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GENERALIZATIONS OF SOME IMO GEOMETRY PROBLEMS

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Abstract: Usually, IMO Geometry problems are of high quality and content, proposing possibilities for further investigations and generalizations. In a corresponding research process accompanying results appear in a natural way too. Some geometric configurations will be considered here in connection with circles and the related generalizations include second-order curves, mainly conics. The software program "THE GEOMETER'S SKETCHPAD" (GSP) is applied as a heuristic tool for the purpose. Two approaches are used: 1) suitable examinations of problem conditions and connected generalizations of basic elements in them; 2) generalizations of the problem solution under consideration.

Key words: triangle, circumscribed conic section, parabola, ellipse, hyperbola, orthocenter, Euler's circle, Euler's line

I. Generalization of basic elements in problem conditions.

I. 1. A problem for the triangle orthocenter. The following problem from the 49-th IMO'2010 paper is considered:

Problem 1. *Given is an acute ΔABC with orthocenter H. The circle through H, whose center is the midpoint of the side BC, meets the sideline BC at A_1 and A_2. Analogously, the circle through H, whose center is the midpoint of the side CA, meets the sideline CA at B_1 and B_2. Also, the circle through H, whose center is the midpoint of the side AB, meets the sideline AB at C_1 and C_2. Prove that the points A_1, A_2, B_1, B_2, C_1 and C_2 are co-cyclic.* (Fig. 1)

Consider an arbitrary ΔABC, for which the points M_a, M_b and M_c are the midpoints of the sides BC, CA and AB, respectively.

I.1.1. Curves, that are generated by an arbitrary point in the plane of a triangle. According to Problem 1, the points A_1, A_2, B_1, B_2, C_1 and C_2 are co-cyclic. Note that the circle is a special second-order curve. Now replace the point H by an arbitrary point P in the plane of ΔABC. Using a similar construction as in Problem 1, we get 6 points A_1, A_2, B_1, B_2, C_1 and C_2, which induce the following assertion (still a hypothesis for the moment) (Fig. 2):

Theorem 1. (analogous to Problem 1) *Given is a triangle ABC and an arbitrary point P in its plane. The circle through P, whose center is the midpoint of the side BC, meets the sideline BC at A_1 and A_2. Analogously, the circle through P, whose center is the midpoint of the side CA, meets the sideline CA at B_1 and B_2. Also, the circle through P, whose center is the midpoint of the side AB, meets the sideline AB at C_1 and C_2. Then, the points A_1, A_2, B_1, B_2, C_1 and C_2 lie on a second-order curve $k(P)$, which turns out to be a circle iff P is the orthocenter of ΔABC.* (Fig. 2)

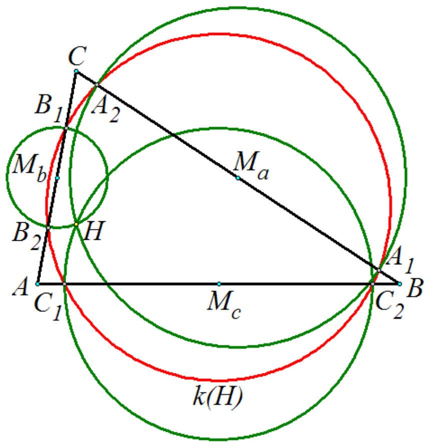

Fig. 1

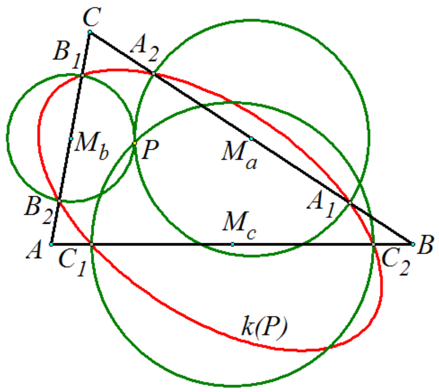

Fig. 2

In what follows the notation $k(X)$ will be used for a second-order curve with center X (if the curve is central). Additionally, experiments with GSP suggest the following:

Corollary 1. *If P is on the circumcircle Γ of ΔABC, then $k(P)$ is a hyperbola or it degenerates into two perpendicular lines (the set of the two lines is a special second-order curve).*

I.1.2. Curves, that are generated by isogonal conjugate points with respect to a triangle. Note that the midpoints M_a, M_b and M_c, which are circle centers in Problem 1, are the orthogonal projections of the circumcenter O onto the sidelines of ΔABC, while the common point of the three circles is the orthocenter H. Let us turn upside down the situation, i. e. let the orthogonal projections of the orthocenter H be centers of the three circles passing through the circumcenter O. Now the corresponding 6 points on the sidelines are co-cyclic. Something more, the GSP experiment shows that the new circle coincides with the previous one. Since H and O are isogonal conjugate with respect to ΔABC, then one could consider any other isogonal conjugate pair. We come to another assertion (again still a hypothesis for the moment):

Theorem 2. *Let P^1 and P^2 be isogonal conjugate points with respect to a given ΔABC, while the points P_a^j, P_b^j and P_c^j be the orthogonal projections of P^j $(j=1,2)$ onto the sidelines BC, CA and AB, respectively. The circle through P^s $(j \neq s = 1,2)$ with center P_a^j meets the sideline BC in A_1^j and A_2^j. The pairs B_1^j, B_2^j and C_1^j, C_2^j are defined on the sidelines CA and AB respectively in a similar way. Then, the points A_1^j, A_2^j, B_1^j, B_2^j, C_1^j and C_2^j $(j=1,2)$ lie on equal circles with centers P^1 and P^2.* (Fig. 3)

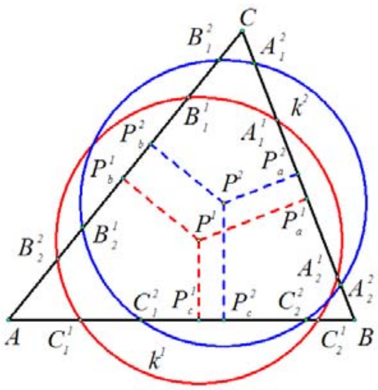

Fig. 3

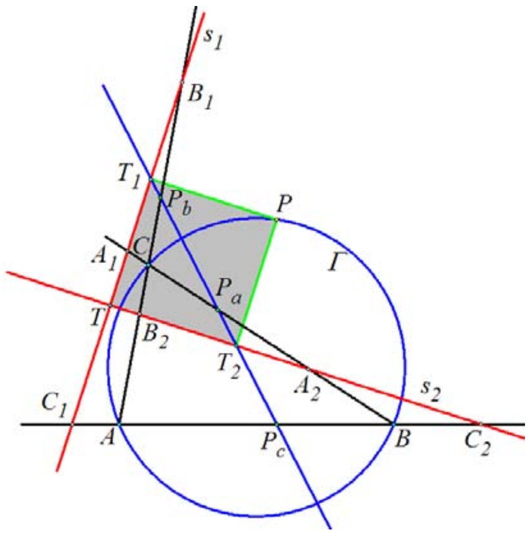

Fig. 4

Of course, the points on the circumcircle Γ of ΔABC (and only they) do not satisfy Theorem 2, since they have no isogonal conjugate ones with respect to ΔABC in the sense of finite points. In this case the "defect" could be removed as in Corollary 1. Construct the circles through $P \in \Gamma$ with centers P_a, P_b and P_c. Consider the intersection points of the circles with the sidelines BC, CA and AB in the way it is accomplished in Problem 1, Theorem 1 and Theorem 2. We state the following:

