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The Circumconics Among Us

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Abstract. We use a computer to determine all circumconics of a triangle that pass through at least three of the first twenty triangle centers listed in the Encyclopedia of Triangle Centers. For each circumconic, we list the center, the fourth point of intersection of the conic with the triangle's circumcircle, and the perspector of the conic.

Keywords. triangle geometry, triangle centers, circumconic, perspector, fourth point of intersection, computer-discovered mathematics.

Mathematics Subject Classification (2020). 51M04, 51-08.

1. Introduction

Let X_n denote the nth named triangle center as cataloged in ETC, the Encyclopedia of Triangle Centers [2]. A *circumconic* of a triangle is a conic that passes through the vertices of the triangle. See figure 1. The circumconic is not necessarily an ellipse; it could be a parabola or a hyperbola, for example. See [7].

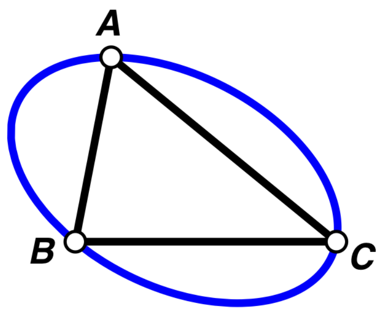

Figure 1. A circumconic of $\triangle ABC$

For five points in the plane (no three in a line), there is always a unique conic through these five points. Given six points, there isn't always a conic that passes through all six points.

We will find all circumconics of a triangle that pass through at least three of the first 20 triangle centers listed in ETC, i.e. members of the set $\{X_1, X_2, \ldots, X_{20}\}$.

In addition to listing which triangle centers the conic passes through, we will also give additional information about points associated with the conic, such as the center of the conic.

The *polar* of a point with respect to a conic is defined in [10]. In the case of a circumconic, the polar of a vertex with respect to the circumconic is the tangent to the conic at that vertex. The polars of the three vertices of a triangle with respect to a circumconic is called the *polar triangle* of that conic. See [4]. This triangle is also known as the *tangential triangle* of that conic. In Figure 2, $\triangle A'B'C'$ is the polar triangle of $\triangle ABC$ with respect to the blue conic.

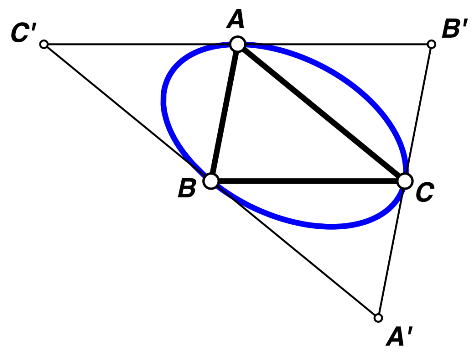

Figure 2. Polar triangle

It is known that a triangle is perspective with its polar triangle. That is, in figure 3, the lines AA', BB', and CC' concur. The point of concurrence is called the perspector of the conic with respect to $\triangle ABC$. See [6].

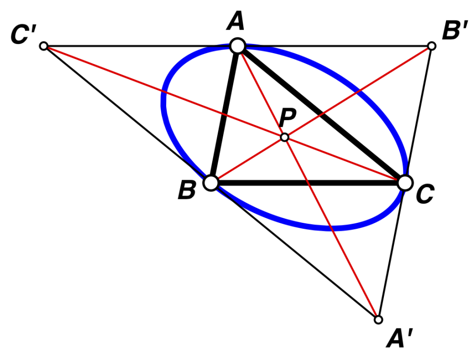

Figure 3. P is the perspector of the conic.

The circumcircle of a triangle will normally meet a circumconic in four points. Three of these points will be the vertices of the triangle. The fourth point is known as the *fourth point of the circumconic with respect to the circumcircle*, or

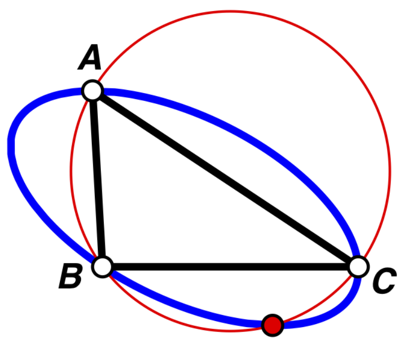

Figure 4. Circumcircle 4th point

more simply as the *circumcircle 4th point*. It is also called the *4th point of the circumconic* in the literature. It is shown as a red dot in Figure 4.

