombus, $AE \perp ED$. By Lemma 7.16, the rectangle with diagonal EI has one-fourth the area of $\triangle AED$. By symmetry, the same is true for the rectangles with diagonals, EF, EG, and EH. Thus $[ABCD] = 4[FGHI]$. ☐

Relationship $[ABCD] = [FGHI]$

The following result is well known [27].

Lemma 7.18. *If r is the inradius of a right triangle with hypotenuse c and legs a and b, then*

$$r = \frac{a+b-c}{2}.$$

Proposition 7.19 (X_{40} Property of a Right Triangle)**.** *Let $\triangle ABC$ be a right triangle with right angle at C. Let F be its X_{40} point. Let $BC = a$, $AC = b$, and $AB = c$. Let the distance from the F to BC be p and let the distance from F to AC be q as shown in Figure 26. Then*

$$p + q = c \qquad \text{and} \qquad 2pq = ab.$$

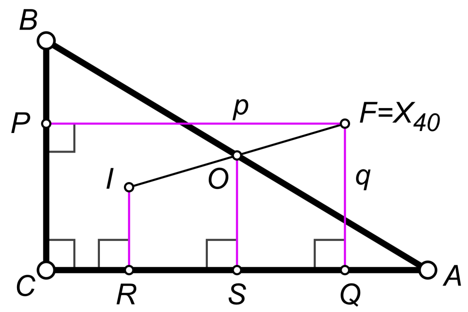

FIGURE 26. X_{40} point of a right triangle

Proof. Let F be the Bevan point of $\triangle ABC$. According to [8], the Bevan point, $F = X_{40}$ is the reflection of $I = X_1$ about $O = X_3$. Let R and S be the projections of I and O, respectively, on AC. Since $\triangle ABC$ is a right triangle, O is the midpoint of AB, and $OS = a/2$. Since I is the center of the incircle of $\triangle ABC$, $IR = r$, where r is the inradius. Since O is the midpoint of IF, we have

$$OS = \frac{IR + FQ}{2}$$

or $a = q + r$. Thus, $q = a - r$. Similarly, $p = b - r$. Thus, using Lemma 7.18, we have

$$p + q = a + b - 2r = c$$

and

$$\begin{aligned}
pq &= (a-r)(b-r) \\
&= \left(a - \frac{a+b-c}{2}\right)\left(b - \frac{a+b-c}{2}\right) \\
&= \left(\frac{a-b+c}{2}\right)\left(\frac{b-a+c}{2}\right) \\
&= \frac{c^2 - (a-b)^2}{4} \\
&= \frac{a^2 + b^2 - (a-b)^2}{4} \\
&= \frac{ab}{2}.
\end{aligned}$$

□

Proposition 7.20 (X_{84} Property of a Right Triangle)**.** *Let $\triangle ABC$ be a right triangle with right angle at C. Let F be its X_{84} point. Let $BC = a$, $AC = b$, and $AB = c$. Let the distance from the F to BC be p and let the distance from F to AC be q as shown in Figure 27. Then*

$$p + q = c \qquad \textit{and} \qquad 2pq = ab.$$

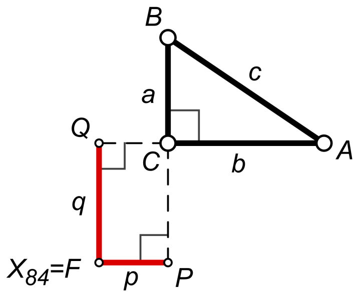

Figure 27. X_{84} point of a right triangle

Proof. According to [10], the barycentric coordinates for the X_{84} point of a triangle are

$$\left(a^3\left(a^2-(b-c)^2\right)^2 - a(b-c)^2\left(a^2-(b+c)^2\right)^2 ::\right).$$

With the condition $a^2 + b^2 = c^2$, this simplifies to

$$(b - c : a - c : c).$$

Note that

$$[BFC] = \frac{1}{2} BC \times PF$$

so

$$p = \frac{2}{a}[BFC].$$

Using the Area Formula, we find that

$$[BFC] = \frac{(c-b)}{(a+b-c)} K$$

where K is the area of $\triangle ABC$ (which, in this case, is $ab/2$). Therefore,

$$p = \frac{2}{a}[BFC] = \frac{2(c-b)}{a(a+b-c)} K = \frac{2(c-b)}{a(a+b-c)}\left(\frac{ab}{2}\right) = \frac{b(c-b)}{a+b-c}.$$

In the same way, we find

$$q = \frac{a(c-a)}{a+b-c}.$$

Thus,

$$p + q = \frac{ac + bc - (a^2+b^2)}{a+b-c} = \frac{ac+bc-c^2}{a+b-c} = c$$

and

$$pq = \frac{ab(c-b)(c-a)}{(a+b-c)^2} = \frac{a^2b^2 - a^2bc - ab^2c + abc^2}{a^2 + 2ab - 2ac + b^2 - 2bc + c^2}$$

which simplifies to $ab/2$ when we substitute $a^2 + b^2$ for c^2. □

Theorem 7.21. *Let E be the diagonal point of rhombus $ABCD$. Let F, G, H, and I be the X_{40} points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively. Let F', G', H', and I' be the X_{84} points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively (Figure 28). Then $FGHI$ and $F'G'H'I'$ are congruent rectangles and the three quadrilaterals have the same perimeter and the same area.*

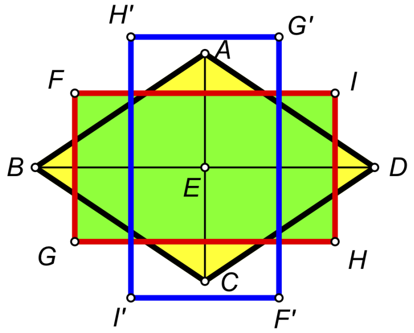

FIGURE 28. $X_{40} \implies \partial ABCD = \partial FGHI$

Proof. By symmetry considerations (or using Theorem 5.50 from [20]), we have that the central quadrilaterals are rectangles with diagonal point E. The sides of these rectangles are parallel to the diagonals of $ABCD$. The diagonals of the rhombus divide each of the three figures into four congruent pieces. We therefore only need to prove the appropriate result for one of these pieces.

Part 1: X_{40}

Let P and Q be the projections of F, the X_{40} point of $\triangle ABE$, on sides BE and CE of right triangle ABE, respectively, as shown in Figure 29.

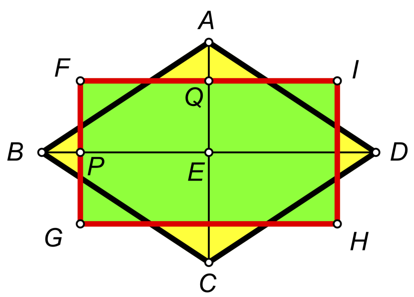

FIGURE 29.

By Proposition 7.19, $FP + FQ = AB$. Therefore, by symmetry, we have

$$FG + GH + HI + IF = AB + BC + CD + DA.$$

Also, $FP \cdot FQ = (BE \cdot AE)/2$. So $[FPEQ] = [ABE]$. Therefore, by symmetry, we have $[FGHI] = [ABCD]$. The rectangle and the rhombus therefore have equal areas and equal perimeters.

Part 2: X_{84}

Let P and Q be the projections of F', the X_{84} point of $\triangle ABE$, on sides BE and CE of right triangle ABE, respectively, as shown in Figure 30.

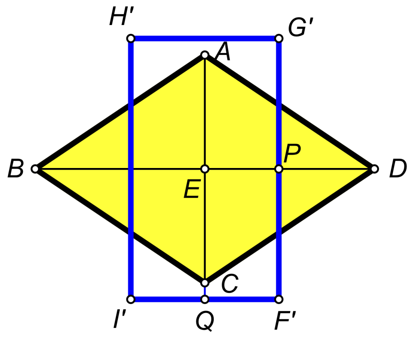

FIGURE 30.

By Proposition 7.20, $F'P + F'Q = AB$. Therefore, by symmetry, we have

$$F'G' + G'H' + H'I' + I'F' = AB + BC + CD + DA.$$

Also, $F'P \cdot F'Q = (BE \cdot AE)/2$. So $[FPEQ] = [ABE]$. Therefore, by symmetry, we have $[F'G'H'I'] = [ABCD]$. The rectangle and the rhombus therefore have equal areas and equal perimeters.

Finally, note that two rectangles that have equal areas and equal perimeters must be congruent, so $ABCD \cong FGHI \cong F'G'H'I'$. $\square$

7.5. **Proofs for Rectangles.**

We now give proofs for some of the results listed in Table 6 for rectangles.

The following result comes from [20, Theorem 5.52].

Lemma 7.22. *For any triangle center, if the reference quadrilateral is a rectangle, then the central quadrilateral is a rhombus. The two quadrilaterals have the same diagonal point. The sides of the rhombus are parallel to diagonals of the rectangle and are bisected by them (Figure 31).*

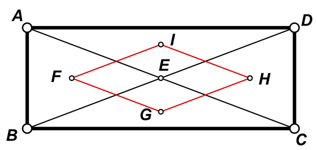

FIGURE 31. rectangle $\implies$ rhombus

Theorem 7.23. *Let ABC be an isosceles triangle with $AB = AC$. Let M be the midpoint of base BC. Then the center X_n lies on the line AM and the ratio of X_nM to AM is a constant for the values shown in Table 9:*

Proof. The ratios X_nM/AM are easily found (by computer) using Lemma 5.10. The resulting ratio is simplified using the constraint that $b = c$. Results where this ratio is not a constant are discarded. $\square$

Table 9. Values of n for which the ratio $X_nM : AM$ is a constant

n	**ratio**
2	$\frac{1}{3}$
148	-1
149	-1
150	-1
290	$-\frac{1}{3}$
402	$\frac{1}{2}$
620	$\frac{1}{2}$
671	$-\frac{1}{3}$
903	$-\frac{1}{3}$

Lemma 7.24. *Let ABC be an isosceles triangle with $AB = AC$. Let M be the midpoint of base BC. Let point X lie on the line AM such that (using signed distances) the ratio*

$$\frac{XM}{AM} = k.$$

Then (still using signed distances),

$$\frac{AX}{AM} = 1 - k.$$

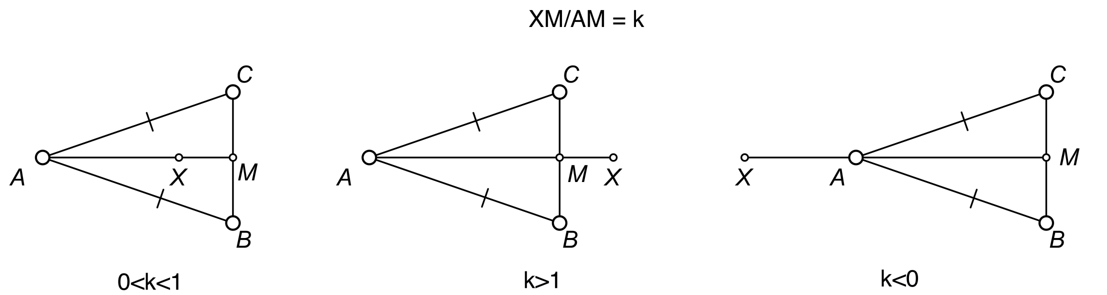

Figure 32. Location of X for various k

Proof. See Figure 32 for the three cases. If $0 < k < 1$, then

$$\frac{AX}{AM} = \frac{AM - XM}{AM} = 1 - k.$$

If $k > 1$, then

$$\frac{AX}{AM} = \frac{AM + MX}{AM} = \frac{AM - XM}{AM} = 1 - k.$$

If $k < 0$, then

$$\frac{AX}{AM} = \frac{MX - MA}{AM} = -\left(\frac{XM - AM}{AM}\right) = -(k - 1) = 1 - k.$$

∎

Relationship $[ABCD] = 8[FGHI]$

Theorem 7.25. *Let E be the diagonal point of rectangle $ABCD$. Let X_n be a triangle center with the property that for all isosceles triangles with vertex V and midpoint of base M, X_nM/VM is a fixed positive constant k. Let F, G, H, and I be the X_n points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively. Then*

$$[ABCD] = \frac{2}{(1-k)^2}[FGHI].$$

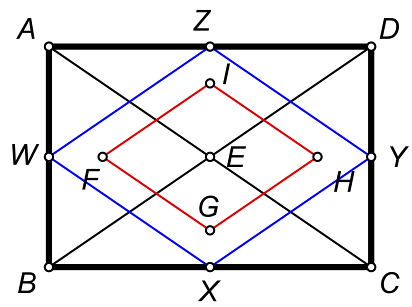

FIGURE 33.

Proof. Since the diagonals of a rectangle are equal and bisect each other, each of the radial triangles is isosceles with vertex E. Let the midpoints of the sides of the rectangle be W, X, Y, and Z as shown in Figure 33. Since F is the X_n point of $\triangle EAB$, by hypothesis,

$$\frac{FW}{EW} = k.$$

Since $k > 0$

$$\frac{EF}{EW} = \frac{EW - FW}{EW} = 1 - \frac{FW}{EW} = 1 - k.$$

Similarly, $EG/EX = 1 - k$, $EH/EY = 1 - k$, and $EI/EZ = 1 - k$. So quadrilaterals $FGHI$ and $WXYZ$ are homothetic, with E the center of similitude and ratio of similarity $1 - k$. Thus

$$[FGHI] = (1-k)^2[WXYZ].$$

But $[WXYZ] = \frac{1}{2}[ABCD]$, so $[FGHI] = \frac{(1-k)^2}{2}[ABCD]$ or, equivalently, $[ABCD] = \frac{2}{(1-k)^2}[FGHI]$. □

When $n = 402$ or $n = 620$, $k = 1/2$ and $[ABCD] = 8[FGHI]$.

Relationship $[ABCD] = 6[FGHI]$

Theorem 7.26. *Let E be the diagonal point of rectangle $ABCD$. Let F, G, H, and I be the X_{395} points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively (Figure 34). Then*

$$[ABCD] = 6[FGHI].$$

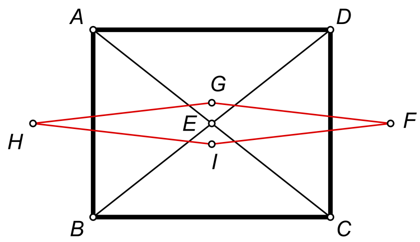

FIGURE 34. rectangle, X_{395} points $\implies$ $[ABCD] = 6[FGHI]$

Proof. From [15], we find that the barycentric coordinates for the X_{395} point of a triangle are

$$(u : v : w) = \left\{\sqrt{3}a^2 - 2S : \sqrt{3}b^2 - 2S : \sqrt{3}c^2 - 2S\right\} \tag{3}$$

where a, b, and c are the lengths of the sides of that triangle and S is twice the area of that triangle.