Theorem 3. *Let P be on the circumcircle Γ of ΔABC, while P_a, P_b and P_c be its orthogonal projections onto the sidelines BC, CA and AB, respectively. The circle through P with center P_a meets the sideline BC in A_1 and A_2. Analogously, the circle through P, with center P_b meets the sideline CA in B_1 and B_2. Also, the circle through P with center P_c meets* *the sideline AB in C_1 and C_2. Then, the points A_1, A_2, B_1, B_2, C_1 and C_2 lie on two perpendicular lines s_1 and s_2.* (Fig. 4)

It is well-known that the points P_a, P_b and P_c from Theorem 3 are collinear and the line s_P on which they lie, is known to be the Simson line. In connection with the Simson line note the following result:

Corollary 2. *The intersection points of s_1, s_2 and s_P together with the generating point P define the vertices of a square.* (Fig. 4)

PROOFS. We have to legalize the formulated assertions in this paragraph by mathematical proofs. It is enough to limit ourselves to Theorem 1, since the applied technique is the same. Details could be found in the book (Grozdev & Nenkov, 2012-[1]). Barycentric coordinates will be used with regard to ΔABC, namely $A(1,0,0), B(0,1,0), C(0,0,1)$. Let $|BC| = a$, $|CA| = b$ and $|AB| = c$. Then $16S^2 = 2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4$, where S is the area of ΔABC. For an arbitrary point $P(\lambda, \mu, \nu)$ $(\lambda + \mu + \nu = 1)$ in the plane of ΔABC consider the notation: $\delta = a^2\mu\nu + b^2\nu\lambda + c^2\lambda\mu$. The point P lies on the circumcircle of ΔABC iff $\delta = 0$. Remind also that if $M_1(x_1, y_1, z_1)$ and $M_2(x_2, y_2, z_2)$ are points in the plane of ΔABC, then:

(1) $$|M_1M_2|^2 = -(y_1 - y_2)(z_1 - z_2)a^2 - (z_1 - z_2)(x_1 - x_2)b^2 - (x_1 - x_2)(y_1 - y_2)c^2$$

If the vectors $\overrightarrow{u_1}(\lambda_1, \mu_1, \nu_1)$ and $\overrightarrow{u_2}(\lambda_2, \mu_2, \nu_2)$ ($\lambda_1 + \mu_1 + \nu_1 = 0,\ \lambda_2 + \mu_2 + \nu_2 = 0$) are coplanar to the plane of ΔABC, then they are perpendicular iff the following equality is verified:

(2) $$(\mu_1\nu_2 + \mu_2\nu_1)a^2 + (\nu_1\lambda_2 + \nu_2\lambda_1)b^2 + (\lambda_1\mu_2 + \lambda_2\mu_1)c^2 = 0.$$

Proof of Theorem 1. It is clear that the theorem is meaningless when P coincides with one of the vertices A, B and C. Such cases are excluded.

Let $p_a = \dfrac{|PM_a|}{a}$, $p_b = \dfrac{|PM_b|}{b}$ and $p_c = \dfrac{|PM_c|}{c}$. The coordinates of the 6 points under consideration are deduced from (1):

$$A_1\left(0, \frac{1}{2} + p_a, \frac{1}{2} - p_a\right),\ A_2\left(0, \frac{1}{2} - p_a, \frac{1}{2} + p_a\right),\ B_1\left(\frac{1}{2} - p_b, 0, \frac{1}{2} + p_b\right),\ B_2\left(\frac{1}{2} + p_b, 0, \frac{1}{2} - p_b\right)$$

$$C_1\left(\frac{1}{2} + p_c, \frac{1}{2} - p_c, 0\right),\ C_2\left(\frac{1}{2} - p_c, \frac{1}{2} + p_c, 0\right).$$

Substitute the coordinates and check the following equation:

(3) $$k(P): \begin{array}{l}(4p_a^2 - 1)(4p_b^2 - 1)(4p_c^2 - 1)(x^2 + y^2 + z^2) + 2(4p_b^2 - 1)(4p_c^2 - 1)(4p_a^2 + 1)yz + \\ + 2(4p_c^2 - 1)(4p_a^2 - 1)(4p_b^2 + 1)zx + 2(4p_a^2 - 1)(4p_b^2 - 1)(4p_c^2 + 1)xy = 0.\end{array}$$

This proves that the points A_1, A_2, B_1, B_2, C_1 and C_2 lie on a second-order curve $k(P)$.

Remark. In the proof it is not used the fact that the circles under consideration are concurrent at a point. This suggests a possibility to examine particular cases: for example the cases when they are tangent to a circle, tangent to a line or another additional condition.

It follows from (1) that:

$$(4)\qquad \begin{aligned} |M_aP|^2 &= a^2p_a^2 = \frac{1}{2}\left(-a^2+b^2+c^2\right)\lambda + \frac{1}{4}a^2 - \delta, \\ |M_bP|^2 &= b^2p_b^2 = \frac{1}{2}\left(a^2-b^2+c^2\right)\mu + \frac{1}{4}b^2 - \delta, \\ |M_cP|^2 &= c^2p_c^2 = \frac{1}{2}\left(a^2+b^2-c^2\right)\nu + \frac{1}{4}c^2 - \delta. \end{aligned}$$

In order to prove the second part of Theorem 1, denote by O_A and O_C the centers of the circumcircles of the triangles $A_1A_2B_1$ and $C_1C_2B_2$, respectively. Further, find (using (2)) the equations of two perpendicular bisectors of each triangle and consider the corresponding system. Thus, the coordinates of O_A and O_C could be determined as it follows:

$$x_{O_A} = -\frac{2a^2\left[4p_{ab} + 2\left(-a^2+b^2+c^2\right)p_b - c^2\right]}{32S^2(1-2p_b)},$$

$$y_{O_A} = \frac{4\left(a^2+b^2-c^2\right)p_{ab} - 4b^2\left(a^2-b^2+c^2\right)p_b + 2a^2b^2 + b^2c^2 + c^2a^2 - a^4 - b^4}{32S^2(1-2p_b)},$$

$$z_{O_A} = \frac{4\left(a^2-b^2+c^2\right)p_{ab} - 4c^2\left(a^2+b^2-c^2\right)p_b + 2a^2b^2 + 3b^2c^2 + c^2a^2 - a^4 - b^4 - 2c^4}{32S^2(1-2p_b)},$$

$$x_{O_C} = \frac{4\left(a^2-b^2+c^2\right)p_{cb} - 4a^2\left(-a^2+b^2+c^2\right)p_b + 2b^2c^2 + 3a^2b^2 + c^2a^2 - b^4 - c^4 - 2a^4}{32S^2(1-2p_b)},$$

$$y_{O_C} = \frac{4\left(-a^2+b^2+c^2\right)p_{cb} - 4b^2\left(a^2-b^2+c^2\right)p_b + a^2b^2 + 2b^2c^2 + c^2a^2 - b^4 - c^4}{32S^2(1-2p_b)},$$

$$z_{O_C} = -\frac{2c^2\left[4p_{cb} + 2\left(a^2+b^2-c^2\right)p_b - a^2\right]}{32S^2(1-2p_b)},$$

where $p_{ab} = a^2p_a^2 - b^2p_b^2$ and $p_{cb} = c^2p_c^2 - b^2p_b^2$.