The circumconic of a triangle whose center coincides with the centroid of the triangle is known as the *Steiner ellipse*. See [9]. It is shown in red in Figure 5 in which the point G is both the center of the ellipse and the centroid of the triangle.

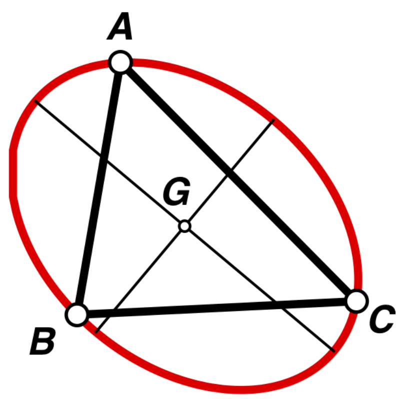

Figure 5. Steiner ellipse.

Given a circumconic of a triangle, it will generally intersect the Steiner ellipse of that triangle in four points. Three of these points will be the vertices of the triangle. The fourth point is known as *the fourth point of the circumconic with respect to the Steiner ellipse*, or more simply as *the Steiner 4th point*. It is shown as a red dot in Figure 6, where the blue conic meets the red Steiner ellipse.

For each of the circumconics that we will find, we will also give information about its center, its perspector, its fourth point, and its Steiner 4th point.

2. Methodology

Searching. We used Mathematica to search for circumconics. We worked primarily using barycentric coordinates. The barycentric coordinates for the vertices of the triangle are $A = (1 : 0 : 0)$, $B = (0 : 1 : 0)$, and $C = (0 : 0 : 1)$. The barycentric coordinates for the first 20 triangle centers were found in [2]. We then formed all triples of unique points (X_i, X_j, X_k) from these points with $i < j < k$.

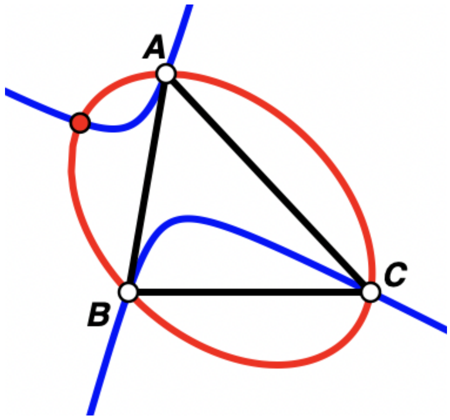

FIGURE 6. Steiner 4th point

Equation of conic. For each triple of points, we found the barycentric equation (in x, y, and z) of the conic that passes through A, B, C, X_i, and X_j using the following result which comes from [1, p. 234] (converted to barycentric coordinates).

Theorem 1. *The barycentric equation for the conic that passes through the five points* $(x_i : y_i : z_i)$, $i = 1, 2, 3, 4, 5$ *is*

$$(1)\qquad \begin{vmatrix} x^2 & y^2 & z^2 & yz & xz & xy \\ x_1^2 & y_1^2 & z_1^2 & y_1z_1 & x_1z_1 & x_1y_1 \\ x_2^2 & y_2^2 & z_2^2 & y_2z_2 & x_2z_2 & x_2y_2 \\ x_3^2 & y_3^2 & z_3^2 & y_3z_3 & x_3z_3 & x_3y_3 \\ x_4^2 & y_4^2 & z_4^2 & y_4z_4 & x_4z_4 & x_4y_4 \\ x_5^2 & y_5^2 & z_5^2 & y_5z_5 & x_5z_5 & x_5y_5 \end{vmatrix} = 0.$$

The following result come from [11, p. 109].

Theorem 2. *The barycentric equation for a circumconic has the form*

$$pyz + qzx + rxy = 0$$

where p, q, *and* r *are functions of* a, b, *and* c.

`Testing the six points.` If substituting the coordinates of X_k for x, y, and z gives an expression that simplifies to 0, then this means that X_k lies on that conic. In other words, it means that all six points lie on a conic.

`The center.` Once we found such a circumconic, we find the center of the conic using the the following result which comes from [1, p. 235] (converted to barycentric coordinates).

Theorem 3. *The center of the circumconic* $pyz + qzx + rxy = 0$ *is the point*

$$\Big(p(-p + q + r) : q(p - q + r) : r(p + q - r)\Big).$$

`The perspector.` To get the perspector of the circumconic we use the following result which comes from [11, p. 114].

Theorem 4. *The perspector of the circumconic* $pyz + qzx + rxy = 0$ *is the point* $(p : q : r)$.