Set up a barycentric coordinate system as shown in Figure 35. Let $BC = a$, $AB = c$, and $AC = b = \sqrt{a^2 + c^2}$. Since ABC is a right triangle, $AE = BE = CE = b/2$. For areas, we have $[ABC] = ac/2$, $[ABE] = ac/4$, and $[BCE] = ac/4$.

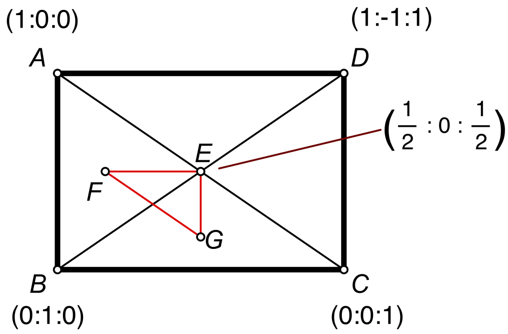

FIGURE 35. barycentric coordinates and rectangle $ABCD$

Since F is the X_{395} point of $\triangle ABE$,

$$F = uA + vB + wE$$

where u, v, w, and S are given by Equation (3), except that the values of a, b, c, and S are the sides and twice the area of $\triangle ABE$. In other words, $a \to BE = b/2$, $b \to AE = b/2$, $c \to AB = c$, and $S \to 2[ABE] = ac/2$. We get

$$F = \left(\frac{1}{4}\left(2c\left(\sqrt{3}c - 3a\right) + \sqrt{3}b^2\right) : \frac{\sqrt{3}b^2}{4} - ac : \frac{1}{2}c\left(\sqrt{3}c - a\right)\right).$$

Similarly, we find the coordinates for G by using Equation (3) applied to $\triangle BCE$, using the substitutions $a \to CE = b/2$, $b \to BE = b/2$, $c \to BC = a$, and $S \to 2[BCE] = ac/2$. We get

$$G = \left(\frac{1}{2}a\left(\sqrt{3}a - c\right) : \frac{\sqrt{3}b^2}{4} - ac : \frac{1}{4}\left(2\sqrt{3}a^2 - 6ac + \sqrt{3}b^2\right)\right).$$

Now we apply the Area Formula (Lemma 5.4) to $\triangle EFG$, to get

$$[EFG] = \frac{K}{12} = \frac{[ABC]}{12}$$

after simplifying and using the fact that $b^2 = a^2 + c^2$ (because $\triangle ABC$ is a right triangle).

By Lemma 7.22, $FGHI$ is a rhombus, so $[EFG] = \frac{1}{4}[FGHI]$. Since $ABCD$ is a rectangle, $[ABCD] = 2[ABC]$. Therefore

$$[ABCD] = 2[ABC] = 2(12[EFG]) = 2(12([FGHI]/4)) = 6[FGHI].$$

□

Theorem 7.27. *Let E be the diagonal point of rectangle $ABCD$. Let F, G, H, and I be the X_{396} points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively. Then*

$$[ABCD] = 6[FGHI].$$

The proof is the same as the proof for Theorem 7.26 and is omitted.

Relationship $[ABCD] = 2[FGHI]$

Theorem 7.28. *Let E be the diagonal point of rectangle $ABCD$. Let n be in the set*

$$\{11, 115, 116, 122, 123, 124, 125, 127, 130, 134, 135, 136$$
$$137, 139, 244, 245, 246, 247, 338, 339, 865, 866, 867, 868\}.$$

Let F, G, H, and I be the X_n points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively (Figure 36). Then F, G, H, and I are the midpoints of the sides of the rectangle and

$$[ABCD] = 2[FGHI].$$

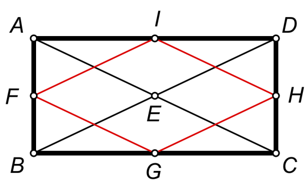

Figure 36. rectangle $\implies$ $[ABCD] = 2[FGHI]$

Proof. Since the diagonals of a rectangle are equal and bisect each other, each of the radial triangles is isosceles with vertex E. Thus, by Lemma 5.12, F, G, H, and I are the midpoints of the sides of the rectangle. Then, by Lemma 5.2, $[ABCD] = 2[FGHI]$. □

This proves all the entries in Table 6 for rectangles with the relationship $[ABCD] = 2[FGHI]$.

Relationship $[ABCD] = \frac{3}{2}[FGHI]$

Theorem 7.29. *Let E be the diagonal point of rectangle $ABCD$. Let F, G, H, and I be the X_{616} points or the X_{617} points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively (Figure 37). Then*

$$[ABCD] = \frac{3}{2}[FGHI].$$

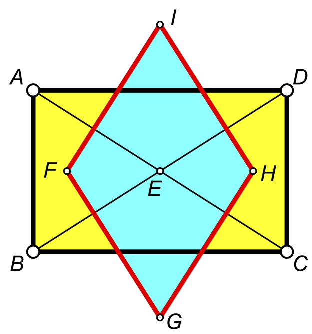

FIGURE 37. rectangle, X_{616} points $\implies$ $[ABCD] = \frac{3}{2}[FGHI]$

Proof. The proof is the same as the proof for Theorem 7.26 and the details are omitted. □

Relationship $[ABCD] = \frac{9}{8}[FGHI]$

Theorem 7.30. *Let E be the diagonal point of rectangle $ABCD$. Let X_n be a triangle center with the property that for all isosceles triangles with vertex A and midpoint of base M, X_nM/AM is a fixed negative constant k. Let F, G, H, and I be the X_n points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively. Then*

$$[ABCD] = \frac{2}{(1-k)^2}[FGHI].$$

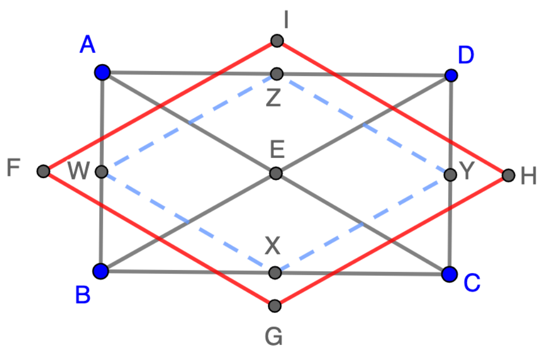

FIGURE 38.

Proof. Since the diagonals of a rectangle are equal and bisect each other, each of the radial triangles is isosceles with vertex E. Let the midpoints of the sides of the rectangle be W, X, Y, and Z as shown in Figure 38. Since F is the X_n point of $\triangle EAB$, by hypothesis,

$$\frac{FW}{EW} = k.$$

Since $k < 0$

$$\frac{EF}{EW} = \frac{EW + WF}{EW} = \frac{EW - FW}{EW} = 1 - \frac{FW}{EW} = 1 - k.$$

Similarly, $EG/EX = 1 - k$, $EH/EY = 1 - k$, and $EI/EZ = 1 - k$. So quadrilaterals $FGHI$ and $WXYZ$ are homothetic, with E the center of similitude and ratio of similarity $1 - k$. Thus

$$[FGHI] = (1 - k)^2[WXYZ].$$

But $[WXYZ] = \frac{1}{2}[ABCD]$, so $[FGHI] = \frac{(1-k)^2}{2}[ABCD]$ or, equivalently, $[ABCD] = \frac{2}{(1-k)^2}[FGHI]$. □

When $n = 290$, $n = 671$, or $n = 903$, $k = -1/3$ and $[ABCD] = \frac{9}{8}[FGHI]$. These values do not appear in Table 6 because the ratios in Table 6 are limited to those with denominators less than 6.

Relationship $[ABCD] = \frac{1}{2}[FGHI]$

Theorem 7.31. *Let E be the diagonal point of rectangle $ABCD$. Let n be 148, 149, or 150. Let F, G, H, and I be the X_n points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively. Then*

$$[ABCD] = \frac{1}{2}[FGHI].$$

Proof. From Table 9 and Theorem 7.30 with $k = -1$, we have $\frac{2}{(1-k)^2} = \frac{1}{2}$. □

Theorem 7.32. *Let E be the diagonal point of rectangle $ABCD$. Let n be 146, 147, 150, 151, 152, or 153. Let F, G, H, and I be the X_n points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively. Then*

$$[ABCD] = \frac{1}{2}[FGHI].$$

The proofs are the same as the proof for Theorem 7.26 and are omitted.

7.6. Proofs for Squares.

Relationship $[ABCD] = 6[FGHI]$

By symmetry (or by Theorem 6.33 of [20]), the central quadrilateral is a square when the reference quadrilateral is a square. It is straightforward to calculate $[ABCD]/[FGHI]$ for various centers. The barycentric coordinates $(u : v : w)$ for the center can be found in [5]. Then the ratio EF/EM can be found by Lemma 7.24 in terms of u, v, and w (Figure 39). Since each radial triangle is an isosceles right triangle, we can replace a, b, and c by $\sqrt{2}$, 1, and 1, respectively, in this ratio. Finally, the ratio of the areas of the squares is the square of the ratio of similitude, so

$$\frac{[ABCD]}{[FGHI]} = \left(\frac{EA}{EF}\right)^2 = \left(\frac{EM\sqrt{2}}{EF}\right)^2 = 2\left(\frac{EM}{EF}\right)^2 = 2\left(\frac{1}{\frac{EF}{EM}}\right)^2.$$

The results are tabulated in the tables on the following pages. We omit the cases where $EF/EM = 0$, but otherwise list all ratios, even when the result is true for quadrilaterals that are ancestors of a square.

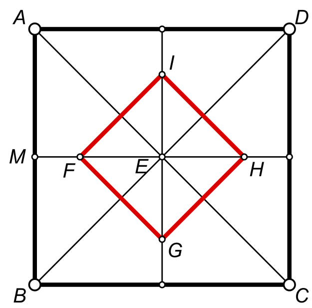

FIGURE 39. A square and its central quadrilateral

We can calculate $[ABCD]/[FGHI]$ even for centers that are not simple algebraic expressions in terms of a, b, and c. For example, the Morley center of a triangle (X_{356}) has barycentric coordinates

$$\left(a\cos\frac{A}{3} + 2a\cos\frac{B}{3}\cos\frac{C}{3} : b\cos\frac{B}{3} + 2b\cos\frac{C}{3}\cos\frac{A}{3} : c\cos\frac{C}{3} + 2c\cos\frac{A}{3}\cos\frac{B}{3}\right).$$

The quantities A, B, and C normally are inverse trigonometric functions of a, b, and c. However, when the triangle is an isosceles right triangle, with right angle at A, we have $A = \pi/2$, $B = \pi/4$, and $C = \pi/4$. Thus, the barycentric coordinates for the Morley center of an isosceles right triangle are

$$\left(\sqrt{2}\cos\frac{\pi}{6} + 2\sqrt{2}\cos\frac{\pi}{12}\cos\frac{\pi}{12} : \cos\frac{\pi}{12} + 2\cos\frac{\pi}{12}\cos\frac{\pi}{6} : \cos\frac{\pi}{12} + 2\cos\frac{\pi}{6}\cos\frac{\pi}{12}\right)$$

$$= \left(\sqrt{2}+\sqrt{6} : \frac{\sqrt{6}+2\sqrt{2}}{2} : \frac{\sqrt{6}+2\sqrt{2}}{2}\right).$$

We therefore have the following theorem.

Theorem 7.33. *Let E be the diagonal point of square $ABCD$. Let F, G, H, and I be the X_{356} points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively. Then*

$$[ABCD]=6[FGHI].$$

Proof. By Lemma 5.10,

$$\frac{FM}{EM}=\frac{u}{u+2v}$$

where $u=\sqrt{2}+\sqrt{6}$ and $v=\sqrt{2}+\sqrt{6}/2$. Thus, $\frac{FM}{EM}=\frac{\sqrt{2}+\sqrt{6}}{3\sqrt{2}+2\sqrt{6}}=1-\frac{1}{\sqrt{3}}$. From Lemma 7.24, we have

$$\frac{EF}{EM}=1-\left(1-\frac{1}{\sqrt{3}}\right)=\frac{1}{\sqrt{3}}.$$

But $EA=EM\sqrt{2}$, so

$$\frac{EA}{EF}=\frac{EM\sqrt{2}}{EF}=\frac{\sqrt{2}}{EF/EM}=\frac{\sqrt{2}}{1/\sqrt{3}}=\sqrt{6}.$$

Since $\frac{[ABCD]}{[FGHI]}=\left(\frac{EA}{EF}\right)^2$, we therefore have

$$\frac{[ABCD]}{[FGHI]}=\left(\sqrt{6}\right)^2=6.$$

□

Relationship $[ABCD]=k[FGHI]$ when X_n is not algebraic

By the same procedure, we get the following result for other values of n for which the barycentric coordinates of X_n are not algebraic.

Theorem 7.34. *Let E be the diagonal point of square $ABCD$. Let F, G, H, and I be the X_n points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively.*

If $n=357$, then $[ABCD]=\frac{1}{3}\left(14+3\sqrt{3}\right)[FGHI]$.

If $n=358$, then $[ABCD]=\left(14-5\sqrt{3}\right)[FGHI]$.

If $n=359$, then $[ABCD]=\frac{9}{2}[FGHI]$.

If $n=360$, then $[ABCD]=8[FGHI]$.

If $n=369$, then $[ABCD]=\frac{1}{4}\left(27-10\sqrt{2}\right)[FGHI]$.