If the circles coincide, then $O_A \equiv O_C$ and we deduce that:

$$-2a^2\left(-a^2+b^2+c^2\right)\lambda + \left(3a^2-b^2+c^2\right)\left(a^2-b^2+c^2\right)\mu - \left(a^2-b^2+c^2\right)\left(a^2+b^2-c^2\right)\nu = 0,$$

$$\left(-a^2+b^2+c^2\right)\left(a^2+b^2-c^2\right)\lambda + 2\left(c^2-a^2\right)\left(a^2-b^2+c^2\right)\mu - \left(a^2+b^2-c^2\right)\left(-a^2+b^2+c^2\right)\nu = 0.$$

Add the condition $\lambda + \mu + \nu = 1$, thus obtaining a linear system of three equations with three unknowns:

$$\lambda_H = \frac{\left(a^2-b^2+c^2\right)\left(a^2+b^2-c^2\right)}{16S^2},$$

$$\mu_H = \frac{(a^2+b^2-c^2)(-a^2+b^2+c^2)}{16S^2},$$
$$\nu_H = \frac{(-a^2+b^2+c^2)(a^2-b^2+c^2)}{16S^2}.$$

The result coincides with the coordinate representation of the orthocenter H. Finally, note that the unique solution of the system, describing the relation $O_A \equiv O_C$, implies that H is unique such that $k(P \equiv H)$ is a circle. This ends the proof.

Further details are included in the book: Grozdev, S., V. Nenkov (2012-[1]). *Around the orthocenter in the plane and the space*. Sofia: Archimedes. ISBN 978-954-779-145-9.

I.2. A problem for cevians and a tangent. Consider the following problem from the 49-th IMO'2010 paper:

Problem 2. *Let P be an interior point of ΔABC. The lines AP, BP and CP meet the circumcircle Γ of ΔABC at K, L and M, respectively. The tangent line to Γ at C meets the sideline AB at S. If $SC = SP$, prove that $MK = ML$.*

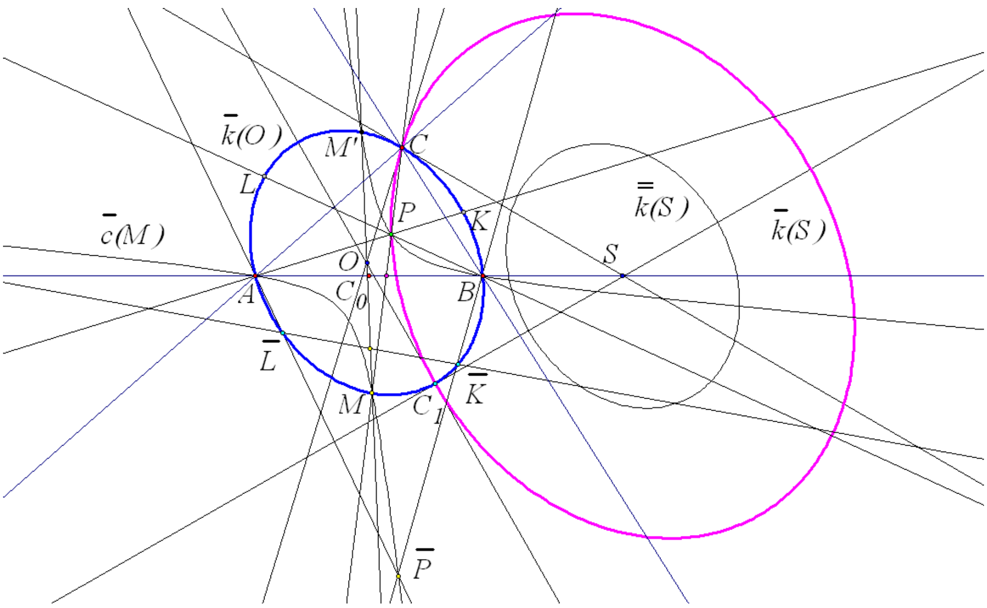

Fig. 5

The investigations are carried out by GSP again. At the beginning it is clear, that the circumcircle Γ could be replaced by a second-order central curve, that is circumscribed to ΔABC. Two cases are possible with respect to the center of such a curve. We will consider only the case when the curves are ellipses or hyperbolas. Denote the center of such a curve by O and the curve itself by $\bar{k}(O)$. In the particular case of Problem 2 the curve $\bar{k}(O)$ is the circle Γ in fact.

Further, look at Problem 2 from the following point of view. The tangent line at C and the point S are fixed and depend on ΔABC only. Thus, the point P is on the circle with center S and radius SC. It follows that a second-order curve $\bar{k}(S)$ through C with center S could be considered instead. But such a curve is not defined in a unique way. For this reason we are forced to examine the condition $MK = ML$, namely to investigate the hidden information in it. What is clear is that the triangle KLM is isosceles and it is inscribed in Γ. Thus, the line defined by M and the center O of Γ passes through the midpoint of the segment KL. When this property is transferred to $\bar{k}(O)$, we get that the line KL is from the conjugate family of the diameter OM of $\bar{k}(O)$ for an arbitrary $M \in \bar{k}(O)$. At the same time the line KL is such that the lines AK, BL and CM are concurrent at P. For this reason we produce experiments by GSP to find the locus of the intersection point of AK and BL, when the line KL runs the set of the conjugate lines to OM. We obtain the following:

Property 1. *Let M be an arbitrary point on $\bar{k}(O)$, $\bar{l}$ be a line from the conjugate family of OM and $\bar{l} \cap \bar{k}(O) = \{\bar{K}, \bar{L}\}$. If $A\bar{K} \cap B\bar{L} = \bar{P}$ and $\bar{l}$ runs the set of the conjugate lines to OM, then $\bar{P}$ describes a second-order curve $\bar{c}(M)$.* (Fig. 5, Fig. 6).