`The circumcircle.` The equation of the circumcircle of the triangle is

$$a^2yz + b^2zx + c^2xy = 0.$$

`The 4th point.` To get the 4th point of intersection of the circumconic and the circumcircle, we solve the two equations for x, y, and z.

The result is the following. See [11, p. 109].

Theorem 5. *The 4th point of intersection of the circumconic $pyz+qzx+rxy = 0$ and the circumcircle is*

$$\left(\frac{1}{b^2r - c^2q} : \frac{1}{c^2p - a^2r} : \frac{1}{a^2q - b^2p}\right).$$

`The Steiner 4th point.` The equation for the Steiner ellipse can be found in [9]. In barycentric coordinates, this equation is

$$yz + zx + xy = 0.$$

To get the 4th point of intersection of the circumconic and the Steiner ellipse, we solve the two equations for x, y, and z.

The result is the following.

Theorem 6. *The 4th point of intersection of the circumconic $pyz+qzx+rxy = 0$ and the Steiner ellipse is*

$$\left(\frac{1}{r - q} : \frac{1}{p - r} : \frac{1}{q - p}\right).$$

`Finding the entry.` Once we have the barycentric coordinates for a point, we can see if that point is listed in ETC by converting the coordinates to normalized trilinear coordinates and then using the procedure described at [3].

3. Results

Using the procedure described above, we found the following results.

Theorem 7. *There are precisely 10 circumconics to a triangle that pass through at least three centers from among the first 20 triangle centers, X_1 through X_{20}. These are listed in Table 1.*

Note that the circumconics that pass through X_4 are rectangular hyerbolas by Feuerbach's conic theorem [8] and has center on the nine-point circle [1, p. 236]. The other conics are not rectangular.

We name each conic with the name "Conic *i*–*j*" where the first two centers on the conic are X_i and X_j.

The ten conics are shown in Figures 7 through 16.

TABLE 1. Circumconics passing through at least three of the 20 triangle centers X_1 through X_{20}.

Circumconics		
conic name	common name	centers on the circumconic
Conic 1–4	Feuerbach hyperbola	X_1, X_4, X_7, X_8, X_9
Conic 1–10		X_1, X_{10}, X_{19}
Conic 2–4	Kiepert hyperbola	X_2, X_4, X_{10}, X_{13}, X_{14}, X_{17}, X_{18}
Conic 2–15		X_2, X_{15}, X_{16}
Conic 3–4	Jerabek hyperbola	X_3, X_4, X_6
Conic 3–15		X_3, X_{15}, X_{17}
Conic 3–16		X_3, X_{16}, X_{18}
Conic 6–9		X_6, X_9, X_{19}
Conic 6–13		X_6, X_{13}, X_{16}
Conic 6–14		X_6, X_{14}, X_{15}