Central Quadrilaterals formed by the Diagonal Point of a Square (part 1)

$[ABCD]/[FGHI]$	n
$\frac{1}{2}\left(7-4\sqrt{3}\right)$	622
$\frac{1}{4}\left(3-2\sqrt{2}\right)$	285, 309, 320, 350, 448
$-2\left(12\sqrt{2}-17\right)$	205, 449, 853
$\frac{1}{2}\left(3-2\sqrt{2}\right)$	189, 239
$-\frac{3}{2}\left(4\sqrt{3}-7\right)$	383
$\frac{1}{8}$	316, 352
$3-2\sqrt{2}$	223, 282, 484, 610, 857, 908, 909, 911, 923
$-\frac{3}{4}\left(\sqrt{3}-2\right)$	299
$\frac{2}{9}$	446
$2-\sqrt{3}$	16
$-2\left(2\sqrt{2}-3\right)$	44, 197, 227, 478, 672, 851, 861, 896, 899, 910
$-\frac{4}{3}\left(\sqrt{3}-2\right)$	624
$-\frac{8}{49}\left(6\sqrt{2}-11\right)$	970
$\frac{1}{2}$	20, 22, 23, 70, 74, 94, 98, 102, 103, 104, 105, 106, 111, 146, 147, 148, 149, 150, 151, 152, 153, 160, 17 5, 253, 280, 323, 325, 347, 385, 401, 477, 586, 638, 675, 697, 699, 701, 703, 705, 707, 709, 711, 713, 715 , 717, 719, 721, 723, 725, 727, 729, 731, 733, 735, 737, 739, 741, 743, 745, 747, 753, 755, 759, 761, 767, 840, 841, 842, 843, 858, 953, 972
$-4\left(2\sqrt{2}-3\right)$	198, 292
$23-16\sqrt{2}+2\sqrt{2\left(58-41\sqrt{2}\right)}$	845
$-3\left(\sqrt{3}-2\right)$	14
$179-126\sqrt{2}$	170
$\frac{1}{4}\left(9-4\sqrt{2}\right)$	88, 411, 416, 673, 897
$931-658\sqrt{2}+4\sqrt{2\left(54146-38287\sqrt{2}\right)}$	167
$\frac{8}{9}$	67, 550, 625, 694
$-\frac{18}{49}\left(6\sqrt{2}-11\right)$	992
1	36, 40, 80, 84, 238, 291, 859
$\frac{1}{6}\left(13-4\sqrt{3}\right)$	634
$43-30\sqrt{2}+4\sqrt{58-41\sqrt{2}}$	844
$\frac{9}{8}$	290, 315, 376, 490, 671, 903
$295-208\sqrt{2}+2\sqrt{2\left(20714-14647\sqrt{2}\right)}$	168
$\frac{4}{49}\left(9+4\sqrt{2}\right)$	963
$\frac{1}{2}\left(11-6\sqrt{2}\right)$	294, 329, 335, 573
$\frac{32}{25}$	486, 548
$\frac{1}{4}\left(14-5\sqrt{3}\right)$	301
$8\left(3-2\sqrt{2}\right)$	199
$5-3\sqrt{2}+2\sqrt{10-7\sqrt{2}}$	504
$\frac{1}{4}\left(3+2\sqrt{2}\right)$	322, 334, 944
$\frac{3}{2}$	616, 617
$9\left(3-2\sqrt{2}\right)$	165
$\frac{1}{2}\left(9-4\sqrt{2}\right)$	144
2	3, 11, 48, 49, 63, 69, 71, 72, 73, 77, 78, 97, 115, 116, 122, 123, 124, 125, 127, 130, 137, 184, 185, 187, 201, 212, 216, 217, 219, 222, 228, 237, 244, 245, 246, 248, 255, 265, 268, 271, 283, 287, 293, 295, 296, 3 04, 305, 306, 307, 326, 328, 332, 336, 337, 338, 339, 343, 345, 348, 382, 394, 399, 408, 417, 418, 426, 44 0, 441, 464, 465, 466, 488, 499, 577, 591, 603, 606, 615, 640, 682, 748, 820, 828, 836, 852, 856, 865, 866 , 867, 868, 895, 902, 974
$\frac{1}{2}\left(995+702\sqrt{2}-4\sqrt{2\left(61466+43463\sqrt{2}\right)}\right)$	400
$121-84\sqrt{2}$	728
$7-4\sqrt{2}+2\sqrt{2\left(10-7\sqrt{2}\right)}$	164
$\frac{9}{4}$	341
$4\left(43-30\sqrt{2}\right)$	480
$\frac{1}{2}\left(33-20\sqrt{2}\right)$	346
$\frac{1}{36}\left(73+12\sqrt{2}\right)$	413
$11-6\sqrt{2}$	191, 200, 979
$3-\sqrt{2}+2\sqrt{3\sqrt{2}-4}$	510
$\frac{1}{2}\left(19-8\sqrt{3}\right)$	627
$\frac{4}{9}\left(3+2\sqrt{2}\right)$	956
$\frac{1}{4}\left(19-6\sqrt{2}\right)$	312, 319, 572
$\frac{49}{18}$	183, 252

Central Quadrilaterals formed by the Diagonal Point of a Square (part 2)

$[ABCD]/[FGHI]$	n
$16\left(3-2\sqrt{2}\right)$	220
$\frac{4}{169}\left(146-17\sqrt{3}\right)$	636
$\frac{1}{16}\left(33+8\sqrt{2}\right)$	314, 561, 679
$\frac{3}{4}\left(2+\sqrt{3}\right)$	298
$\frac{72}{25}$	549, 599, 626
$\frac{1}{2}\left(3+2\sqrt{2}\right)$	8, 313, 501, 947, 962
$\frac{1}{3}\left(14-3\sqrt{3}\right)$	18
$19-10\sqrt{2}-4\sqrt{2\left(10-7\sqrt{2}\right)}$	166
$\frac{1}{98}\left(561-184\sqrt{2}\right)$	391
$18\left(3-2\sqrt{2}\right)$	210
$\frac{25}{8}$	76, 95, 277, 333, 353, 492, 631, 801
$\frac{1}{144}\left(347+78\sqrt{2}\right)$	871
$\frac{8}{49}\left(11+6\sqrt{2}\right)$	596, 960, 993
$\frac{1}{4}\left(27-10\sqrt{2}\right)$	369
$12\left(2-\sqrt{3}\right)$	618
$\frac{1}{4}\left(10-\sqrt{2}+4\sqrt{2\left(2-\sqrt{2}\right)}\right)$	556
$\frac{1}{98}\left(233+60\sqrt{2}\right)$	330
$\frac{1}{6}\left(13+4\sqrt{3}\right)$	633
$9-4\sqrt{2}$	9, 35, 321
$\frac{1}{32}\left(83+18\sqrt{2}\right)$	261, 310
$\frac{1}{4}\left(26-7\sqrt{3}\right)$	302
$\frac{2}{49}\left(43+30\sqrt{2}\right)$	958
$\frac{1}{2}\left(41-24\sqrt{2}\right)$	452
$\frac{32}{9}$	140, 141, 182, 566, 641
$2\left(81-56\sqrt{2}\right)$	594
$\frac{1}{4}\left(9+4\sqrt{2}\right)$	21, 75, 371, 497, 775, 991, 997
$2+\sqrt{3}$	15
$\frac{2}{49}\left(123-22\sqrt{2}\right)$	949, 965
$\frac{121}{32}$	308
$11-6\sqrt{2}+4\sqrt{10-7\sqrt{2}}$	503
$\frac{18}{289}\left(33+20\sqrt{2}\right)$	392
$\frac{1}{2}\left(6-\sqrt{2}+4\sqrt{2-\sqrt{2}}\right)$	188
$\frac{1}{49}\left(163+18\sqrt{2}\right)$	87, 936
$\frac{4}{121}\left(146-17\sqrt{3}\right)$	629
$\frac{1}{98}\left(193+132\sqrt{2}\right)$	390
$\frac{1}{8}\left(51-14\sqrt{2}\right)$	966
$\frac{98}{25}$	574
$2\left(3249-2296\sqrt{2}\right)$	762
$\frac{1}{18}\left(51+14\sqrt{2}\right)$	257
4	10, 55, 241, 405, 500, 582, 950
$\frac{2}{3}\left(13-4\sqrt{3}\right)$	398
$\frac{1}{16}\left(51+10\sqrt{2}\right)$	274
$\frac{200}{49}$	632
$31-20\sqrt{2}+2\sqrt{2\left(194-137\sqrt{2}\right)}$	258
$\frac{4}{169}\left(146+17\sqrt{3}\right)$	635
$\frac{1}{18}\left(41+24\sqrt{2}\right)$	585
$\frac{1}{2}\left(154+105\sqrt{2}-8\sqrt{2\left(338+239\sqrt{2}\right)}\right)$	557
$\frac{1}{49}\left(137+48\sqrt{2}\right)$	982
$\frac{1}{8}\left(17+12\sqrt{2}\right)$	1000
$25\left(3-2\sqrt{2}\right)$	380, 846
$\frac{2}{9}\left(11+6\sqrt{2}\right)$	38
$\frac{64}{49}\left(9-4\sqrt{2}\right)$	45
$\frac{1}{4}\left(8+3\sqrt{2}+2\sqrt{2\left(2+\sqrt{2}\right)}\right)$	260
$\frac{1}{49}\left(113+72\sqrt{2}\right)$	984
$3+2\sqrt{2\left(3\sqrt{2}-4\right)}$	364
$2\left(121-84\sqrt{2}\right)$	756
$\frac{9}{2}$	2, 54, 284, 311, 349, 359, 569, 570, 581, 637, 639, 943
$\frac{1}{64}\left(209+60\sqrt{2}\right)$	873
$\frac{1}{36}\left(107+42\sqrt{2}\right)$	870
$\frac{2}{49}\left(57+40\sqrt{2}\right)$	996
$23-8\sqrt{2}-2\sqrt{2\left(50-31\sqrt{2}\right)}$	363
$33-20\sqrt{2}$	964, 986
$4-\sqrt{2}+2\sqrt{2\left(2-\sqrt{2}\right)}$	259
$\frac{16}{49}\left(9+4\sqrt{2}\right)$	496, 613, 988
$\frac{1}{4}\left(11+6\sqrt{2}\right)$	968

Central Quadrilaterals formed by the Diagonal Point of a Square (part 3)

$[ABCD]/[FGHI]$	n
$2\left(8+5\sqrt{2}-2\sqrt{2\left(10+7\sqrt{2}\right)}\right)$	178
$\frac{1}{9}\left(33+8\sqrt{2}\right)$	256
$\frac{4}{3}\left(2+\sqrt{3}\right)$	623
$2\left(11-6\sqrt{2}\right)$	37, 41, 442, 498, 584, 601, 976
$\frac{1}{36}\left(97+60\sqrt{2}\right)$	409
$\frac{1}{2}\left(4+4\sqrt[4]{2}+\sqrt{2}\right)$	366
$\frac{36}{529}\left(41+24\sqrt{2}\right)$	551
$5-\sqrt{2}+2\sqrt{2-\sqrt{2}}$	362
$\frac{128}{25}$	575, 695
$\frac{1}{8}\left(27+10\sqrt{2}\right)$	60, 86, 443
$\frac{2}{961}\left(2217+188\sqrt{2}\right)$	980
$\frac{841}{162}$	592
$\frac{1}{2}\left(18-\sqrt{2}-8\sqrt{2-\sqrt{2}}\right)$	236
$\frac{1}{2}\left(19-6\sqrt{2}\right)$	612, 894
$\frac{4}{49}\left(51+10\sqrt{2}\right)$	142, 474, 939, 954, 975
$14-5\sqrt{3}$	61, 358
$2\left(11+8\sqrt{2}-2\sqrt{2\left(24+17\sqrt{2}\right)}\right)$	367
$\frac{1}{4}\left(33-8\sqrt{2}\right)$	941
$\frac{50}{9}$	39, 233, 493, 567, 590
$\frac{1}{16}\left(73+12\sqrt{2}\right)$	751
$\frac{1}{4}\left(14+5\sqrt{3}\right)$	300
$\frac{18}{289}\left(123-22\sqrt{2}\right)$	374
$\frac{4}{121}\left(146+17\sqrt{3}\right)$	630
$3+2\sqrt{2}$	1, 377
$\frac{288}{49}$	547, 597
6	395, 396
$\frac{2}{49}\left(233-60\sqrt{2}\right)$	750
$\frac{49}{8}$	83, 288, 327, 588
$36\left(3-2\sqrt{2}\right)$	375
$\frac{2}{49}\left(123+22\sqrt{2}\right)$	940
$\frac{1}{2}\left(5+\sqrt{2}+4\sqrt{1+\sqrt{2}}\right)$	508
$\frac{16}{49}\left(11+6\sqrt{2}\right)$	495, 611
$\frac{1}{3}\left(14+3\sqrt{3}\right)$	17, 357
$\frac{1}{2}\left(27-10\sqrt{2}\right)$	938, 955
$\frac{1}{2}\left(10-\sqrt{2}+4\sqrt{2\left(2-\sqrt{2}\right)}\right)$	483
$\frac{162}{25}$	373
$\frac{4}{9}\left(9+4\sqrt{2}\right)$	893
$\frac{\left(-2+2\sqrt{2}+\left(2-\sqrt{2}\right)^{3/4}\right)^2}{2\left(\sqrt{2}-1\right)^2}$	507
$32-19\sqrt{2}+2\sqrt{2\left(194-137\sqrt{2}\right)}$	289
$\frac{1}{16}\left(129-16\sqrt{2}\right)$	981
$2\left(9-4\sqrt{2}\right)$	12, 42, 863, 922
$\frac{121}{18}$	384
$2+\sqrt{2}+2\ 2^{3/4}$	365
$\frac{\left(1+2^{2/3}\sqrt[3]{\sqrt{2}-1}\right)^2}{\sqrt[3]{2}\left(\sqrt{2}-1\right)^{2/3}}$	506
$\frac{1}{4}\left(19+6\sqrt{2}\right)$	81, 85, 386, 404, 977
$\frac{1}{2}\left(7+4\sqrt{3}\right)$	621
$41-24\sqrt{2}$	171
$\frac{18}{49}\left(11+6\sqrt{2}\right)$	354, 969
$\frac{1}{2}\left(227-154\sqrt{2}+28\sqrt{2\left(58-41\sqrt{2}\right)}\right)$	179
$\frac{1}{2}\left(21+9\sqrt{3}-2\sqrt{62+35\sqrt{3}}\right)$	559
$\frac{1}{2}\left(9+4\sqrt{2}\right)$	176, 192, 387, 989
$\frac{1}{2}\left(6+\sqrt{2}+4\sqrt{2+\sqrt{2}}\right)$	174
$\frac{25}{49}\left(9+4\sqrt{2}\right)$	987
$\frac{1}{49}\left(1169-672\sqrt{2}+8\sqrt{33178-23201\sqrt{2}}\right)$	180
$\frac{3}{2}\left(5+3\sqrt{2}-\sqrt{3\left(3+2\sqrt{2}\right)}\right)$	554
$\frac{1}{8}\left(33+20\sqrt{2}\right)$	995

Central Quadrilaterals formed by the Diagonal Point of a Square (part 4)