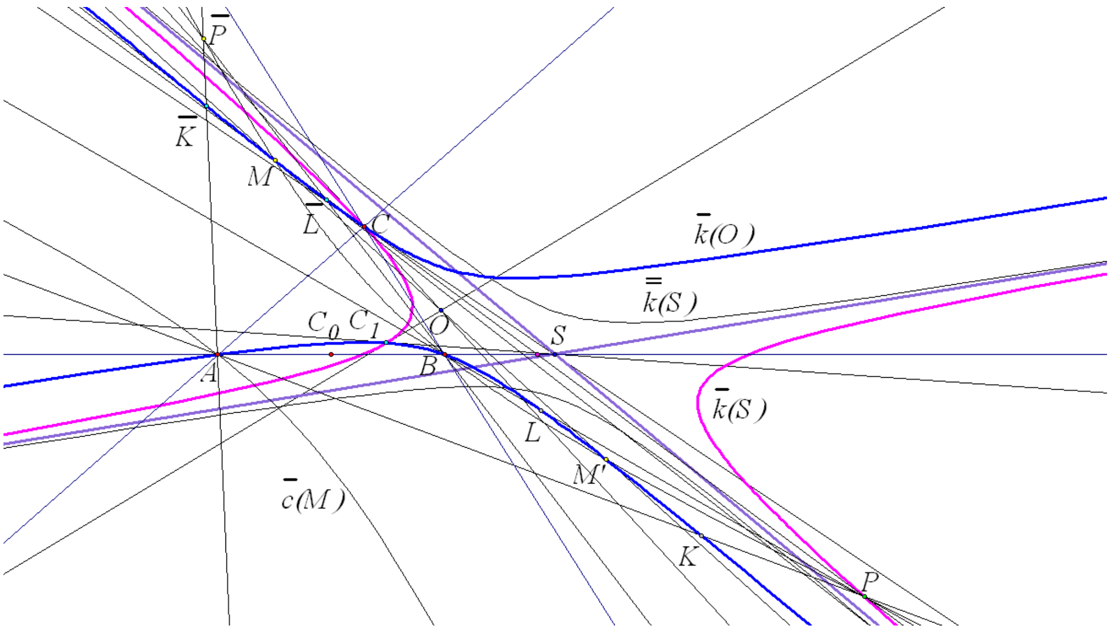

Fig. 6

Observations on $\bar{c}(M)$ by GSP induce the following:

Property 2. *The curve* $\bar{c}(M)$ *is with center* C_0 *and passes through* A, B, M *and* M', *where* M' *is symmetric to* M *with respect to* O (Fig. 5, Fig. 6).

Property 2 allows an easier construction of $\bar{c}(M)$ by GSP.

Property 3. *For an arbitrary point* M *on* $\bar{k}(O)$ *the point* P *is defined as the second intersection point of the line* CM *and the curve* $\bar{c}(M)$ (Fig. 5, Fig. 6).

Property 4. *When* M *runs* $\bar{k}(O)$, *the point* P *describes a second-order curve* $\bar{k}(S)$, *which passes through the vertex* C.

Property 5. *If the tangent to* $\bar{k}(O)$ *at the point* C *meets the line* AB *at* S, *then* S *is the center of* $\bar{k}(S)$ (Fig. 5, Fig. 6).

The results are combined in the following:

Theorem 4. *Let the conic* $\bar{k}(O)$ *with center* O *be circumscribed to* ΔABC *and* M *be an arbitrary point on* $\bar{k}(O)$. *If* K *and* L *are such points on* $\bar{k}(O)$, *that the line* KL *is from the conjugate family of the diameter* OM *and the lines* AK, BL *and* CM *are concurrent at* P, *then* P *describes a conic* $\bar{k}(S)$, *when* M *runs* $\bar{k}(O)$ *and the center* S *of this conic is the intersection point of the tangent to* $\bar{k}(O)$ *at* C *and the sideline* AB (Fig. 5, Fig. 6).

Here are some of the accompanying properties:

Property 6. *The central line* OS *of the curves* $\bar{k}(O)$ *and* $\bar{k}(S)$ *passes through the midpoint of their common chord* CC_1.

Property 7. *Simultaneously, the curves* $\bar{k}(O)$ *and* $\bar{k}(S)$ *are ellipses or hyperbolas* (Fig. 5, Fig. 6).

Property 8. *If the curves* $\bar{k}(O)$ *and* $\bar{k}(S)$ *are ellipses, then they are homothetic* (Fig. 5).

Property 9. *If the curves* $\bar{k}(O)$ *and* $\bar{k}(S)$ *are hyperbolas, then they have parallel asymptotes* (Fig. 6).

Property 10. *If the curves* $\bar{k}(O)$ *and* $\bar{k}(S)$ *are hyperbolas, then the translation, defined by* $\overrightarrow{OS}$, *transforms the hyperbola* $\bar{k}(O)$ *to a hyperbola* $\bar{\bar{k}}(S)$, *which is positioned in the angle area between the asymptotes of* $\bar{k}(S)$ *not containing* $\bar{k}(S)$ (Fig. 6).

When the curve $\bar{k}(O)$ is a parabola, the considerations are similar.

PROOFS. What follows are legalizations of the formulated assertions in this paragraph by mathematical proofs. Again, we limit ourselves to Theorem 4 only, since the applied technique is the same. Details could be found in the paper (Grozdev & Nenkov, 2012-[2]).

Property 5 grounds the use of the points S and C_1 in the geometric determination of $\bar{k}(O)$ and $\bar{k}(S)$, because the second tangent to $\bar{k}(O)$ through S passes through the other intersection point C_1 of $\bar{k}(O)$ and $\bar{k}(S)$. Also, the tangents to $\bar{k}(S)$ through the center O of $\bar{k}(O)$ pass through the common points C and C_1 of the two curves. Thus, the tangents through the center of the one curve to the other one pass through their common points and vice versa.

Take an arbitrary point $S(s,1-s,0)$ on the sideline AB and define $\bar{k}(O)$ as a curve passing through A, B, C, an arbitrary $C_1(\lambda,\mu,\nu)$ and being tangent to the line CS. The equation of the curve is the following:

(1) $$\bar{k}(O):\ s\lambda\mu\, yz-(1-s)\lambda\mu\, zx-[s\mu\nu-(1-s)\nu\lambda]xy=0\,.$$

Determine the coordinates of C_1 by (1) and the equation of SC_1 in such way that the line SC_1 is tangent to $\bar{k}(O)$:

(2) $$C_1\left(\lambda,-\frac{(1-s)\lambda}{s},\frac{(1-2s)\lambda+s}{s}\right).$$

From (1) and (2) we obtain:

(3) $$\bar{k}(O):\ s\lambda\, yz-(1-s)\lambda\, zx-2[(1-2s)\lambda+s]xy=0\,.$$

If $M(x_M,y_M,z_M)$ is an arbitrary point on $\bar{k}(O)$, while $M_0(m,1-m,0)$ is the intersection point of CM and $\bar{k}(O)$, then from (3) we have, that:

(4) $$x_M=\frac{\lambda m(s-m)}{\theta(m)},\ y_M=\frac{\lambda(1-m)(s-m)}{\theta(m)},\ z_M=\frac{2m(1-m)((1-2s)\lambda+s)}{\theta(m)},$$

where $\theta(m)=-2((1-2s)\lambda+s)m^2+((1-4s)\lambda+2s)m+s\lambda$.

We need the coordinates of the center $O(x_0,y_0,z_0)$ and they could be determined by the equations of the diameters d_a and d_b, which are conjugate to the sidelines BC and CA, respectively. For the purpose, find the intersection point A' of the line through A, which is parallel to BC. Now, the line d_a is defined by the midpoint of AA' and the point A_0. The line d_b is defined analogously. The parametric representations of the lines under consideration are the following:

(5) $$\begin{aligned} d_a:&\quad x=2s\lambda t_1,\quad y=\frac{1}{2}-((1-2s)\lambda+2s)t_1,\quad z=\frac{1}{2}+((1-4s)\lambda+2s)t_1,\\ d_b:&\quad x=\frac{1}{2}-((1-2s)\lambda+2s)t_2,\quad y=2(1-s)\lambda t_2,\quad z=\frac{1}{2}+((3-4s)\lambda+2s)t_2.\end{aligned}$$

Solving the corresponding system, we get:

$$(6)\quad x_0=\frac{s\lambda((3-4s)\lambda+2s)}{\tau(s)},\ y=\frac{(s-1)\lambda((1-4s)\lambda+2s)}{\tau(s)},\ z=\frac{2((1-2s)\lambda+s)((1-2s)\lambda+2s)}{\tau(s)},$$

where $\tau(s)=\lambda^2+4s(1-2s)\lambda+4s^2$.