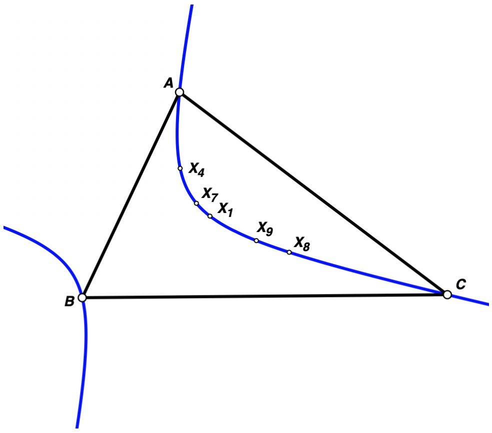

FIGURE 7. Conic 1-4: Feuerbach hyperbola

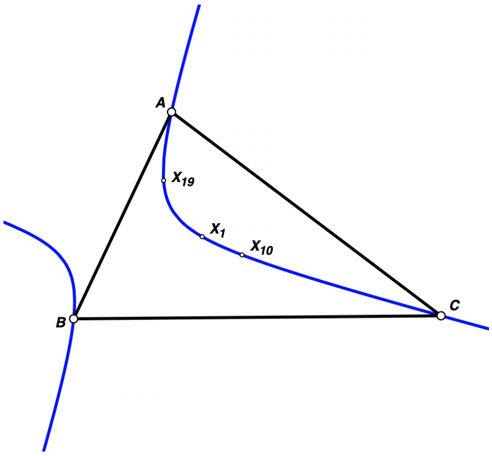

FIGURE 8. Conic 1-10

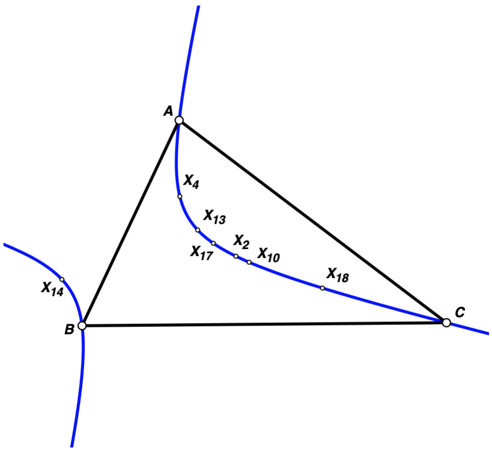

FIGURE 9. Conic 2-4: Kiepert hyperbola

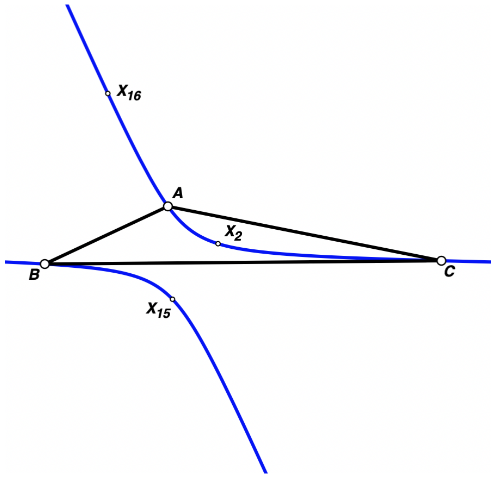

FIGURE 10. Conic 2-15

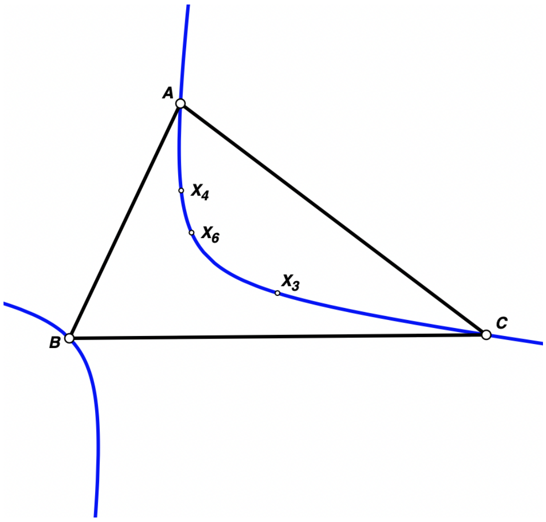

FIGURE 11. Conic 3-4: Jerabek Hyperbola

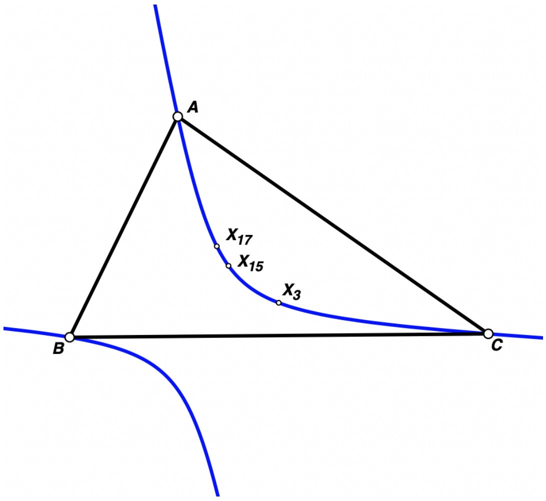

FIGURE 12. Conic 3-15

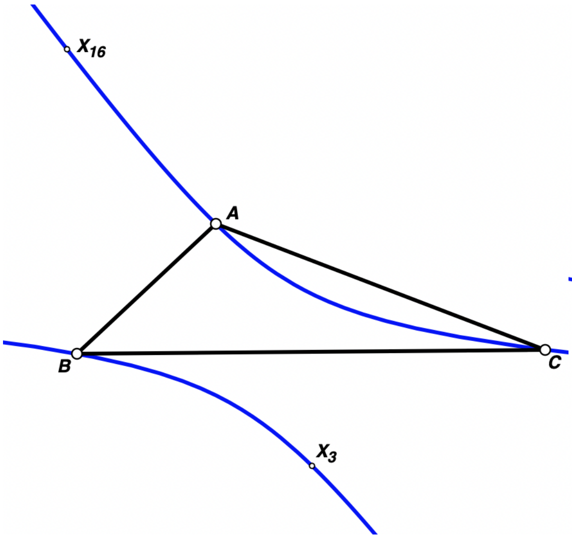

FIGURE 13. Conic 3-16

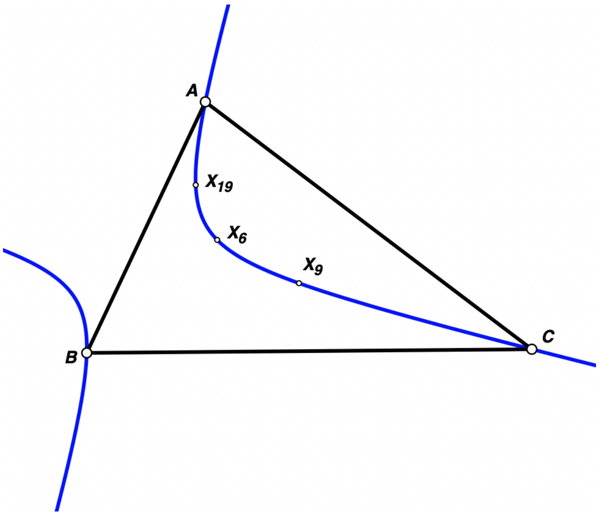

FIGURE 14. Conic 6-9

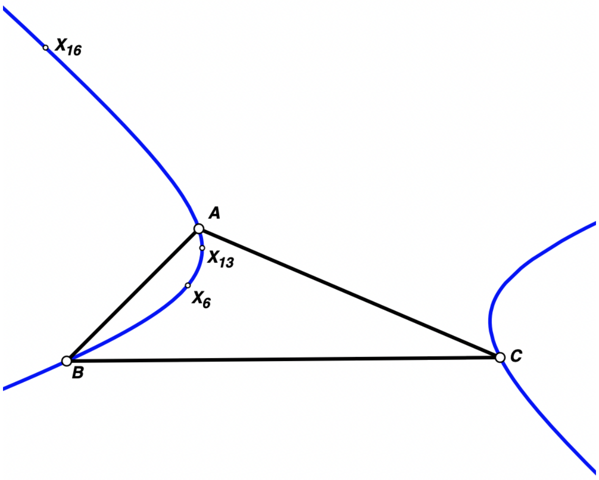