$[ABCD]/[FGHI]$	n
$\frac{2}{49}\left(163+18\sqrt{2}\right)$	869
$18\left(57-40\sqrt{2}\right)$	872
8	5, 6, 169, 226, 344, 360, 389, 402, 482, 485, 494, 578, 620, 642, 942
$\frac{9}{16}\left(9+4\sqrt{2}\right)$	757
$\frac{1}{9}\left(41+24\sqrt{2}\right)$	985
$2\left(13+9\sqrt{2}-2\sqrt{58+41\sqrt{2}}\right)$	177
$\frac{1}{98}\left(561+184\sqrt{2}\right)$	89
$10+6\sqrt{2}-\sqrt{3\left(17+12\sqrt{2}\right)}$	202
$\frac{1}{4}\left(17+12\sqrt{2}\right)$	82, 388
$\frac{2}{49}\left(113+72\sqrt{2}\right)$	999
$3+\sqrt{2}+2\sqrt{2\left(1+\sqrt{2}\right)}$	509
9	983
$\frac{2}{9}\left(27+10\sqrt{2}\right)$	904
$7+2\sqrt{2\left(2-\sqrt{2}\right)}$	173
$2\left(33-20\sqrt{2}\right)$	213, 605
$\frac{1}{4}\left(26+7\sqrt{3}\right)$	303
$\frac{1}{2}\left(11+6\sqrt{2}\right)$	7, 58, 614, 951
$4\left(11-6\sqrt{2}\right)$	172
$\frac{1}{64}\left(387+182\sqrt{2}\right)$	763
$\frac{81}{8}$	598
$\frac{2}{49}\left(177+52\sqrt{2}\right)$	967
$19-6\sqrt{2}$	43, 937
$4+\sqrt{2}+2\sqrt{2\left(2+\sqrt{2}\right)}$	266
$\frac{36}{49}\left(9+4\sqrt{2}\right)$	553
$\frac{1}{4}\left(33+8\sqrt{2}\right)$	948
$3\left(2+\sqrt{3}\right)$	13
$-2\left(-34-23\sqrt{2}+4\sqrt{2\left(58+41\sqrt{2}\right)}\right)$	234
$2\left(3+2\sqrt{2}\right)$	31, 65, 355, 602, 774, 855, 945, 998
$\frac{1}{8}\left(57+28\sqrt{2}\right)$	593
$97-60\sqrt{2}$	609
$\frac{25}{2}$	194, 251, 262
$\frac{9}{4}\left(3+2\sqrt{2}\right)$	229, 959
$\frac{1}{4}\left(18+7\sqrt{2}+4\sqrt{2\left(10+7\sqrt{2}\right)}\right)$	555
$\frac{2}{3}\left(13+4\sqrt{3}\right)$	397
$2\left(41-24\sqrt{2}\right)$	181
$9+4\sqrt{2}$	57, 79, 379
$\frac{25}{32}\left(11+6\sqrt{2}\right)$	552
$\frac{1}{2}\left(19+8\sqrt{3}\right)$	628
$11+2\sqrt{2}+4\sqrt{2-\sqrt{2}}$	361
$\frac{1}{2}\left(17+12\sqrt{2}\right)$	145, 595, 961, 994
$\frac{1}{2}\left(18+\sqrt{2}+8\sqrt{2+\sqrt{2}}\right)$	558
18	32, 51, 195, 218, 381, 568, 800, 864, 973
$\frac{1}{4}\left(41+24\sqrt{2}\right)$	849
$14+5\sqrt{3}$	62
$4\left(3+2\sqrt{2}\right)$	56, 214, 215, 481, 502, 946, 990
$10+6\sqrt{2}+\sqrt{3\left(17+12\sqrt{2}\right)}$	203
$2\left(9+4\sqrt{2}\right)$	560, 678, 854
$\frac{1}{2}\left(33+20\sqrt{2}\right)$	279
32	143, 263, 546, 576
$2\left(11+6\sqrt{2}\right)$	583, 604
$\frac{1}{2}\left(43+30\sqrt{2}\right)$	957
$12\left(2+\sqrt{3}\right)$	619
$9\left(3+2\sqrt{2}\right)$	267, 269, 978
$2\left(139-66\sqrt{2}\right)$	600
$\frac{1}{2}\left(131+90\sqrt{2}\right)$	479
$47+32\sqrt{2}+6\sqrt{2\left(58+41\sqrt{2}\right)}$	505
$121+84\sqrt{2}$	738

8. Poncelet point

In this section, we examine central quadrilaterals formed from the Poncelet point of the reference quadrilateral.

The *Poncelet point* (sometimes called the Euler-Poncelet point) of a quadrilateral is the common point of the nine-point circles of the component triangles (half-triangles) of the quadrilateral. A triangle formed from three vertices of a quadrilateral is called a *component triangle* of that quadrilateral. The *nine-point circle* of a triangle is the circle through the midpoints of the sides of that triangle.

Figure 40 shows the Poncelet point of quadrilateral $ABCD$. The yellow points represent the midpoints of the sides and diagonals of the quadrilateral. The component triangles are BCD, ACD, ABD, and ABC. The blue circles are the nine-point circles of these triangles. The common point of the four circles is the Poncelet point (shown in green).

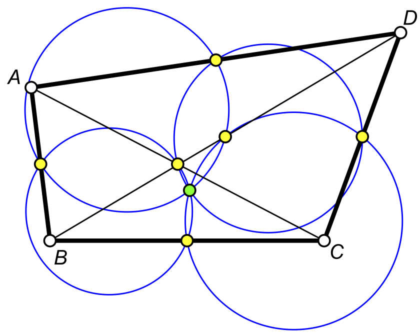

Figure 40. The Poncelet point of quadrilateral $ABCD$

Proposition 8.1. *The Poncelet point of a parallelogram coincides with the diagonal point.*

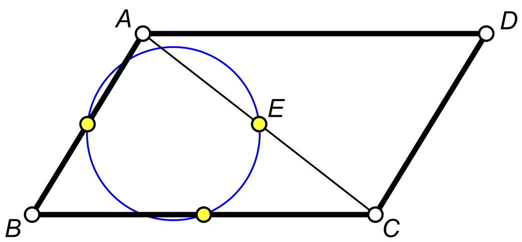

Figure 41. Nine-point circle of component triangle ABC

Proof. Since the diagonals of a parallelogram bisect each other, the diagonal point E, is the midpoint of side AC of component triangle ABC. Thus, the nine-point circle of $\triangle ABC$ passes through E (Figure 41). Similarly, all the nine-point circles of the other component triangles pass through E. Hence the diagonal point is common to all four of these circles and is therefore the Poncelet point of the quadrilateral. □

The following result is well known.

Lemma 8.2. *The nine-point circle of a triangle passes through the feet of the altitudes.*

Proposition 8.3. *The Poncelet point of an orthodiagonal quadrilateral coincides with the diagonal point.*

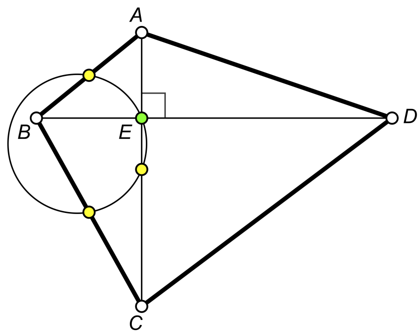

Figure 42. Nine-point circle of component triangle ABC

Proof. Let $ABCD$ be an orthodiagonal quadrilateral with diagonal point E. Then BE is an altitude of component triangle ABC. By Lemma 8.2, the nine point circle of $\triangle ABC$ passes through E (Figure 42). Similarly, the nine-point circles of the other component triangles also pass through E. Thus, E is the Poncelet point of quadrilateral $ABCD$. □

Our computer study examined the central quadrilaterals formed by the Poncelet point. Since the Poncelet point coincides with the diagonal point of an orthodiagonal quadrilateral, we omit results for orthodiagonal quadrilaterals. Since the Poncelet point coincides with the diagonal point of a parallelogram, we omit results for parallelograms. We checked the central quadrilateral for all the first 1000 triangle centers (omitting points at infinity) and all reference quadrilateral shapes listed in Table 1.

The results found are listed in Table 10.

Table 10.

Central Quadrilaterals formed by the Poncelet Point		
Quadrilateral Type	**Relationship**	**centers**
Hjelmslev	$[ABCD] = \frac{1}{2}[FGHI]$	20

The following result is well known, [26].

Lemma 8.4. *The nine-point circle of a triangle bisects any line from the orthocenter to a point on the circumcircle.*

Lemma 8.5. *Let E be the Poncelet point of quadrilateral $ABCD$ that has right angles at B and D. Then E is the midpoint of BD.*

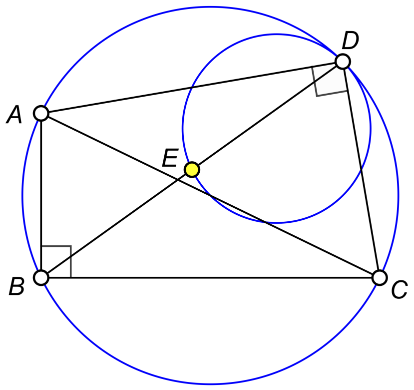

FIGURE 43. Poncelet point of a Hjelmslev quadrilateral

Proof. Since $\angle ABC$ is a right angle, AC is a diameter of the circumcircle of $\triangle ABC$. Since $\angle ADC$ is a right angle, D lies on the circumcircle of $\triangle ABC$. Since $\triangle ADC$ is a right triangle, its orthocenter is point D. By Lemma 8.4, the nine-point circle of $\triangle ADC$ bisects BD (Figure 43). Let E be the midpoint of BD. Similarly, the nine-point circle of $\triangle ABC$ bisects BD. So both nine-point circles pass through E. By definition, the nine-point circles of triangles ABD and CBD pass through E. Thus, E is a common point of the nine-point circles of all four component triangles of quadrilateral $ABCD$. Therefore, E is the Poncelet point of $ABCD$. $\square$

Relationship $[ABCD] = \frac{1}{2}[FGHI]$

Theorem 8.6. *Let E be the Poncelet point of a Hjelmslev quadrilateral $ABCD$. Let F, G, H, and I be the de Longchamps points (X_{20} points) of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively (Figure 44). Then*

$$[ABCD] = \frac{1}{2}[FGHI].$$

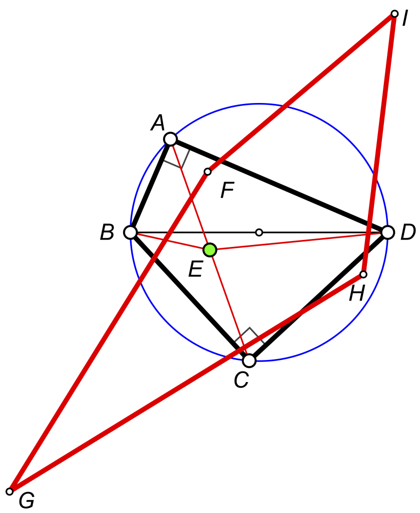

FIGURE 44. Hjelmslev, X_{20} points $\implies$ $[ABCD] = \frac{1}{2}[FGHI]$

Proof. Recall that a Hjelmslev quadrilateral is a quadrilateral with right angles at two opposite vertices. Call the quadrilateral $ABCD$ with right angles at B and D (Figure 45). We set up a barycentric coordinate system using $\triangle ABC$ as the reference triangle, so that

$$A = (1 : 0 : 0)$$
$$B = (0 : 1 : 0)$$
$$C = (0 : 0 : 1).$$

We let the barycentric coordinates of D be $(p : q : r)$ with $p + q + r = 1$ and without loss of generality, assume $p > 0$, $q < 0$, and $r > 0$. Let E be the Poncelet point of quadrilateral $ABCD$. By Lemma 8.5, E is the midpoint of BD, so has normalized barycentric coordinates $E = (\frac{p}{2} : \frac{q+1}{2} : \frac{r}{2})$ as shown in Figure 45.

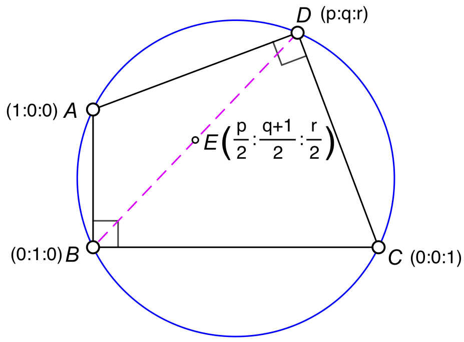

Figure 45. Coordinate system for a Hjelmslev quadrilateral

Using the Distance Formula (Lemma 5.3), we can compute the distances between the various points. We get

$$\begin{aligned}
AB &= c\\
BC &= a\\
AC &= b\\
AD &= \sqrt{-a^2qr + b^2r(q+r) + c^2q(q+r)}\\
BD &= \sqrt{a^2r(p+r) - b^2pr + c^2p(p+r)}\\
CD &= \sqrt{a^2q(p+q) + b^2p(p+q) - c^2pq}\\
AE &= \frac{1}{2}\sqrt{-a^2(q+1)r + b^2r(q+r+1) + c^2(q+1)(q+r+1)}\\
BE &= \frac{1}{2}\sqrt{a^2r(p+r) - b^2pr + c^2p(p+r)}\\
CE &= \frac{1}{2}\sqrt{a^2(q+1)(p+q+1) + b^2p(p+q+1) - c^2p(q+1)}\\
DE &= \frac{1}{2}\sqrt{a^2r(p+r) - b^2pr + c^2p(p+r)}.
\end{aligned}$$

Using the Area Formula (Lemma 5.4), we can compute the areas of triangles ABC and ACD. We find

$$[ABC] = K \qquad \text{and} \qquad [ACD] = -qK$$

so that

$$[ABCD] = K(1 - q).$$

Recall that q is negative, so these areas are positive.

From [7], we find that the barycentric coordinates for the X_{20} point of a triangle with sides of lengths a, b, and c are $(x : y : z)$ where

$$\begin{aligned}x &= 3a^4 - 2a^2b^2 - 2a^2c^2 - b^4 + 2b^2c^2 - c^4\\ y &= -a^4 - 2a^2b^2 + 2a^2c^2 + 3b^4 - 2b^2c^2 - c^4\\ z &= -a^4 + 2a^2b^2 - 2a^2c^2 - b^4 - 2b^2c^2 + 3c^4.\end{aligned}$$

We can use the Change of Coordinates Formula (Lemma 5.5) to find the coordinates for point F, the X_{20} point of $\triangle ABE$, by substituting the lengths of BE, AE, and AB for a, b, and c in the expression for the normalized barycentric coordinates for X_{20}. In the same manner, we can find the barycentric coordinates for G, H, and I.

These barycentric coordinates are very complicated, but can be simplified using the fact that quadrilateral $ABCD$ is a Hjelmslev quadrilateral.

Since $\triangle ABC$ is a right triangle, we have the relationship

$$a^2 + c^2 = b^2. \tag{4}$$

Since $\triangle ADC$ is a right triangle, we have the relationship

$$AD^2 + CD^2 = AC^2.$$

In terms of a, c, p, and q, this is equivalent to

$$c^2 = \frac{a^2\left(p^2 + 2pq - p + q^2 - q\right)}{(1-p)p} \tag{5}$$

where we have eliminated b and r since $b = \sqrt{a^2 + c^2}$ and $r = 1 - p - q$.