We continue with the determination of the equation of $\bar{k}(S)$. Let $C'(2s,2(1-s),-1)$ be the symmetric point to C with regard to S. Consider the coordinate system defined by $\Delta C_1C'C$ with $C_1(1,0,0)$, $C'(0,1,0)$, $C(0,0,1)$. If a point has coordinates (x',y',z') with respect to $\Delta C_1C'C$ and coordinates (x,y,z) with respect to ΔABC, then:

$$(7)\qquad x=\lambda x'+2sy',\ y=-\frac{1-s}{s}\lambda x'+2(1-s)y',\ z=\frac{(1-2s)\lambda+s}{s}x'-y'+z'.$$

From (7) we obtain:

$$(8)\qquad \begin{aligned} &x'=\frac{1}{2\lambda}x-\frac{s}{2(1-s)\lambda}y,\quad y'=\frac{1}{4s}x+\frac{1}{4(1-s)}y,\\ &z'=-\frac{(1-4s)\lambda+2s}{4s\lambda}x+\frac{(3-4s)\lambda+2s}{4(1-s)\lambda}y+z.\end{aligned}$$

From (6) and (8) we determine the coordinates of O with respect to $\Delta C_1C'C$:

$$(9)\qquad x_0'=\frac{2s((1-2s)\lambda+s)}{\tau(s)},\ y_0'=\frac{\lambda^2}{\tau(s)},\ z_0'=\frac{1}{2}.$$

The third equation in (9) implies that O lies on the line through the midpoints of the segments CC' and CC_1. Property 6 follows directly from here.

Determine the curve $\bar{k}(S)$ passing through the points C_1, C', C and tangent to the lines OC_1 and OC:

$$(10)\qquad x_0'y'z'-y_0'z'x'+z_0'x'y'=0.$$

Replacing (9) in (10), we get the equation of $\bar{k}(S)$ with respect to ΔABC:

$$(11)\qquad \begin{aligned}\bar{k}(S):\ &2(1-s)((1-2s)\lambda+s)x^2+2s((1-2s)\lambda+s)y^2+\\ &+s((3-4s)\lambda+2s)yz+(1-s)((1-4s)\lambda+2s)zx=0.\end{aligned}$$

If P is an arbitrary point on $\bar{k}(S)$ and the line CP intersects AB in $P_0(p,1-p,0)$, then the equations of CP and (11) for the coordinates of P give:

$$(12)\qquad \begin{aligned} x_P&=\frac{((8s^2-8s+1)\lambda+2s(1-2s))p+s((3-4s)\lambda+2s)}{\theta(p)}p,\\ y_P&=\frac{((8s^2-8s+1)\lambda+2s(1-2s))p+s((3-4s)\lambda+2s)}{\theta(p)}(1-p),\\ z_P&=\frac{2((1-2s)\lambda+s)(-p^2+2sp-s)}{\theta(p)},\end{aligned}$$

where $\theta(p) = -2((1-2s)\lambda + s)p^2 + ((1-4s)\lambda + 2s)p + s\lambda$.

To determine the type of he curves $\bar{k}(O)$ and $\bar{k}(S)$, we need conditions for the number of their common points with the infinity line, whose equation is $x + y + z = 0$. From equations (3) and (11) we find that such a condition is the sign of the expression $\tau(s) = \lambda^2 + 4s(1-2s)\lambda + 4s^2$. It follows that the curves are ellipses or hyperbolas simultaneously, which proves the next property.

Now, we go back to Properties 1 – 3.

If M is an infinite point (this is possible only when $\bar{k}(O)$ is hyperbola and such a case is realized when m is one of the zeros of the equation $\vartheta(m) = 0$), then the line $\bar{l}$ from the conjugate family of OM (which is asymptote to $\bar{k}(O)$) is parallel to OM. Then $\bar{l} \cap \bar{k}(O) = \{\overline{K}, \overline{L} \equiv M\} = \{\overline{K} \equiv M, \overline{L}\}$. Since BM and AM are fixed lines, then the relations $A\overline{K} \cap BM = \overline{P}$ and $AM \cap B\overline{L} = \overline{P}$ mean, that the point $\overline{P}$ describes the parallel lines AM and BM, which identify a second-order curve $\bar{c}(M)$. Thus, Property 1 is proved. Additionally, C_0 is the center of $\bar{c}(M)$ and assuming that $M' \equiv M$, i. e. that M is symmetric to itself with respect to O, we could consider Property 2 as proved too. In such a case the line CM intersects $\bar{c}(M)$ twice at its infinity point M. This leads to Property 3. Thus, we establish, that M is a point on the hyperbola $\bar{k}(S)$.

Consider further the case, when M is a finite point ($\vartheta(m) \neq 0$). Take a coordinate system with regard to ΔABM, where $A(1,0,0)$, $B(0,1,0)$, $M(0,0,1)$. If the coordinates of a point are (x'', y'', z'') with respect to ΔABM and (x, y, z) with respect to ΔABC, then:

$$(13) \qquad x = x'' + x_M z'', \; y = y'' + y_M z'', \; z = z_M z''.$$

From (13) we obtain:

$$(14) \qquad x'' = x - \frac{x_M}{z_M} z, \; y'' = y - \frac{y_M}{z_M} z, \; z'' = \frac{1}{z_M} z.$$

The coordinates of O with respect to ΔABM are determined from (6) and (14):

$$(15) \qquad \begin{aligned} x_0'' &= \frac{(1-s)\lambda}{(1-m)\tau(s)}(((1-4s)\lambda + 2s)m + 2s\lambda), \\ y_0'' &= \frac{s\lambda}{m\tau(s)}(((3-4s)\lambda + 2s)m - ((1-2s)\lambda + 2s)), \\ z_0'' &= \frac{((1-2s)\lambda + 2s)\theta(m)}{m(1-m)\tau(s)}. \end{aligned}$$

Define a curve $\bar{c}(M)$ passing through the points $A(1,0,0)$, $B(0,1,0)$, $M(0,0,1)$, $M'(2x_0'', 2y_0'', 2z_0'' - 1)$ and $M''(1,1,-1)$. We have:

$$(16) \qquad \bar{c}(M):\ (1-2x_0'')x_0'' y'' z'' - (1-2y_0'')y_0'' z'' x'' + (y_0'' - x_0'')(1-2z_0'')x'' y'' = 0.$$

Check by (16), that the center of $\bar{c}(M)$ is the point C_0. It follows that $\bar{c}(M)$ satisfies Property 2.

Also, from (15) and (16) we obtain the equation:

$$(17)\qquad \bar{c}(M): (1-s)m^2 y''z'' + s(1-m)^2 z''x'' + \left(m^2 - 2sm + s\right)x''y'' = 0.$$

Take an arbitrary point $\overline{K}(x_1'', y_1'', z_1'')$ on the curve $\bar{k}(O)$, whose equation with respect to ΔABM is the following:

$$(18)\qquad \bar{k}(O): (1-2x_0'')x_0''y''z'' + (1-2y_0'')y_0''z''x'' + (1-2z_0'')z_0''x''y'' = 0.$$

The line $\bar{l}$, determined by the equations

$$(19)\qquad \bar{l}: x'' = x_1'' - x_0''(2x_0'' - 1)l,\ y'' = y_1'' + y_0''(2y_0'' - 1)l,\ z'' = z_1'' + (y_0'' - x_0'')(2z_0'' - 1)l,$$

is conjugate to the diameter OM and passes through $\overline{K}$.