FIGURE 15. Conic 6-13

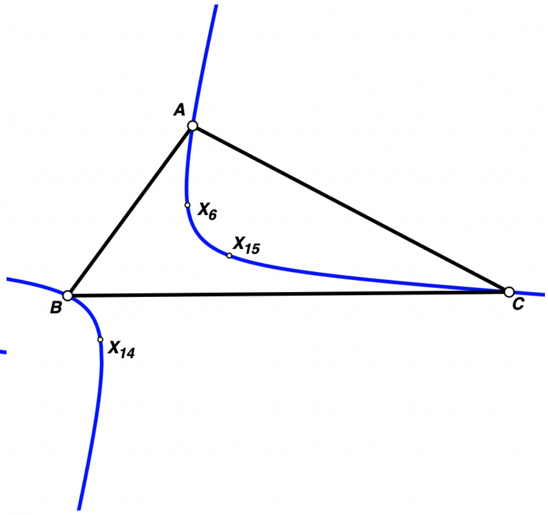

FIGURE 16. Conic 6-14

4. Equations of the Circumconics

For each circumconic found in Section 3, we found its equation in barycentric coordinates. We give the equation below.

We use the cyclic summation symbol as follows.

$$\sum_{\text{cyclic}} f(a,b,c)yz \equiv f(a,b,c)yz + f(b,c,a)zx + f(c,a,b)xy.$$

We use the symbol S to denote $2K$ where $K = \sqrt{s(s-a)(s-b)(s-c)}$ and $s = (a+b+c)/2$. We use the symbol S_A to represent $(b^2+c^2-a^2)/2$ and we use the symbol S_ω to represent $(a^2+b^2+c^2)/2$.