Simplifying the formulas for the barycentric coordinates for F, G, H, and I taking relationships (4) and (5) into account, we find that

$$\begin{aligned}F &= \left(1 - \frac{p}{2} : \frac{p(-q) + p + 3q + 1}{2(p-1)} : \frac{p^2 + p(q-2) - 3q - 1}{2(p-1)}\right)\\ G &= \left(-\frac{p^2 + pq - 2(q+1)}{2(p+q)} : -\frac{p(q-1) + q^2 + q + 2}{2(p+q)} : \frac{1}{2}(p+q+1)\right)\\ H &= \left(\frac{p}{2} + q + 1, \frac{p(q-1) + 2q^2 + q + 1}{2(p-1)}, \frac{p^2 + 3pq - 2p + 2q^2 - q + 1}{2 - 2p}\right)\\ I &= \left(\frac{p^2 - pq + 2q}{2(p+q)}, \frac{p(q-1) - q(q+3)}{2(p+q)}, \frac{1}{2}(-p+q+3)\right).\end{aligned}$$

Using the Area Formula, we can compute the areas of triangles FGH and HIF. We get

$$\begin{aligned}[FGH] &= \frac{K\left(p(q+3) + q^2 + q - 2\right)}{p-1}\\ [HIF] &= -\frac{K(3pq + p + (q-1)q)}{p-1}\end{aligned}$$

so that

$$[FGHI] = [FGH] + [HIF] = 2K(1-q).$$

Thus, $[FGHI] = 2[ABCD]$. $\square$

9. Circumcenter

In this section, we examine central quadrilaterals formed from the circumcenter of the reference quadrilateral. Note that only cyclic quadrilaterals have circumcenters. The *circumcenter* of a cyclic quadrilateral is the center of the circle through the vertices of the quadrilateral.

Our computer study examined the central quadrilaterals formed by the circumcenter. Since the circumcenter of a rectangle coincides with the diagonal point of the rectangle, we omit results for rectangles. We checked the central quadrilateral for all the first 1000 triangle centers (omitting points at infinity) and all reference quadrilateral shapes listed in Table 1 that are cyclic.

The results found are listed in Table 11.

Table 11.

Central Quadrilaterals formed by the Circumcenter		
Quadrilateral Type	**Relationship**	**centers**
cyclic	$[ABCD] = 8[FGHI]$	402, 620
	$[ABCD] = 2[FGHI]$	11, 115, 116, 122–125, 127, 130, 134-137, 139, 244–247, 338, 339, 865–868
	$[ABCD] = \frac{3}{2}[FGHI]$	616, 617
	$[ABCD] = \frac{9}{8}[FGHI]$	290, 671, 903
	$[ABCD] = \frac{1}{2}[FGHI]$	148–150

Theorem 9.1. *Let E be the circumcenter of cyclic quadrilateral $ABCD$. Let X_n be a triangle center with the property that for all isosceles triangles with vertex V and midpoint of base M, X_nM/VM is a fixed positive constant k. Let F, G, H, and I be the X_n points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively. Then*

$$[ABCD] = \frac{2}{(1-k)^2}[FGHI].$$

Proof. The proof is the same as the proof of Theorem 7.25. $\square$

Relationship $[ABCD] = 8[FGHI]$

Theorem 9.2. *Let E be the circumcenter of cyclic quadrilateral $ABCD$. Let F, G, H, and I be the X_{402} points or the X_{620} points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively (Figure 46). Then*

$$[ABCD] = 8[FGHI].$$

Proof. Since E is the center of the circle through points A, B, C, and D, each of the radial triangles is isosceles with vertex E. Let the midpoints of the sides of the quadrilateral be W, X, Y, and Z as shown in Figure 46. From Theorem 7.10 and Table 7, we find that for $n = 402$ and $n = 620$, the ratio X_nM/AM is a

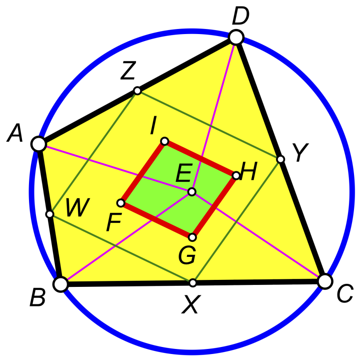

FIGURE 46. X_{402} points $\implies [ABCD] = 8[FGHI]$

constant, $\frac{1}{2}$, for all isosceles triangles with vertex A and midpoint of opposite side M. Therefore, by Theorem 9.1, with $k = \frac{1}{2}$, we must have

$$[ABCD] = \frac{2}{(1-k)^2}[FGHI] = 8[FGHI].$$

□

Relationship $[ABCD] = 2[FGHI]$

Theorem 9.3. *Let E be the circumcenter of cyclic quadrilateral $ABCD$. Let n be in the set*

$$\{11, 115, 116, 122, 123, 124, 125, 127, 130, 134, 135, 136$$
$$137, 139, 244, 245, 246, 247, 338, 339, 865, 866, 867, 868\}.$$

Let F, G, H, and I be the X_n points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively (Figure 47). Then F, G, H, and I are the midpoints of the sides of the quadrilateral and

$$[ABCD] = 2[FGHI].$$

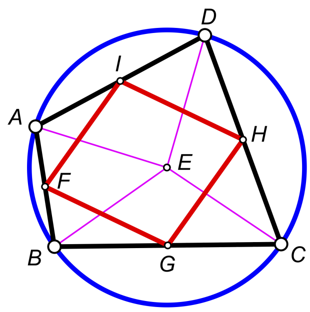

FIGURE 47. $[ABCD] = 2[FGHI]$

Proof. Since E is the center of the circle through points A, B, C, and D, each of the radial triangles is isosceles with vertex E. Thus, by Lemma 5.12, F, G, H, and I are the midpoints of the sides of the quadrilateral. Then, by Lemma 5.2, $[ABCD] = 2[FGHI]$. □

$\boxed{\text{Relationship } [ABCD] = \frac{3}{2}[FGHI]}$

Proposition 9.4 (X_{616} Property of an Isosceles Triangle)**.** *Let $\triangle ABC$ be an isosceles triangle with $AB = AC$. If F is the X_{616} point of $\triangle ABC$, then $AF \perp BC$ and $AF = BC/\sqrt{3}$ (Figure 48).*

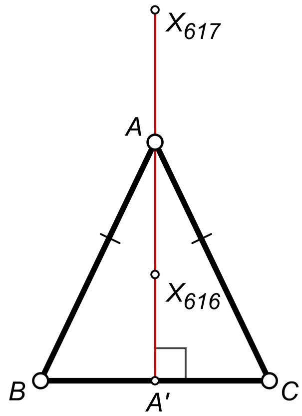

Figure 48. X_{616} and X_{617} points of an isosceles triangle

Proof. We use barycentric coordinates with respect to $\triangle ABC$. Let A' be orthogonal projection of A on the side BC. Since $\triangle ABC$ is isosceles, A' is the midpoint of BC, so $A' = (0 : 1 : 1)$. The barycentric coordinates of F are

$$F = \left(5a^4 - a^2\left(4b^2 + 4c^2 - 2\sqrt{3}S\right) - b^4 + 2b^2\left(c^2 - \sqrt{3}S\right) - c^4 - 2\sqrt{3}c^2S :: \right)$$

where S denotes twice the area of $\triangle ABC$. Using the fact that $b = c$ we get

$$F = \Bigg(5a^4 - a^2\left(8b^2 - 2\sqrt{3}S\right) - 2b^4 - 2\sqrt{3}b^2S + 2b^2\left(b^2 - \sqrt{3}S\right) :$$
$$- a^4 - 2a^2\left(b^2 + \sqrt{3}S\right) : -a^4 - 2a^2\left(b^2 + \sqrt{3}S\right)\Bigg)$$

A simple calculation shows that the point F lies on the line AA' (which has equation $y = z$), so $AF \perp BC$.

Using the distance formula to get the length of AF and substituting $c = b$ and $S = \frac{1}{4}a\sqrt{4b^2 - a^2}$, we get

$$AF^2 = \frac{a^2}{3} = \frac{BC^2}{3}.$$

Thus, $AF = BC/\sqrt{3}$. □

Proposition 9.5 (X_{617} Property of an Isosceles Triangle)**.** *Let $\triangle ABC$ be an isosceles triangle with $AB = AC$. If F is the X_{617} point of $\triangle ABC$, then $AF \perp BC$ and $AF = BC/\sqrt{3}$.*

Proof. The proof is similar to the proof of Proposition 9.4, so the details are omitted. □

Theorem 9.6. *Let E be the circumcenter of cyclic quadrilateral $ABCD$. Let F, G, H, and I be the X_{616} points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively (Figure 49). Then*

$$[ABCD] = \frac{3}{2}[FGHI].$$

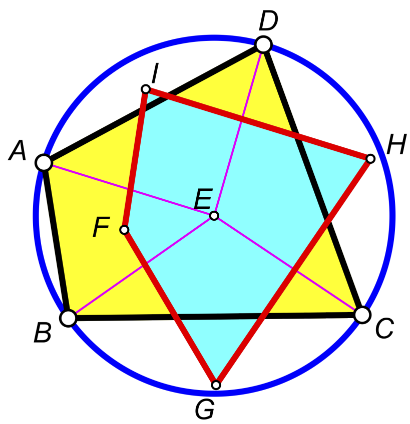

FIGURE 49. X_{616} points $\Longrightarrow$ $[ABCD] = \frac{3}{2}[FGHI]$

Proof. Let $AB = a$, $BC = b$, $CD = c$, and $DA = d$. From Lemma 9.4, we have $EF = a/\sqrt{3}$ and $EG = b/\sqrt{3}$ (Figure 50).

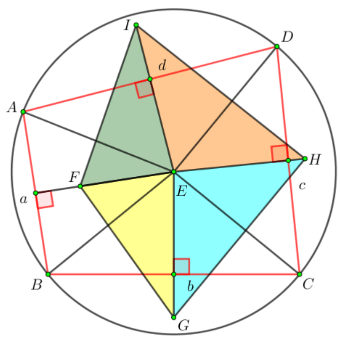

FIGURE 50.

Therefore,

$$\begin{aligned}
[EFG] &= \frac{1}{2} \cdot EF \cdot EG \cdot \sin(\angle FEG) \\
&= \frac{1}{2} \cdot \frac{a}{\sqrt{3}} \cdot \frac{b}{\sqrt{3}} \cdot \sin(180^\circ - B) \\
&= \frac{1}{6} ab \sin B \\
&= \frac{1}{3}[ABC].
\end{aligned}$$

Similarly, $[EGH] = \frac{1}{3}[BCD]$, $[EHI] = \frac{1}{3}[CDA]$, $[EIF] = \frac{1}{3}[DAB]$. Therefore,

$$\begin{aligned}[EFGH] &= [EFG] + [EGH] + [EHI] + [EIF] \\ &= \frac{1}{3}\left([ABC] + [BCD] + [CDA] + [DAB]\right)\end{aligned}$$

from which, using the relations $[ABCD] = [ABC] + [CDA] = [BCD] + [DAB]$, we get the desired result. □

Note. Theorem 7.29 is a special case of Theorem 9.6 since all rectangles are cyclic.

Theorem 9.7. *Let E be the circumcenter of cyclic quadrilateral $ABCD$. Let F, G, H, and I be the X_{617} points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively. Then*

$$[ABCD] = \frac{3}{2}[FGHI]$$

Proof. The proof is similar to the proof of Theorem 9.6, so the details are omitted. □

Relationship $[ABCD] = \frac{9}{8}[FGHI]$

Theorem 9.8. *Let E be the circumcenter of cyclic quadrilateral $ABCD$. Let n be 290, 671, or 903. Let F, G, H, and I be the X_n points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively (Figure 51). Then*

$$[ABCD] = \frac{9}{8}[FGHI].$$

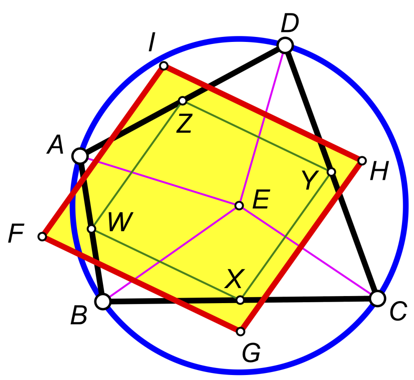

Figure 51. X_{290} points $\implies$ $[ABCD] = \frac{9}{8}[FGHI]$

Proof. The proof is the same as the proof of Theorem 9.2, except $k = -\frac{1}{3}$ and $\frac{2}{(1-k)^2} = \frac{9}{8}$. □

Relationship $[ABCD] = \frac{1}{2}[FGHI]$

Theorem 9.9. *Let E be the circumcenter of cyclic quadrilateral $ABCD$. Let n be 148, 149, or 150. Let F, G, H, and I be the X_n points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively (Figure 52). Then*

$$[ABCD] = \frac{1}{2}[FGHI].$$

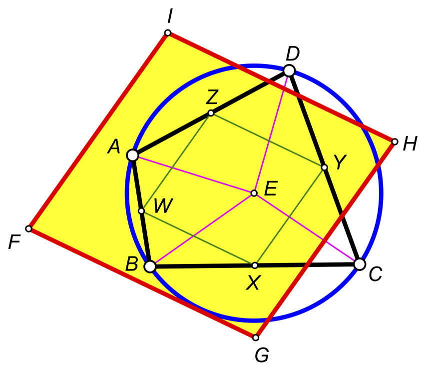

FIGURE 52. X_{149} points $\implies [ABCD] = \frac{1}{2}[FGHI]$

Proof. The proof is the same as the proof of Theorem 9.2, except $k = -1$ and $\frac{2}{(1-k)^2} = \frac{1}{2}$. □

10. Steiner point

In this section, we examine central quadrilaterals formed from the Steiner point of the reference quadrilateral.

A *midray circle* of a quadrilateral is the circle through the midpoints of the line segments joining one vertex of the quadrilateral to the other vertices (Figure 53).

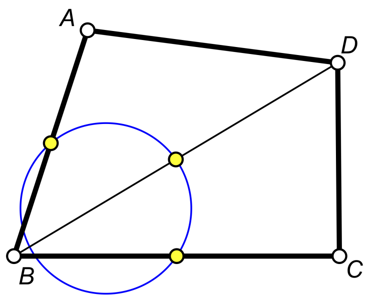

Figure 53. Midray circle of quadrilateral $ABCD$ relative to vertex B

The *Steiner point* (sometimes called the Gergonne-Steiner point) of a quadrilateral is the common point of the midray circles of the quadrilateral.

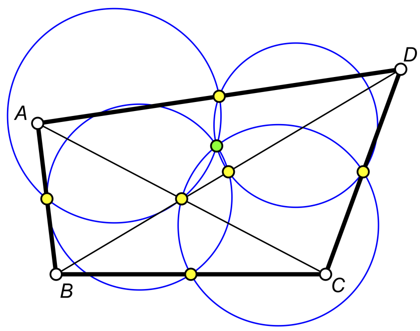

Figure 54. Steiner point of quadrilateral $ABCD$

Figure 54 shows the Steiner point of quadrilateral $ABCD$. The yellow points represent the midpoints of the sides and diagonals of the quadrilateral. The blue circles are the midray circles. The common point of the four circles is the Steiner point (shown in green).