The coordinates of the second intersection point $\overline{L}(x_2'', y_2'', z_2'')$ of $\bar{l}$ and $\bar{k}(O)$ are determined by (18) and (19):

$$(20)\qquad \begin{aligned} x_2'' &= \frac{(x_0'' - y_0'')y_0''x_1'' + (1-2x_0'')x_0''y_1''}{y_0''z_0''},\\ y_2'' &= \frac{(1-2y_0'')y_0''x_1'' + (y_0'' - x_0'')x_0''y_1''}{z_0''x_0''},\\ z_2'' &= \frac{(y_0'' - x_0'')(1-2z_0'')}{x_0''y_0''z_0''}(y_0''x_1'' - x_0''y_1'') + z_1''. \end{aligned}$$

If $A\overline{K} \cap B\overline{L} = \overline{P}(x_3'', y_3'', z_3'')$, then:

$$(21)\qquad x_3'' = \frac{x_2''z_1''}{(y_1'' + z_1'')z_2'' + x_2''z_1''},\ y_3'' = \frac{y_1''z_2''}{(y_1'' + z_1'')z_2'' + x_2''z_1''},\ z_3'' = \frac{z_1''z_2''}{(y_1'' + z_1'')z_2'' + x_2''z_1''}.$$

Substitute (21) to the left hand side of (16) and apply (20) and (18). In such a way (16) is verified, which means that $\overline{P}$ lies on $\bar{c}(M)$. Consequently, the curve $\bar{c}(M)$ satisfies Property 1 and we get the curve $\bar{c}(M)$, which has been forecasted by the GSP experiments.

The next step is to find the second intersection point P of CM and $\bar{c}(M)$, using (17). We have:

$$(22)\qquad x_P'' = \frac{(1-2s)m + s}{1-m},\ y_P'' = \frac{(1-2s)m + s}{m},\ z_P'' = -\frac{m^2 - 2sm + s}{m(1-m)}.$$

When $m = p$, we obtain (12) from (13) and (22). This proves Property 3 leading to the curve $\bar{k}(S)$ by Property 4.

Remark. Note that the point M could be determined by the infinite point $M_0(1,-1,0)$. In this concrete case, (4), (15), (17) and (22) have special representations, which could be obtained by exporting the highest power of m and putting $m \to \infty$ in the remaining expression (the same refers to $M_0(1,-1,0)$, which is obtained from $\overline{M}_0\left(1,\frac{1}{m}-1,0\right)$ when $m\to\infty$). Proceed in a similar way for the point P in (12). All final results remain under the new (4), (15), (17), (22) and (12).

According to the Remark, the construction by GSP, which leads to Theorem 4, is grounded and this ends the proof of the theorem.

Further details are included in the paper: Grozdev, S., V. Nenkov (2012-[2]). Loci generated by conjugate lines and cevians. *Mathematics and Informatics*, 6, pp. 562 – 577. ISSN 1310-2230.

I. 3. A problem for inscribed and circumscribed circles. The following problem from the 51-st IMO'2012 paper is considered:

Problem 3. *Let I be the incenter of ΔABC and Γ be its circumcircle. The line AI meets Γ for a second time at D. The point E on the arc $\overset{\frown}{BDC}$ and the point F on the side BC are such that $\sphericalangle BAF=\sphericalangle CAE<\frac{1}{2}\sphericalangle BAC$. If G is the midpoint of the segment IF, prove that the intersection point of the lines DG and EI lies on Γ.*

Problem 3 includes two special curves for ΔABC. One of the possibilities for a generalization of this problem is to replace the inscribed circle and the circumscribed one by suitable central conics (ellipses and hyperbolas). To keep an analogy, the new curves should be connected in a similar way as the inscribed circle and the circumscribed one are with respect to ΔABC. The main difficulty is to discover a class of pairs of inscribed and circumscribed curves, which have similar properties as in Problem 3.

Consider barycentric coordinates with regard to ΔABC with $A(1,0,0)$, $B(0,1,0)$ and $C(0,0,1)$. Denote by $A_0\left(0,\frac{1}{2},\frac{1}{2}\right)$, $B_0\left(\frac{1}{2},0,\frac{1}{2}\right)$ and $C_0\left(\frac{1}{2},\frac{1}{2},0\right)$ the midpoints of the sides BC, CA and AB, respectively. An arbitrary point $I(x_I,y_I,z_I)$ $(x_I+y_I+z_I=1)$, which does not lie on the sidelines BC, CA, AB, B_0C_0, C_0A_0 and A_0B_0, is a center of an in-conic $k(I)$ for ΔABC. The points $I_A\left(-\frac{x_I}{1-2x_I},\frac{y_I}{1-2x_I},\frac{z_I}{1-2x_I}\right)$, $I_B\left(\frac{x_I}{1-2y_I},-\frac{y_I}{1-2y_I},\frac{z_I}{1-2y_I}\right)$ and $I_C\left(\frac{x_I}{1-2z_I},\frac{y_I}{1-2z_I},-\frac{z_I}{1-2z_I}\right)$ define $\Delta I_AI_BI_C$, which is known to be conjugate to I with respect to ΔABC. The points I_A, I_B and I_C are centers of exscribed conics $k(I_A)$, $k(I_B)$ and $k(I_C)$ of ΔABC. The midpoints of the segments II_A, II_B, II_C, I_BI_C, I_CI_A and I_AI_B lie on a conic $\overline{k}(O)$ with center O. Each of the points I and O defines the other one uniquely and for this reason we name the curves $\bar{k}(O)$ and $k(I)$ to be associated to ΔABC. The associated curves have similar properties to these of the inscribed circle and the circumscribed one of ΔABC.