Theorem 8. *The barycentric equation for Conic 1-4 is*

$$\sum_{\text{cyclic}} a(a-b-c)(b-c)yz = 0.$$

Theorem 9. *The barycentric equation for Conic 1-10 is*

$$\sum_{\text{cyclic}} a(b^2-c^2)yz = 0.$$

Theorem 10. *The barycentric equation for Conic 2-4 is*

$$\sum_{\text{cyclic}} (b^2-c^2)yz = 0.$$

Theorem 11. *The barycentric equation for Conic 2-15 is*

$$\sum_{\text{cyclic}} a^2(b^2-c^2)\left(4S_A^2 - b^2c^2\right)yz = 0.$$

Theorem 12. *The barycentric equation for Conic 3-4 is*

$$\sum_{\text{cyclic}} a^2 S_A(b^2-c^2)yz = 0.$$

Theorem 13. *The barycentric equation for Conic 3-15 is*

$$\sum_{\text{cyclic}} a^2(b^2-c^2)S_A\left(S_A\sqrt{3}+S\right)yz = 0.$$

Theorem 14. *The barycentric equation for Conic 3-16 is*

$$\sum_{\text{cyclic}} a^2(b^2-c^2)S_A\left(S_A\sqrt{3}-S\right)yz = 0.$$

Theorem 15. *The barycentric equation for Conic 6-9 is*

$$\sum_{\text{cyclic}} a^2(a-b-c)(b-c)yz = 0.$$

Theorem 16. *The barycentric equation for Conic 6-13 is*

$$\sum_{\text{cyclic}} a^2\left(b^2-c^2\right)\left(b^2c^2-4S_A^2\right)\left(a^2S_\omega - \left(b^2-c^2\right)^2 + \sqrt{3}a^2S\right)yz = 0.$$

Theorem 17. *The barycentric equation for Conic 6-14 is*

$$\sum_{\text{cyclic}} a^2\left(b^2-c^2\right)\left(b^2c^2-4S_A^2\right)\left(a^2S_\omega - \left(b^2-c^2\right)^2 - \sqrt{3}a^2S\right)yz = 0.$$

5. Centers and Perspectors

For each circumconic found in Section 3, we used the equation found in Section 4 to find the barycentric coordinates for the center of the conic, using Theorem 3. We then used Theorem 4 to find the perspector of the circumconic.

The results are given below. The notation

$$f(a,b,c) ::$$

means

$$\Big(f(a,b,c) : f(b,c,a) : f(c,a,b)\Big).$$

Theorem 18. *The center of Conic 1-4 is*

$$(b-c)^2(a-b-c) ::$$

and the perspector is

$$a(a-b-c)(b-c) ::$$

Theorem 19. *The center of Conic 1-10 is*

$$a(b-c)^2(a+b)(a+c)(b+c) ::$$

and the perspector is

$$a(b^2-c^2) ::$$

Theorem 20. *The center of Conic 2-4 is*

$$(b^2-c^2)^2 ::$$

and the perspector is

$$b^2-c^2 ::$$

Theorem 21. *The center of Conic 2-15 is*

$$a^4(b^2-c^2)^2(b^2c^2-4S_A^2)^2 ::$$

and the perspector is

$$a^2\left(b^2-c^2\right)\left(b^2c^2-4S_A^2\right) ::$$

Theorem 22. *The center of Conic 3-4 is*

$$(b^2-c^2)^2S_A ::$$

and the perspector is

$$a^2(b^2-c^2)S_A ::$$

Theorem 23. *The center of Conic 3-15 is*

$$a^2\left(b^2-c^2\right)^2S_A\left(\sqrt{3}S_A+S\right)\left(2\sqrt{3}a^2S_A^2+b^2c^2S\right) ::$$

and the perspector is

$$a^2\left(b^2-c^2\right)S_A\left(\sqrt{3}S_A+S\right) ::$$

Theorem 24. *The center of Conic 3-16 is*

$$a^2\left(b^2-c^2\right)^2S_A\left(\sqrt{3}S_A-S\right)\left(2\sqrt{3}a^2S_A^2-b^2c^2S\right) ::$$

and the perspector is

$$a^2\left(b^2-c^2\right)S_A\left(\sqrt{3}S_A-S\right) ::$$

Theorem 25. *The center of Conic 6-9 is*

$$a^2(b-c)^2(a-b-c)\left(abc-(b+c)(2a^2+bc)+2aS_\omega\right)::$$

and the perspector is

$$a^2(b-c)(a-b-c)::$$

Theorem 26. *The center of Conic 6-13 is*

$$a^2\left(b^2-c^2\right)^2\left(b^2c^2-4S_A^2\right)\left(a^2S_\omega+\sqrt{3}a^2S-\left(b^2-c^2\right)^2\right)\times$$