Proposition 10.1. *The Steiner point of a parallelogram coincides with the diagonal point.*

Proof. The diagonals of a parallelogram bisect each other. Every midray circle passes through the midpoint of a diagonal. Therefore all midray circles pass through the diagonal point of the quadrilateral. □

Proposition 10.2. *The Steiner point of a cyclic quadrilateral coincides with the circumcenter of that quadrilateral.*

Proof. Let $ABCD$ be a cyclic quadrilateral with circumcenter O. Let X, Y, and Z be the midpoints of AB, AC, and AD, respectively. Then the circle through

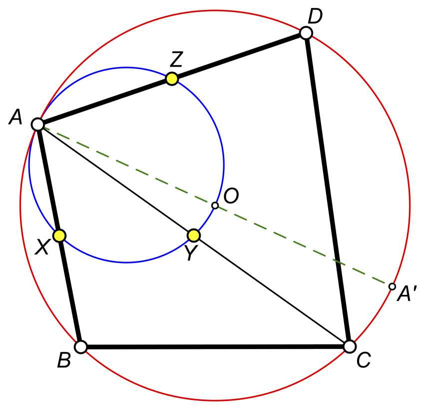

FIGURE 55. Relationship between midray circle and circumcircle

X, Y, and Z is a midray circle of quadrilateral $ABCD$. Let A' be the point on the circumcircle diametrically opposite A (Figure 55). The homothety with center A and ratio of similarity $\frac{1}{2}$ maps B into X, C into Y, D into Z, and A' into O. This homothety therefore maps the circumcircle into the midray circle. Thus, the circumcenter of the quadrilateral, O, lies on the midray circle. Since this is true for all the midray circles, the point of intersection of the midray circles (the Steiner point) must be O. $\square$

The following result comes from [21].

Lemma 10.3. *Let P be any point on altitude AH of $\triangle ABC$. Let X and X' be the midpoints of AB and AC, respectively. Let ω be the circumcircle of $\triangle XPX'$. Let PP' be the chord of ω through P that is parallel to BC. Let H' be the orthogonal projection of P' on BC. Then H' is the midpoint of BC.*

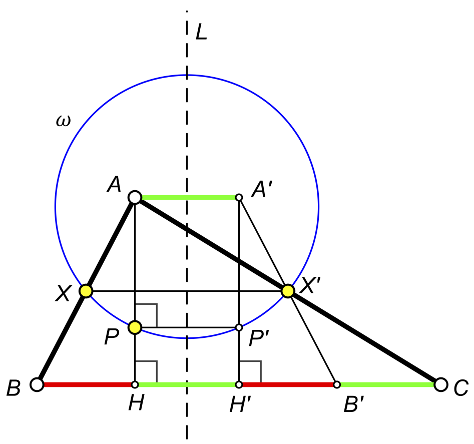

FIGURE 56.

Proof. Let L be the perpendicular bisector of XX'. Let A' be the reflection of A about L. Let B' be the reflection of B about L. Since X' is the reflection of X about L, $A'B'$ passes through X' (Figure 56). Then $\triangle AX'A' \cong \triangle CX'B'$. Thus, $AA' = B'C$. Also, $HH' = AA'$, so $HH' = B'C'$. By symmetry, $BH = H'B'$. Hence, $BH' = BH + HH' = HB' + B'C = H'C$ and H' is the midpoint of BC. $\square$

Proposition 10.4. *The Steiner point of an orthodiagonal quadrilateral coincides with the point of intersection of the perpendicular bisectors of the diagonals.*

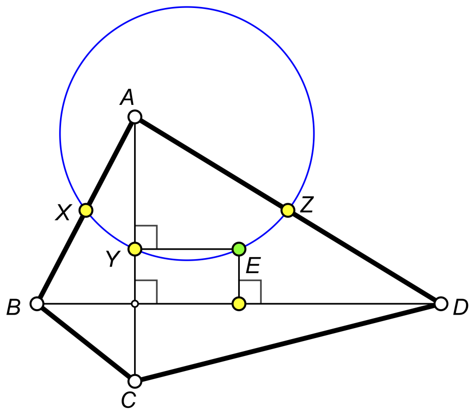

Figure 57. Steiner point of an orthodiagonal quadrilateral

Proof. Let the orthodiagonal quadrilateral be $ABCD$ and let X, Y, and Z be the midpoints of AB, AC, and AD, respectively. Let E be the intersection point of the perpendicular bisectors of diagonals AC and BD (Figure 57). From Lemma 10.3, we can conclude that the midray circle through X, Y, and Z passes through E. Similarly, the other midray circles pass through E. Thus, E is the Steiner point of quadrilateral $ABCD$. □

Note. The point of intersection of the perpendicular bisectors of the diagonals of a quadrilateral is known as the *quasi circumcenter* (QG-P5 in [22]) of the quadrilateral.

When the orthodiagonal quadrilateral is a kite, we get the following result.

Corollary 10.5. *Let $ABCD$ be a kite in which BD bisects AC. Then the Steiner point of $ABCD$ is the midpoint of BD (Figure 58).*

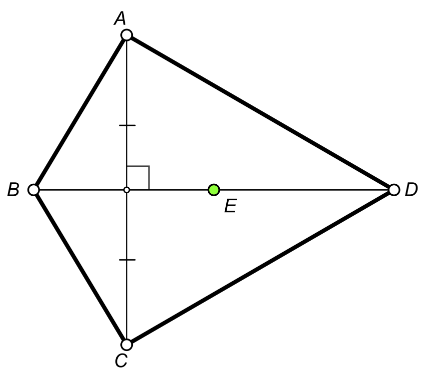

Figure 58. Steiner point E of kite is midpoint of BD

Our computer study examined the central quadrilaterals formed by the Steiner point. Since the Steiner point coincides with the diagonal point of a parallelogram, we omit results for parallelograms. Since the Steiner point of a cyclic quadrilateral coincides with the circumcenter of that quadrilateral, we omit results for cyclic quadrilaterals. We checked the central quadrilateral for all the first 1000 triangle centers (omitting points at infinity) and all reference quadrilateral shapes listed in Table 1.

The results found are listed in Table 12.

TABLE 12.

Central Quadrilaterals formed by the Steiner Point		
Quadrilateral Type	**Relationship**	**centers**
equidiagonal kite	$[ABCD] = 8[FGHI]$	642
	$[ABCD] = 2[FGHI]$	486

Relationship $[ABCD] = 8[FGHI]$

The following result by Peter Moses comes from [19].

Lemma 10.6. *Erect squares inwards on the sides of triangle $\triangle ABC$. The circumcenter of the centers of the squares is the center X_{642} of $\triangle ABC$.*

Theorem 10.7. *Let E be the Steiner point of equidiagonal kite $ABCD$ with $AB = BC$ and $AD = CD$. Let F, G, H, and I be the X_{642} points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively (Figure 59). Then $FGHI$ is a square homothetic to the Varignon parallelogram of $ABCD$ and*

$$[ABCD] = 8[FGHI].$$

The two squares have the same diagonal point.

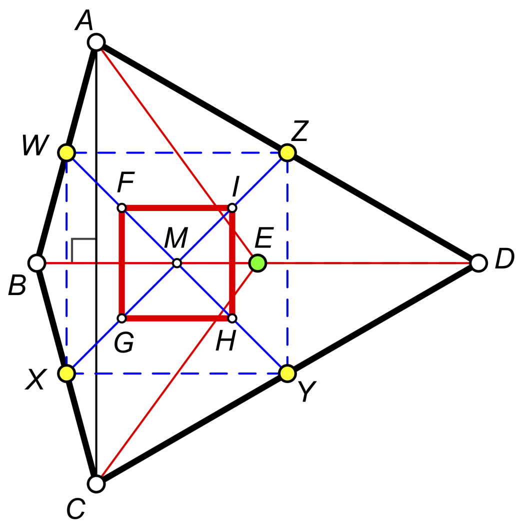

FIGURE 59. Equidiagonal kite, X_{642} points $\implies$ $[ABCD] = 8[FGHI]$

Proof. We use Cartesian coordinates with the origin at point E, with x-axis the line BD. Without loss of generality, assume that $AC = BD = 2$ and $AB < AD$. Since $ABCD$ is a kite, $BD \perp AC$ and BD bisects AC. By Corollary 10.5, we have $B = (-1, 0)$ and $D = (1, 0)$. Let K be the point of intersection of AC and BD. Since $ABCD$ is equidiagonal, $AK = CK = 1$ and we can let $K = (-u, 0)$, $A = (-u, 1)$, and $C = (-u, -1)$, with $u > 0$ as shown in Figure 60.

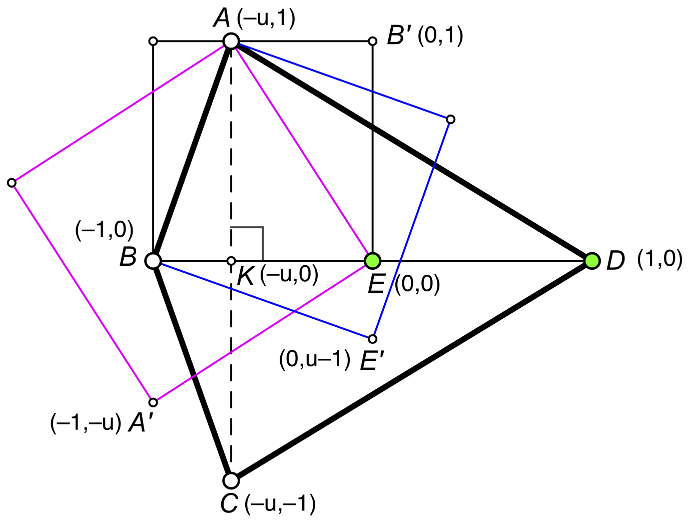

FIGURE 60. Coordinate set-up for an equidiagonal kite

In order to get the coordinates of point F, we use Lemma 10.6. See Figure 61.

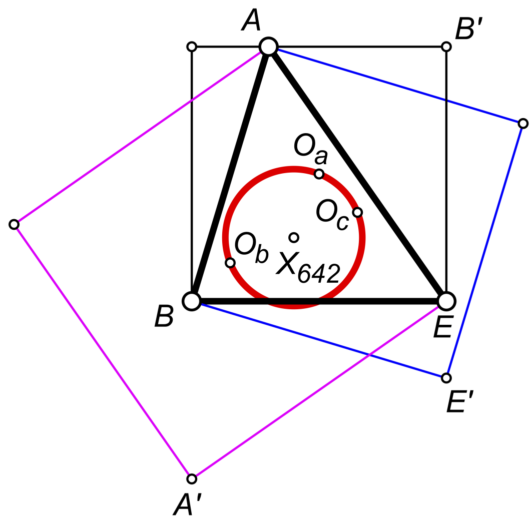

FIGURE 61.

The center of the square constructed inwards on the side BE is the midpoint of the segment BB' where $B' = (0, 1)$. Therefore, $O_a = \left(-\frac{1}{2}, \frac{1}{2}\right)$.

The center of the square constructed inwards on the side AE is the midpoint of the segment AA' where $A' = (-1, -u)$ since $\triangle EBA' \cong \triangle EB'A$. Therefore, $O_b = \left(\frac{-u-1}{2}, \frac{-u+1}{2}\right)$.

The center of the square constructed inwards on the side AB is the midpoint of the segment AE' where $E' = (0, u - 1)$ since $\triangle BEE' \cong \triangle ABK$, so $EE' = BK$. Therefore, $O_c = \left(-\frac{u}{2}, \frac{u}{2}\right)$.

The point F is the circumcenter of $\triangle O_aO_bO_e$. It coincides with the point of intersection of the perpendicular bisectors of O_aO_b and O_aO_c. The midpoint of O_aO_b is easily found, as well as the slope of O_aO_b. The slope of the perpendicular bisector of O_aO_b is the negative reciprocal of the slope of O_aO_b. Using the Point Slope Formula, we find that the equation of the perpendicular bisector of O_aO_b is $2x + 2y + u = 0$. Similarly, the equation of the perpendicular bisector of O_aO_c is $2x - 2y + u = 0$.

Solving these two equations gives us the coordinates for point F. The point G is the reflection of F with respect to BD. The coordinates of F and G are therefore

$$F = \left(\frac{-2u-1}{4}, \frac{1}{4}\right), \qquad G = \left(\frac{-2u-1}{4}, -\frac{1}{4}\right).$$

In the same manner, we find that the coordinates of H and I are

$$H = \left(\frac{-2u+1}{4}, -\frac{1}{4}\right), \qquad I = \left(\frac{-2u+1}{4}, \frac{1}{4}\right).$$

Therefore, $FG = FI = \frac{1}{2}$, so $FGHI$ is a square (notice that $FGHI$ is a parallelogram by construction). The center of $FGHI$ is the midpoint $M = \left(-\frac{u}{2}, 0\right)$ of FH. The coordinates for the midpoints of the sides AB, BC, CD, and DA are

$$W = \left(\frac{-u-1}{2}, \frac{1}{2}\right), \qquad X = \left(\frac{-u-1}{2}, -\frac{1}{2}\right),$$

$$Y = \left(\frac{-u+1}{2}, -\frac{1}{2}\right), \qquad Z = \left(\frac{-u+1}{2}, \frac{1}{2}\right).$$

The center of $WXYZ$ is $\left(-\frac{u}{2}, 0\right)$. The square $FGHI$ and the Varignon parallelogram $WXYZ$ of $ABCD$ have the same center and parallel sides, so they are homothetic. The ratio of similarity is $k = \frac{1}{2}$.

Finally, since $[ABCD] = 2$ and $[FGHI] = \frac{1}{4}$, we have $[ABCD] = 8[FGHI]$. $\square$

Relationship $[ABCD] = 2[FGHI]$

Lemma 10.8. *Let P be any point on side AD of square $ABCD$. Then the inner Vecten point (X_{486} point) of $\triangle PBC$ coincides with the diagonal point of the square (Figure 62).*

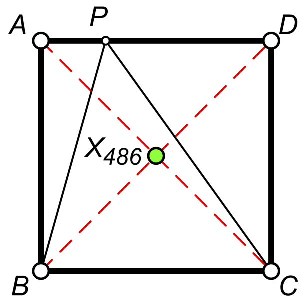

FIGURE 62. Vecten point of $\triangle PBC$ in square $ABCD$

Proof. From [17] we know that the inner Vecten point is the intersection of PO_a and CO_c where O_a is the center of the square erected internally on side BC of $\triangle PBC$ and O_c is the center of the square erected internally on side BP of $\triangle PBC$ (Figure 63).