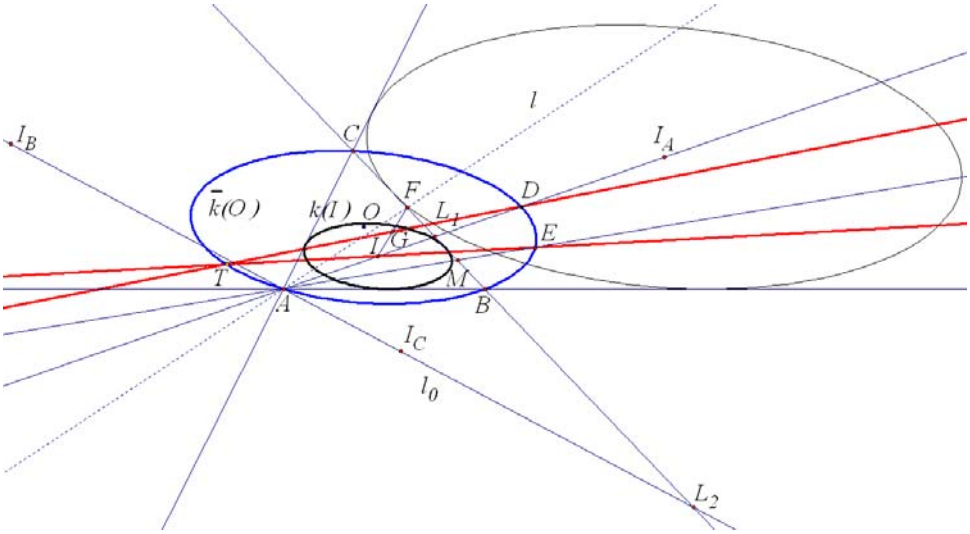

Fig. 7

The equation $\sphericalangle BAF = \sphericalangle CAE$ from Problem 3 could be considered as a symmetry between the lines AE and AF with respect to AI. Now, let E be a point on $\bar{k}(O)$ and $F \in CA$ be such that the lines AE and AF are symmetric with respect to AI (I is the center of the associated curve $k(I)$). However, experiments by GSP show that the lines DG and EI do not intersect on $\bar{k}(O)$. For this reason another observation is needed: the equation $\sphericalangle BAF = \sphericalangle CAE$ means also that the lines AE and AF are harmonic conjugate with respect to the lines AI and $I_B I_C$. Now, the experiments by GSP lead to the hypothesis that the intersection point of the lines DG and EI lies on $\bar{k}(O)$. We have:

Theoreme 5. *Let $\bar{k}(O)$ and $k(I)$ be associated circumcurve and incurve of ΔABC, respectively. Let the line AI meet $\bar{k}(O)$ for a second time at the point D, the line l_0 be harmonic conjugate to AI with respect to AB and AC, while E be an arbitrary point on $\bar{k}(O)$. If the line l, which is harmonic conjugate to AE with respect to AI and l_0, meets the sideline BC at F, while G is the midpoint of IF, then the intersection point of the lines DG and EI lies on $\bar{k}(O)$* (Fig. 7).

Proof: Let $AI \cap BC = L_1\left(0, \frac{y_I}{1-x_I}, \frac{z_I}{1-x_I}\right)$, $l_0 \equiv I_B I_C \cap BC = L_2\left(0, \frac{y_I}{y_I - z_I}, \frac{z_I}{z_I - y_I}\right)$, $AE \cap BC = M(0, m, 1-m)$ and $l \cap BC = F(0, f, 1-f)$. It follows from the harmonic property, that

$s=\frac{\overline{L_1F}}{\overline{L_2F}}=-\frac{\overline{L_1M}}{\overline{L_2M}}$, hence $\overrightarrow{OF}=\frac{\overrightarrow{OL_1}-s.\overrightarrow{OL_2}}{1-s}$ and $\overrightarrow{OM}=\frac{\overrightarrow{OL_1}+s.\overrightarrow{OL_2}}{1+s}$. The first equation gives $s=\frac{y_I-z_I}{y_I+z_I}.\frac{(y_I+z_I)f-y_I}{(y_I-z_I)f-y_I}$, which together with the first defines:

(1) $$m=\frac{y_I^2(f-1)}{(y_I^2-z_I^2)f-y_I^2}.$$

Since the $D\left(-\frac{x_I^2}{1-2x_I},\frac{(1-x_I)y_I}{1-2x_I},\frac{(1-x_I)z_I}{1-2x_I}\right)$ is the midpoint of the segment II_A, then DG is the middle segment in ΔI_AFA. Thus, the line DG is determined by the point D and a vector, which is collinear with FI_A. We have:

(2) $$DG:\ \begin{aligned} x&=-\frac{x_I^2}{1-2x_I}-x_I\,t,\\ y&=\frac{(1-x_I)y_I}{1-2x_I}+[y_I-(1-2x_I)f]t,\\ z&=\frac{(1-x_I)z_I}{1-2x_I}+[x_I-y_I+(1-2x_I)f]t.\end{aligned}$$

Additionally, the curve $\bar{k}(O)$ is represented by the equation:

(3) $$\bar{k}(O):\ x_I^2yz+y_I^2zx+z_I^2xy=0.$$

From (2) and (3) we deduce the coordinates of the second intersection point T of DG and $\bar{k}(O)$:

(4) $$\begin{aligned} x_T&=\left(-x_I^2f^2+x_I(1-2y_I)f+y_I(1-y_I)\right).\frac{x_I}{\tau_1},\\ y_T&=\left(x_I(1-x_I)f^2+y_I(1-2x_I)f-y_I^2\right).\frac{y_I}{\tau_1},\\ z_T&=\left(x_I(1-x_I)f^2-(z_I+2x_Iy_I)f+y_I(1-y_I)\right).\frac{z_I}{\tau_1},\end{aligned}$$

where $\tau_1=x_I(1-2x_I)f^2-(1-2x_I)(1-2y_I)f+y_I(1-2y_I)$.

The coordinates of the second intersection point E of the line AM and the curve $\bar{k}(O)$ are:

(5) $$x_E=-\frac{f(f-1)x_I^2}{\tau_2},\ y_E=\frac{(f-1)y_I^2}{\tau_2},\ z_E=-\frac{fz_I^2}{\tau_2},$$

where $\tau_2=-x_I^2f^2+(x_I^2+y_I^2-z_I^2)f-y_I^2$.

Note that the collinearity of the points $M_1(x_1,y_1,z_1)$, $M_2(x_2,y_2,z_2)$ and $M_3(x_3,y_3,z_3)$ is equivalent to the following condition:

$$(6)\qquad \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0.$$

Now, using (4) and (5), check that the coordinates of I, T and E satisfy (6). Hence, the point T lies on the line IE.

Consideration of three more cases is needed for a complete proof:

1) $AE \parallel BC$; 2) DG is parallel to the asymptote of $\bar{k}(O)$, i. e. T is the infinite point for $\bar{k}(O)$; 3) IE is parallel to the asymptote of $\bar{k}(O)$, i. e. E is the infinite point for $\bar{k}(O)$.

1) Let $AE \parallel BC$. Then, F is the midpoint of L_1L_2 and $f = \dfrac{y_I^2}{y_I^2 - z_I^2}$, while the coordinates of the point E are: $x_E = 1$, $y_E = \dfrac{z_I^2 - y_I^2}{x_I^2}$, $z_E = \dfrac{y_I^2 - z_I^2}{x_I^2}$. The coordinates could be deduced from (5), when $\tau_2 = -\left(\dfrac{x_I y_I z_I}{y_I^2 - z_I^2}\right)^2$. The equations (5) describe all finite points of $\bar{k}(O)$.

Cases 2) and 3) are meaningful when $\bar{k}(O)$ is a hyperbola only. If $\bar{k}(O)$ is a hyperbola, these two cases correspond to the conditions $\tau_1 = 0$ and $\tau_2 = 0$, respectively. Note that the double equality $\tau_1 = \tau_2 = 0$ is impossible. Additionally, we have:

$$(7)\qquad \tau_2 = \tau_1 - x_I(1 - x_I)f^2 + 2x_I y_I f - y_I(1 - y_I).$$

2) Let $\tau_1 = 0$. From here and (2) it follows, that the vector

$$\vec{u}\left(1 - 2y_I, (1 - 2x_I)f^2, (2x - 1)f^2 + 2y_I - 1\right)$$

is collinear with the line DG. Also, by the coordinates of E and (7) we get, that the line IE is collinear with $\vec{u}$.