$$\Big(-a^{10}+4a^8\left(b^2+c^2\right)-a^6\left(6b^4+7b^2c^2+6c^4\right)+a^4\left(4b^6-b^4c^2-b^2c^4+4c^6\right)$$

$$-a^2\left(b^8-7b^6c^2+6b^4c^4-7b^2c^6+c^8\right)-3b^2c^2\left(b^2-c^2\right)^2\left(b^2+c^2\right)+2\sqrt{3}SW\Big)$$

where $W =$

$$a^8-a^6\left(b^2+c^2\right)-a^4\left(b^4+b^2c^2+c^4\right)+a^2\left(b^6+b^4c^2+b^2c^4+c^6\right)-b^2c^2\left(b^2-c^2\right)^2$$

and the perspector is

$$a^2\left(b^2-c^2\right)\left(b^2c^2-4S_A^2\right)\left(a^2S_\omega+\sqrt{3}a^2S-\left(b^2-c^2\right)^2\right)::$$

Theorem 27. *The center and perspector of Conic 6-14 is the same as those of Conic 6-13 with $\sqrt{3}$ replaced by $-\sqrt{3}$.*

We summarize the results in Table 2. Each of these points are triangle centers themselves. We give the index of the center in ETC. If the center is not cataloged in ETC (as of December 17, 2021), then we give the first normalized trilinear coordinate of the center, rounded to 20 significant digits.

Table 2. Centers and Perspectors

Centers and Perspectors		
conic name	conic center	perspector
Conic 1–4	X_{11}	X_{650}
Conic 1–10	X_{244}	X_{661}
Conic 2–4	X_{115}	X_{523}
Conic 2–15	X_{18334}	X_{526}
Conic 3–4	X_{125}	X_{647}
Conic 3–15	3.4250735301223642904	−24.956806005202353349
Conic 3–16	3.3144761882667538523	14.803093134358701555
Conic 6–9	X_{38991}	X_{663}
Conic 6–13	X_{38994}	X_{6138}
Conic 6–14	X_{38993}	X_{6137}

6. Fourth Points

For each circumconic found in Section 3, we found the coordinates for the center of the conic, the perspector, the 4th point of the intersection of the conic and the circumcircle, and the 4th point of intersection of the conic and the Steiner ellipse.

Theorem 28. *The 4th point of intersection of Conic 1-4 with the circumcircle is*

$$\frac{a}{b^3+c^3-(a^2+bc)(b+c)+2abc}::$$

and the 4th point of intersection of Conic 1-4 with the Steiner ellipse is

$$\frac{bc}{b^2+c^2-ab-ac}::$$

Theorem 29. *The 4th point of intersection of Conic 1-10 with the circumcircle is*

$$\frac{1}{bc(b+c)\left(2S_A-bc\right)}::$$

and the 4th point of intersection of Conic 1-10 with the Steiner ellipse is

$$\frac{1}{(b+c)\left(a^2-bc\right)}::$$

Theorem 30. *The 4th point of intersection of Conic 2-4 with the circumcircle is*

$$\frac{1}{a^2b^2+a^2c^2-b^4-c^4}::$$

and the 4th point of intersection of Conic 2-4 with the Steiner ellipse is

$$\frac{1}{2a^2-b^2-c^2}::$$

Theorem 31. *The 4th point of intersection of Conic 2-15 with the circumcircle is*

$$\frac{a^2}{-2a^6+2a^4\left(b^2+c^2\right)-a^2\left(b^4+c^4\right)+\left(b^2-c^2\right)^2\left(b^2+c^2\right)}::$$

and the 4th point of intersection of Conic 2-15 with the Steiner ellipse is

$$\frac{1}{a^6\left(b^2+c^2\right)-2a^4\left(b^4+c^4\right)+a^2\left(b^6+c^6\ \right)-2b^2c^2\left(b^2-c^2\right)^2}::$$