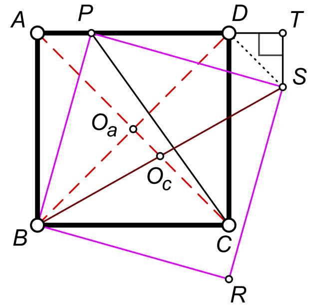

FIGURE 63. Squares erected on sides PB and BC

The line PO_a clearly passes through O_a, so we need only show that CO_c also passes through O_a. Let $PBRS$ be the square erected internally on side BP of $\triangle PBC$. Drop a perpendicular from S to AD meeting it at T. Right triangles BAP and PTS have equal hypotenuses. Angles $\angle PBA$ and $\angle SPT$ are equal because they are both complementary to $\angle APB$. Thus, $\triangle BAP \cong \triangle PTS$. Hence $PT = AB$ and $TS = AP$. Since $AB = AD$, this implies that $DT = PT - PD = AB - PD = AD - PD = AP$. Because $DT = TS$, $\angle SDT = 45°$. But $\angle CAT = 45°$, so $DS \parallel AC$. In $\triangle BDS$, O_a is the midpoint of BD and O_c is the midpoint of BS, so $O_aO_c \parallel DS$. Thus, O_aO_c coincides with AC and hence CO_c passes through O_a. $\square$

Theorem 10.9. *Let E be the Steiner point of equidiagonal kite $ABCD$ with $AB = BC$ and $AD = CD$. Let F, G, H, and I be the inner Vecten points (X_{486} points) of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively (Figure 64). Then $FGHI$ is a square congruent to the Varignon parallelogram of $ABCD$ and*

$$[ABCD] = 2[FGHI].$$

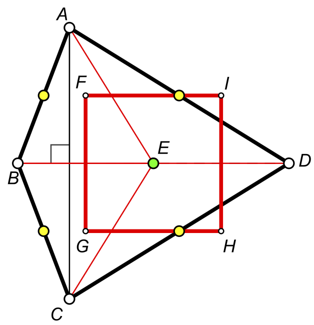

FIGURE 64. Equidiagonal kite, X486 points $\implies$ $[ABCD] = 2[FGHI]$

Proof. Since $ABCD$ is a kite, $BD \perp AC$ and BD and BD bisects AD. By Corollary 10.5, E is the midpoint of BD. Let K be the point of intersection of AC and BD. Erect perpendiculars to BD at B, E and D. Erect perpendiculars to AC at A, and C. The points of intersection of these perpendiculars are P, Q, R, S, T, and U as shown in Figure 65.

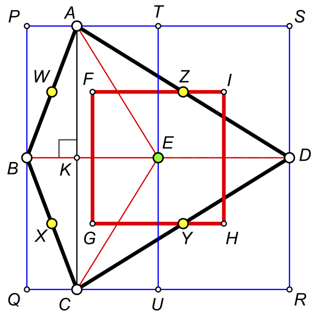

Figure 65.

Since $ABCD$ is equidiagonal, $AC = BD$ or $AK = KC$. This implies that quadrilaterals $PBET$, $TEDS$, $BQUE$, and $EURD$ are all squares.

By Lemma 10.8, F, G, H, and I are the centers of these squares, from which it follows that $FGHI$ is a square congruent to each of these squares. Square $FGHI$ has center at E and side of length equal to BE. If W, X, Y, and Z are the midpoints of the sides of $ABCD$, then $WXYZ$ is the Varignon parallelogram of $ABCD$. Since $WZ = \frac{1}{2}BD = BE$, square $FGHI \cong WXYZ$.

Finally, since $[ABCD] = 2[WXYZ]$ and $[FGHI] = [WXYZ]$, we have $[ABCD] = 2[FGHI]$. $\square$

11. Centroid

In this section, we examine central quadrilaterals formed from the centroid of the reference quadrilateral.

A *bimedian* of a quadrilateral is the line segment joining the midpoints of two opposite sides (Figure 66).

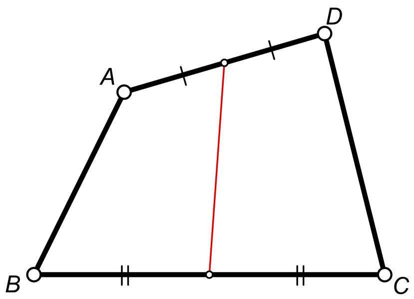

Figure 66. Bimedian of a quadrilateral

The *centroid* (or vertex centroid) of a quadrilateral is the point of intersection of the bimedians (Figure 67). The centroid bisects each bimedian.

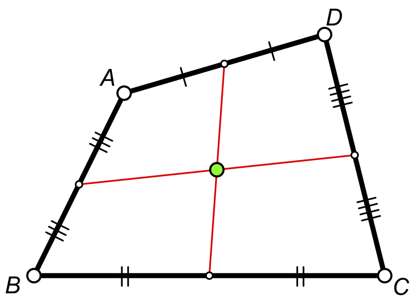

Figure 67. Centroid of a quadrilateral

Our computer study examined the central quadrilaterals formed by the centroid. Since it is easy to prove that the centroid coincides with the diagonal point of a parallelogram, we omit results for parallelograms. We checked the central quadrilateral for all the first 1000 triangle centers (omitting points at infinity) and all reference quadrilateral shapes listed in Table 1.

The results found are listed in the following table.

Table 13.

Central Quadrilaterals formed by the Centroid		
Quadrilateral Type	**Relationship**	**centers**
bicentric trapezoid	$[ABCD]=8[FGHI]$	402
	$[ABCD]=2[FGHI]$	122, 123, 127, 339
	$[ABCD]=\frac{1}{2}[FGHI]$	74, 477

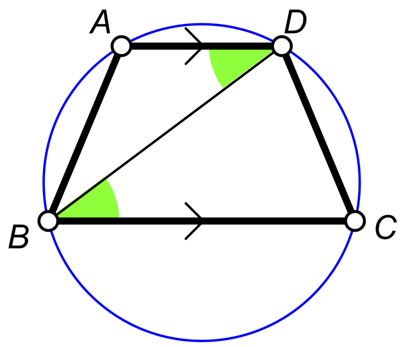

Figure 68. $AB = CD$

Lemma 11.1. *A cyclic trapezoid is isosceles.*

Proof. Let $ABCD$ be a cyclic trapezoid with $AD \parallel BC$ (Figure 68). Since $AD \parallel BC$, we must have $\angle ADB = \angle DBC$. Arcs intercepted by equal inscribed angles have the same measure, so minor arcs AB and CD are congruent. Equal arcs have equal chords, so $AB = CD$. □

Lemma 11.2. *Let $ABCD$ be a tangential trapezoid with $AD \parallel BC$. Let E be the incenter of $ABCD$. Then $AE \perp BE$ (Figure 69).*

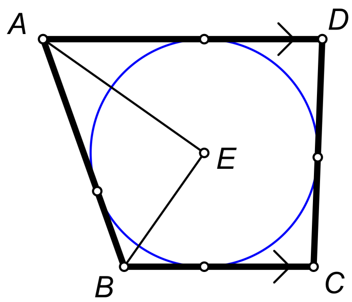

Figure 69. $AE \perp BE$

Proof. Since $AD \parallel BC$, $\angle BAD + \angle CBA = 180°$. Since E is the incenter, AE bisects $\angle BAD$ and BE bisects $\angle CBA$. Therefore,

$$\angle EBA + \angle BAE = \frac{1}{2}(\angle CBA + \angle BAD) = \frac{1}{2}(180°) = 90°.$$

The sum of the angles of $\triangle AEB$ is 180°. Hence $\angle AEB = 90°$, so $AE \perp BE$. □

Lemma 11.3. *Let $ABCD$ be a bicentric trapezoid with $AD \parallel BC$. Let E be the incenter of $ABCD$. Then $AE = DE$ and $BE = CD$ (Figure 70).*

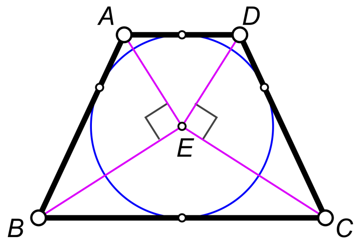

Figure 70. $BE = CE$

Proof. By Lemma 11.2, $AE \perp BE$ and $DE \perp CE$. By Lemma 11.1, $AB = DC$. Thus $\triangle AEB \cong \triangle DEC$ and so $BE = CE$. Similarly, $AE = DE$. □

Lemma 11.4. *The centroid of a bicentric trapezoid coincides with its incenter.*

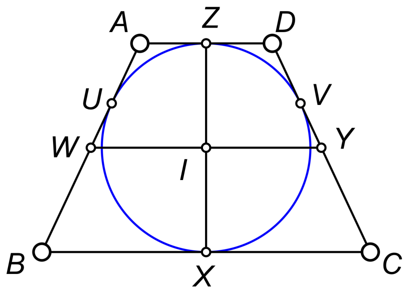

Figure 71. I is the centroid of bicentric trapezoid $ABCD$

Proof. Let the bicentric trapezoid be $ABCD$ with $AD \parallel BC$. Let the midpoints of the sides be W, X, Y, and Z as shown in Figure 71. Since a bicentric quadrilateral is cyclic, by Lemma 11.1, $AB = DC$. By definition, the centroid of quadrilateral $ABCD$ is point I, the intersection of WY and XZ. Since $WY \parallel AD \parallel BC$, and WY bisects both AB and CD, it must also bisect XZ. Therefore, $IX = IZ$. Since a bicentric quadrilateral is tangential, by Lemma 11.2, $AI \perp BI$. Hence $\triangle AIB$ is a right triangle and IW is the median to the hypotenuse. Thus, $IW = \frac{1}{2}AB$. Similarly, $IY = \frac{1}{2}CD$. Consequently, $IW = IY$.

Let IU be the altitude to the hypotenuse of right triangle AIB. Similarly, $\triangle CID$ is a right triangle and let IV be the altitude to its hypotenuse. Since XZ is the perpendicular bisector of AD and BC, we can conclude that $IA = ID$ and $IB = IC$. Thus, $\triangle AIB \cong \triangle DIC$. Corresponding parts of congruent figures are congruent, so $IU = IV$. The two tangents to a circle from an external point are equal, so $AU = AZ$. Triangles AIU and AIZ are congruent since $\angle IUA = \angle AZI = 90°$ and $AU = AZ$. Hence, $IU = IZ$.

We have now shown that $IU = IZ = IV = IX$, so I is the incenter of $ABCD$. □

Relationship $[ABCD] = 8[FGHI]$

Theorem 11.5. *Let E be the centroid of a bicentric trapezoid $ABCD$. Let F, G, H, and I be the X_{402} points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively (Figure 72). Then*

$$[ABCD] = 8[FGHI].$$

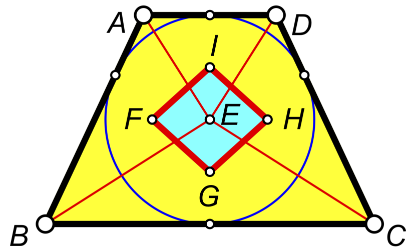

FIGURE 72. bicentric trapezoid, $X_{402} \implies [ABCD] = 8[FGHI]$

Proof. Let W, X, Y, and Z be the midpoints of the sides of the bicentric trapezoid as shown in Figure 73.

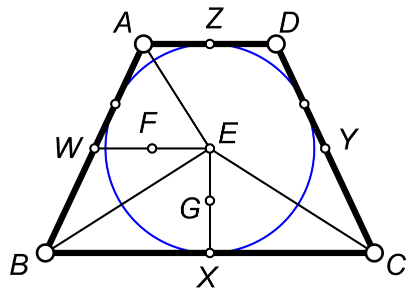

FIGURE 73.

Since E is the centroid of $ABCD$, it lies on the bimedian XZ which is the perpendicular bisector of BC. Therefore $EB = EC$ and $\triangle EBC$ is isosceles. By Theorem 7.23 and Table 9, G is the midpoint of EX. By Lemma 11.2, $EA \perp EB$, so $\triangle AEB$ is a right triangle. Note that W is the midpoint of the hypotenuse. By Lemma 7.5, the X_{402} point, F, coincides with the X_5 point. By Lemma 5.7, the X_5 point coincides with the midpoint of the median to the hypotenuse. Therefore, F is the midpoint of EW.

Similarly, H is the midpoint of EY and I is the midpoint of EZ. Thus, quadrilateral $FGHI$ is homothetic to quadrilateral $WXYZ$ with ratio of similarity $\frac{1}{2}$. Thus $[WXYZ] = 4[FGHI]$. But $WXYZ$ is the Varignon parallelogram of $ABCD$, so $[ABCD] = 2[WXYZ]$. Consequently, $[ABCD] = 2[WXYZ] = 8[FGHI]$. □

Relationship $[ABCD] = \frac{1}{2}[FGHI]$

Proposition 11.6 (X_{74} Property of a Right Triangle)**.** *Let ABC be a right triangle with right angle at C. Let P be the X_{74} point of $\triangle ABC$. Then $BCAP$ is an orthogonal kite and $CP = 2ab/c$ (Figure 74).*

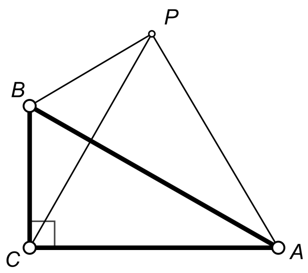

Figure 74. X_{74} point of a right triangle

Proof. According to [9], the barycentric coordinates for the X_{74} point of a triangle are

$$\left(a^2\left(a^4-2a^2b^2+a^2c^2+b^4+b^2c^2-2c^4\right)\left(a^4+a^2b^2-2a^2c^2-2b^4+b^2c^2+c^4\right)::\right).$$

With the condition that $a^2+b^2=c^2$, this simplifies to

$$P=\left(2a^2:2b^2:c^2\right).$$

Using the Distance Formula and the condition $a^2+b^2=c^2$, we find that $BP=a$. Similarly, $AP=b$. Thus, $BCAP$ is an orthogonal kite. The length CP is therefore twice the length of the altitude from A, which has length ab/c. □

Proposition 11.7 (X_{74} Property of an Isosceles Triangle)**.** *Let ABC be an isosceles triangle with $AC = BC$. Let P be the X_{74} point of $\triangle ABC$. Then $BCAP$ is a cyclic kite and $CP = b^2/CM$ where M is the midpoint of AB(Figure 75).*

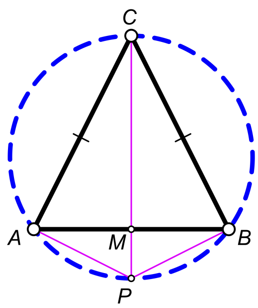

Figure 75. X_{74} point of a right triangle

Proof. The barycentric coordinates for the X_{74} point of a triangle were given in the proof of Proposition 11.6. With the condition that $a=b$, this simplifies to

$$P=\left(2b^2,2b^2,-c^2\right).$$

Let M be the midpoint of AB, so that $M = (1 : 1 : 0)$ and $AM = c/2$. Using the Distance Formula and the condition $a^2 + b^2 = c^2$, we find that $CM \times MP$ simplifies to $c^2/4$. But this is equal to $AM \times BM$. Thus, $BCAP$ is cyclic. □

Note that all kites are tangential, so $BCAP$ is actually a bicentric kite.

Theorem 11.8. *Let E be the centroid of a bicentric trapezoid $ABCD$. Let F, G, H, and I be the X_{74} points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively (Figure 76). Then $FGHI$ is a kite and*

$$[ABCD] = \frac{1}{2}[FGHI].$$

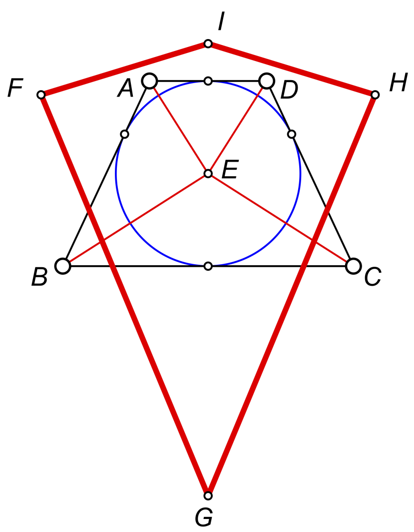

FIGURE 76. Bicentric trapezoid, X_{74} points $\implies$ $[ABCD] = \frac{1}{2}[FGHI]$

Proof. Let $BC = 2a$, $AD = 2b$, and let r and p be the inradius and the semiperimeter of $ABCD$. Let X, Y, Z, and T be the points where the incircle touches the sides of $ABCD$ as shown in Figure 77.

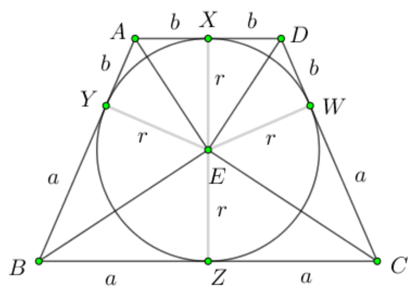

FIGURE 77. bicentric trapezoid and incircle

We have $[ABCD] = rp = 2r(a + b)$ and $r^2 = ab$. We will now calculate the area of $FGHI$. See Figure 78.

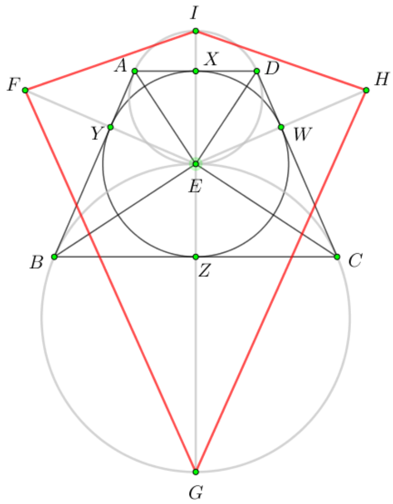

Figure 78. Bicentric trapezoid and incircle

By Proposition 11.7, we have $I \in \odot(AED)$ and $G \in \odot(BCE)$. Therefore, by the intersecting chords theorem, we get

$$IX \cdot XE = AX \cdot XD \quad \Rightarrow \quad IX = \frac{AX \cdot XD}{EX} = \frac{b^2}{r}$$

and

$$GZ \cdot ZE = BZ \cdot ZC \quad \Rightarrow \quad GZ = \frac{BZ \cdot ZC}{ZE} = \frac{a^2}{r}.$$

Hence,

$$EI = EX + XI = r + \frac{b^2}{r} = \frac{r^2 + b^2}{r} = \frac{ab + b^2}{r}$$

and

$$EG = EZ + ZG = r + \frac{a^2}{r} = \frac{r^2 + a^2}{r} = \frac{ab + a^2}{r}.$$

By Proposition 11.6, we have $EF = 2r$, so

$$[IEF] = \frac{1}{2} EI \cdot EF \cdot \sin(\angle IEF) = \frac{1}{2}\left(\frac{ab + b^2}{r}\right) \cdot 2r \sin B = b(a+b)\sin(\angle IEF).$$

Since $\angle BYE = \angle EZB = 90°$, quadrilateral $BYEZ$ is cyclic and so $\angle IEF = \angle B$. The value $\sin B$ is the height of A above BC divided by AB, so $\sin B = 2r/(a+b)$. Using this relation, we obtain

$$[IEF] = b(a+b) \cdot \frac{2r}{a+b} = 2rb.$$

Similarly, we have $[FEG] = 2ra$. Finally,

$$[FGHI] = 2\left([IEF] + [FEG]\right) = 2\left(2rb + 2ra\right) = 4r(a+b) = 2[ABCD],$$

and we are done. $\square$

Lemma 11.9. *The X_{477} point of an isosceles triangle coincides with its X_{74} point.*

Proof. With the condition that $a = b$, the barycentric coordinates for the X_{477} point simplifies to

$$P = \left(2b^2, 2b^2, -c^2\right).$$

These are the same coordinates as the X_{74} point in an isosceles triangle. $\square$

Lemma 11.10. *The X_{477} point of right triangle is the reflection of its X_{74} point about the median to the hypotenuse (Figure 79).*

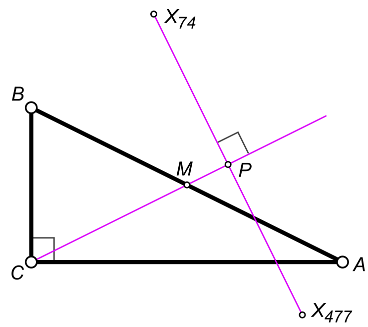

FIGURE 79. X_{74} and X_{477} points in a right triangle

Proof. Let the right triangle be $\triangle ABC$ with right angle at C. Let $M = (1 : 1 : 0)$ be the midpoint of the hypotenuse. The equation of line CM is $x = y$. The distance formula shows that $CX_{74} = 2ab/c$ and $CX_{477} = 2ab/c$, so $CX_{74} = CX_{477}$. We need only show that the midpoint, P, of $X_{74}X_{477}$ lies on CM. Calculating the midpoint of X_{74} and X_{477} subject to the condition $a^2 = b^2 + c^2$, we find that

$$P = \left(4a^2b^2 : 4a^2b^2 : a^4 - 6a^2b^2 + b^4\right).$$

Since the x and y components are equal, this proves that P lies on CM. □

Theorem 11.11. *Let E be the centroid of a bicentric trapezoid $ABCD$. Let F, G, H, and I be the X_{477} points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively (Figure 80). Then $FGHI$ is a kite with $FI = HI$, $FG = HG$, and*

$$[ABCD] = \frac{1}{2}[FGHI].$$

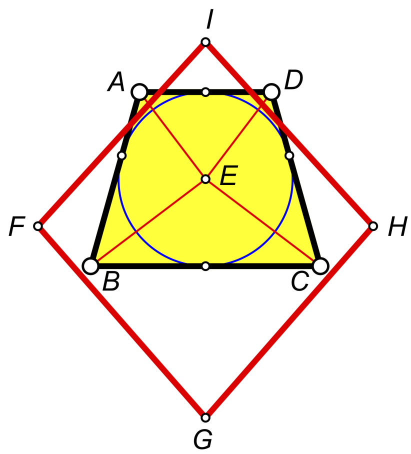

FIGURE 80. Bicentric trapezoid, X_{477} points $\implies$ $[ABCD] = \frac{1}{2}[FGHI]$

Proof. Let L be the line through E parallel to BC. Note that $\triangle AEB$ is a right triangle and the median to the hypotenuse is parallel to BC. Let F', G', H', and I' be the X_{74} points of the radial triangles (Figure 81).

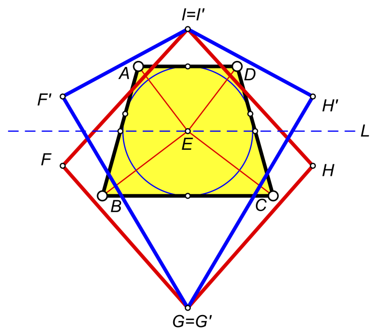

Figure 81. Bicentric trapezoid with X_{477} points and X_{74} points

By Lemma 11.9, G' coincides with G and I' coincides with I. By Lemma 11.10, F' is the reflection of F about L and H' is the reflection of H about L. Thus, $FH = F'H'$. By Lemma 7.11,

$$[FGHI] = \frac{1}{2}FH \cdot GI = \frac{1}{2}F'H' \cdot G'I' = [F'G'H'I'].$$

Thus,

$$[ABCD] = \frac{1}{2}[F'G'H'I'] = \frac{1}{2}[FGHI]$$

by Theorem 11.8. □

Relationship $[ABCD] = 2[FGHI]$

Theorem 11.12. *Let E be the centroid of a bicentric trapezoid $ABCD$. Let n be 122, 123, 127, or 339. Let F, G, H, and I be the X_n points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively. Then $FGHI$ is the Varignon parallelogram of $ABCD$ (Figure 82) and*

$$[ABCD] = 2[FGHI].$$

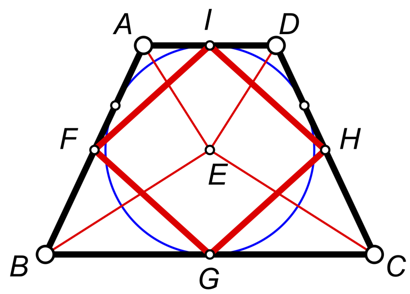

Figure 82. Bicentric trapezoid, X_{127} points $\implies$ $[ABCD] = \frac{1}{2}[FGHI]$

Proof. By Lemma 11.2, $\triangle AEB$ is a right triangle. By Lemma 7.8, F is the midpoint of AB. Similarly, H is the midpoint of CD. By Lemma 5.12, $\triangle BEC$ is isosceles. By Lemma 5.12, G is the midpoint of BC. Similarly, I is the midpoint of AD. Hence $FGHI$ is the Varignon parallelogram of $ABCD$ and

$$[ABCD] = 2[FGHI].$$

□

12. ANTICENTER

In this section, we examine central quadrilaterals formed from the anticenter of the reference quadrilateral. Note that only cyclic quadrilaterals have anticenters. A *maltitude* of a quadrilateral is a line from the midpoint of one side perpendicular to the opposite side (Figure 83).

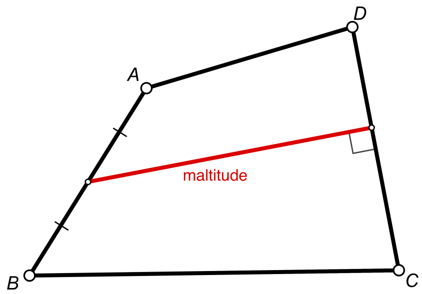

FIGURE 83. Maltitude of a quadrilateral

The four maltitudes of a cyclic quadrilateral concur at a point called the *anticenter* of the quadrilateral (Figure 84).

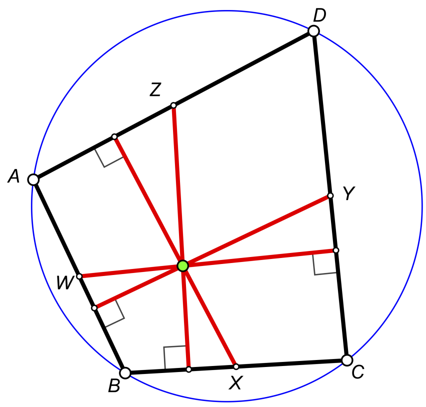

FIGURE 84. Anticenter of a cyclic quadrilateral

The following result is well known, [25].

Lemma 12.1 (Brahmagupta's Theorem)**.** *The anticenter of a cyclic orthodiagonal quadrilateral coincides with the diagonal point (Figure 85).*

The following result is well known, [24].

Lemma 12.2. *The anticenter of a cyclic quadrilateral coincides with the Poncelet point.*

Our computer study examined the central quadrilaterals formed by the anticenter. Since the anticenter of a cyclic quadrilateral coincides with the Poncelet point, we omit results for cyclic quadrilaterals. We checked the central quadrilateral for all the first 1000 triangle centers (omitting points at infinity) and all reference quadrilateral shapes listed in Table 1 that are cyclic.

No other results were found.

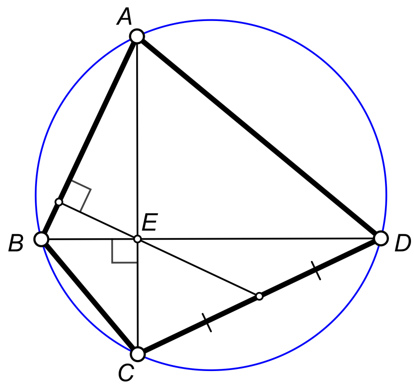

Figure 85. Brahmagupta's Theorem

Table 14.

Central Quadrilaterals formed by the Anticenter		
Quadrilateral Type	**Relationship**	**centers**
No relationships were found		

13. Orthocenter

In this section, we examine central quadrilaterals formed from the orthocenter of the reference quadrilateral. Note that only cyclic quadrilaterals have orthocenters. We define an *altitude* of a quadrilateral to be a line from a vertex to the orthocenter of the triangle formed by the other three vertices. The four altitudes of a cyclic quadrilateral concur at a point that we will call the *orthocenter* of the quadrilateral. (These are nonstandard definitions.)

Our computer study examined the central quadrilaterals formed by the orthocenter. Since the orthocenter of a rectangle coincides with the diagonal point, we omit results for rectangles. We checked the central quadrilateral for all the first 1000 triangle centers (omitting points at infinity) and all reference quadrilateral shapes listed in Table 1 that are cyclic.

No other results were found

Table 15.

Central Quadrilaterals formed by the Orthocenter		
Quadrilateral Type	**Relationship**	**centers**
No relationships were found.		

14. Incenter

In this section, we examine central quadrilaterals formed from the incenter of the reference quadrilateral. Note that only tangential quadrilaterals have incenters. A *tangential quadrilateral* in one in which a circle can be inscribed, touching all four sides. The center of this circle is called the *incenter* of the quadrilateral. The circle is called the *incircle*.

Our computer study examined the central quadrilaterals formed by the incenter. Since the incenter of a rhombus coincides with the diagonal point, we omit results for rhombi. In a bicentric trapezoid, the incenter coincides with the centroid (Lemma 11.4), so we have excluded results for bicentric trapezoids that are true when the radiator is the centroid. We checked the central quadrilateral for all the first 1000 triangle centers (omitting points at infinity) and all reference quadrilateral shapes listed in Table 1 that are tangential.

No other results were found.

Table 16.

Central Quadrilaterals formed by the Incenter		
Quadrilateral Type	**Relationship**	**centers**
No relationships were found.		

15. Midpoint of the 3rd Diagonal

In this section, we examine central quadrilaterals formed from the midpoint of the 3rd diagonal of the reference quadrilateral. If $ABCD$ is a convex quadrilateral with no two sides parallel, let AB meet CD at P and let BC meet DA at Q. Then line segment PQ is called the *3rd diagonal* of $ABCD$. Note that this line segment only exists when the reference quadrilateral is not a trapezoid.

Our computer study examined the central quadrilaterals formed by the midpoint of the 3rd diagonal. We checked the central quadrilateral for all the first 1000 triangle centers (omitting points at infinity) and all reference quadrilateral shapes listed in Table 1 that are not trapezoids.

No other results were found.

Table 17.

Central Quadrilaterals formed by the midpoint of the 3rd diagonal		
Quadrilateral Type	**Relationship**	**centers**
No relationships were found.		

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