Theorem 32. *The 4th point of intersection of Conic 3-4 with the circumcircle is*

$$\frac{a^2}{-2a^4+a^2\left(b^2+c^2\right)+\left(b^2-c^2\right)^2}::$$

and the 4th point of intersection of Conic 3-4 with the Steiner ellipse is

$$\frac{1}{a^2\left(a^2\left(b^2+c^2\right)-b^4-c^4\right)}::$$

Theorem 33. *The 4th point of intersection of Conic 3-15 with the circumcircle is*

$$\frac{a^2}{-2a^6+a^4\left(b^2+c^2\right)+\left(b^2-c^2\right)^2\left(b^2+c^2\right)\sqrt{3}+2S\left(-2a^4+a^2\left(b^2+c^2\right)+\left(b^2-c^2\right)^2\right)}::$$

and the 4th point of intersection of Conic 3-15 with the Steiner ellipse is

$$\frac{1}{2a^2S\left(-a^2\left(b^2+c^2\right)+b^4+c^4\right)+U\sqrt{3}}::$$

where $U=$

$$-a^6\left(b^2+c^2\right)+2a^4\left(b^4-b^2c^2+c^4\right)-a^2\left(b^2-c^2\right)^2\left(b^2+c^2\right)+2b^2c^2\left(b^2-c^2\right)^2.$$

Theorem 34. *The 4th points of intersection of Conic 3-16 with the circumcircle and the Steiner ellipse are the same as those for Conic 3-15 with* $\sqrt{3}$ *replaced by* $-\sqrt{3}$.

Theorem 35. *The 4th point of intersection of Conic 6-9 with the circumcircle is*

$$\frac{a^2}{-2a^2+a(b+c)+(b-c)^2}::$$

and the 4th point of intersection of Conic 6-9 with the Steiner ellipse is

$$\frac{1}{a^2b^2+a^2c^2-ab^3-ac^3+b^3c-2b^2c^2+bc^3}::$$

Theorem 36. *The 4th point of intersection of Conic 6-13 with the circumcircle is*

$$\frac{a^2}{V+2\sqrt{3}SW}::$$

where

$$\begin{aligned}V=&-4a^{10}+9a^8\left(b^2+c^2\right)-a^6\left(5b^4+14b^2c^2+5c^4\right)+2\left(b^2-c^2\right)^4\left(b^2+c^2\right)\\&+a^4\left(b^6+4b^4c^2+4b^2c^4+c^6\right)-a^2\left(b^2-c^2\right)^2\left(3b^4-b^2c^2+3c^4\right)\end{aligned}$$

and

$$W=a^6\left(b^2+c^2\right)-2a^4\left(b^4+c^4\right)+a^2\left(b^6+c^6\right)-2b^2c^2\left(b^2-c^2\right)^2$$

and the 4th point of intersection of Conic 6-13 with the Steiner ellipse is

$$\frac{1}{X+2\sqrt{3}SY}::$$

where

$$\begin{aligned}X=&-a^8\left(5b^4+8b^2c^2+5c^4\right)+3a^6\left(b^2+c^2\right)^3+a^4\left(b^8-6b^6c^2-6b^2c^6+c^8\right)\\&+2a^{10}\left(b^2+c^2\right)-a^2\left(b^2-c^2\right)^2\left(b^6-2b^4c^2-2b^2c^4+c^6\right)-b^2c^2\left(b^2-c^2\right)^4\end{aligned}$$

and

$$Y=-a^6\left(b^4+c^4\right)+2a^4\left(b^6+c^6\right)-a^2\left(b^8+c^8\right)+b^2c^2\left(b^2-c^2\right)^2\left(b^2+c^2\right).$$

Theorem 37. *The 4th points of intersection of Conic 6-14 with the circumcircle and the Steiner ellipse are the same as those for Conic 6-13 with* $\sqrt{3}$ *replaced by* $-\sqrt{3}$.

We summarize the results in Table 3. Each of these points are triangle centers themselves. We give the index of the center in ETC. If the center is not cataloged in ETC (as of December 17, 2021), then we give the first normalized trilinear coordinate of the center, rounded to 20 significant digits.

7. Epilog

The program used to find the circumconics among the first 20 Kimberling centers can just as easily find all circumconics passing through at least three centers from among the first "n" Kimberling centers, for $n > 20$. There are still many new circumconics to be discovered!

TABLE 3. 4th Points

4th points		
conic name	circumcircle 4th point	Steiner 4th point
Conic 1–4	X_{104}	X_{2481}
Conic 1–10	X_{759}	X_{18827}
Conic 2–4	X_{98}	X_{671}
Conic 2–15	X_{842}	5.3837381795689501633
Conic 3–4	X_{74}	X_{290}
Conic 3–15	13.556237473111978264	4.5532074323737705821
Conic 3–16	14.053539976756462913	3.3208364046894082094
Conic 6–9	X_{2291}	1.4762967908330227918
Conic 6–13	X_{2379}	7.8685588733658168253
Conic 6–14	X_{2378}	2.9881191186676777831

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