3) Let $\tau_2 = 0$. From here and the coordinates of T, using (7) we obtain, that vector $\vec{v}\left(x_I^2 f(1 - f), y_I^2 (f - 1), -z_I^2 f\right)$ is collinear with the line IT. Substitute and check, that the coordinates of $\vec{v}$ satisfy (3). Consequently, this vector is collinear with the asymptote of $\bar{k}(O)$.

This ends the proof.

Further details are included in the paper: Grozdev, S., V. Nenkov (2011). A property of central conics associated with a triangle. *Mathematics and Mathematics Education*. 40, pp. 394-399. ISSN 1313-3330.

II. Generalizations of problem solutions. Typical examples of this approach are the geometric problems from the 52-nd IMO'2013 paper.

II.1. A problem for a circle with a special position of its center.

Problem 4. *The excircles* $\Gamma_a(I_a)$, $\Gamma_b(I_b)$ *and* $\Gamma_c(I_c)$ *of* ΔABC *are tangent to the sides* BC, CA *and* AB *at the points* A_1, B_1 *and* C_1, respectively. *Prove that if the circumcenter of* $\Delta A_1B_1C_1$ *lies on the circumcenter of* ΔABC, *then* ΔABC *is a right-angled.*

Denote by k and $\Gamma(O)$ the circumcircles of $\Delta A_1B_1C_1$ and ΔABC, respectively. Let ABC be arbitrary triangle and S_c be the second intersection point of the line I_aI_b and $\Gamma(O)$ (Fig. 8, Fig. 9). As usually, the lengths of the sides BC, CA and AB are denoted by a, b and c, respectively, while $p=\dfrac{a+b+c}{2}$ is the semi-perimeter of ΔABC. Additionally, let $\Gamma_a(I_a)$, $\Gamma_b(I_b)$ and $\Gamma(I)$ be tangent to the lines CA, BC and AB at the points B', A' and C', respectively (Fig. 8, Fig. 9).

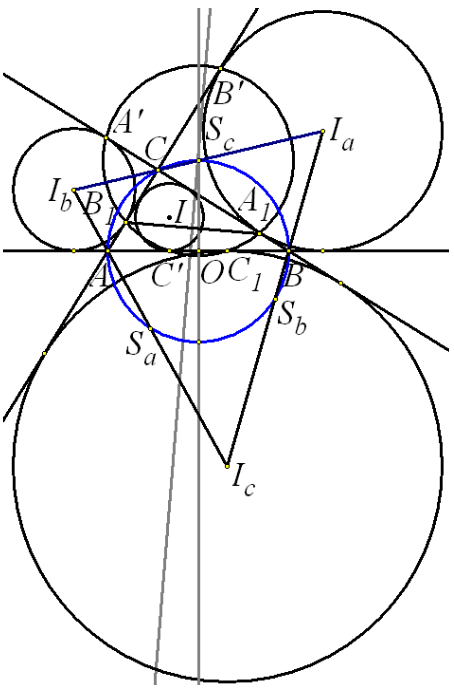

Fig. 8

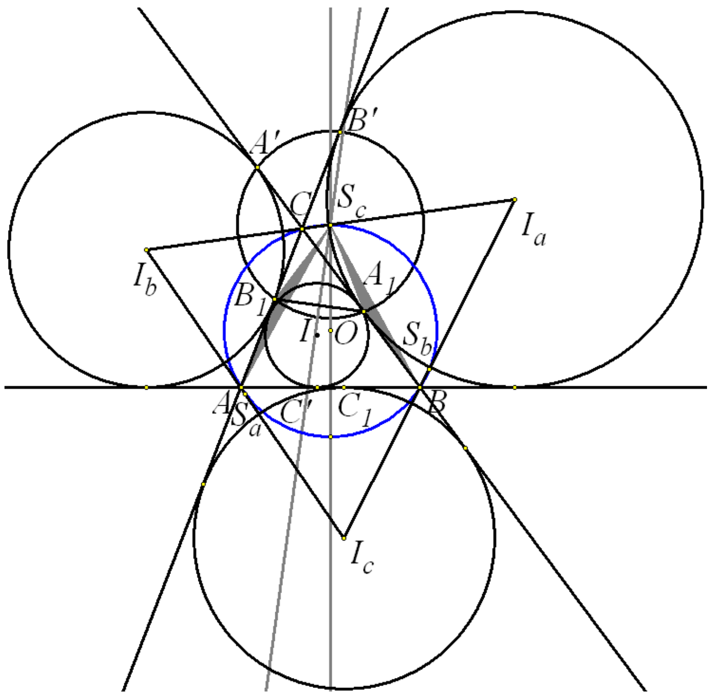

Fig. 9

An elegant solution of Problem 4 is based on 3 fundamental properties of an arbitrary triangle.

1) The point S_c *lies on the perpendicular bisector of the segment* A_1B_1.

Since the perpendicular bisector of AB passes through S_c, then $AS_c=BS_c$. For the tangents AB_1 and BA_1 we have $AB_1=BA_1=p-c$. Additionally, $\measuredangle CAS_c=\measuredangle CBS_c=\dfrac{\overset{\frown}{CS_c}}{2}$ (Fig. 8, Fig. 9). Consequently $\Delta AB_1S_c\cong\Delta BA_1S_c$. It follows that $B_1S_c=A_1S_c$.

2) The points A_1, B_1, A' *and* B' *are co-cyclic. They lie on the circle with center* S_c.

This is so, because B' and A' are symmetric to A_1 and B_1 with respect to the line I_aI_b, respectively. Also $B_1S_c = A_1S_c$ (Fig. 8, Fig. 9).

3) The points C_1 *and* C' *are equidistant from* S_c.

It follows from the equalities $AC' = BC_1 = p - a$ that the points C' and C_1 are symmetric with respect to the perpendicular bisector of the segment AB. Consequently $S_cC' = S_cC_1$ (Fig. 8, Fig. 9).

The conclusion is that S_c is a candidate for a center of k (property *1)*). If this is the case, then the points A', B' and C' lie on k (Fig. 9). Now, applying the secant property to k and A, we get $AB'.AB_1 = AC_1.AC'$ (Fig. 9). On the other hand $AB' = p$, $AB_1 = p - c$, $AC_1 = p - b$ and $AC' = p - a$, thus obtaining the relation $c^2 = a^2 + b^2$. According to the Pythagoras theorem, $\sphericalangle ACB = 90°$ and this ends the solution.

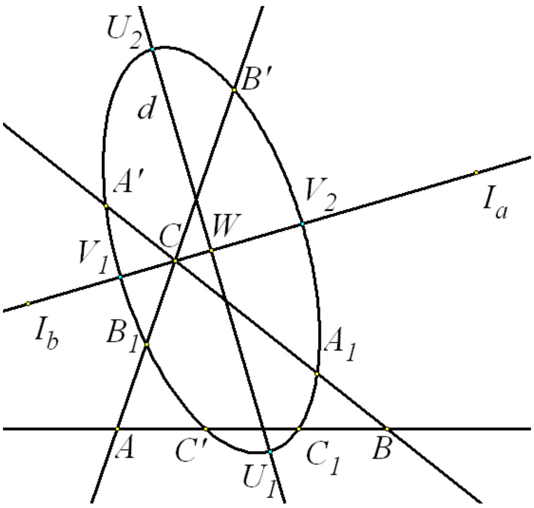

Fig. 10

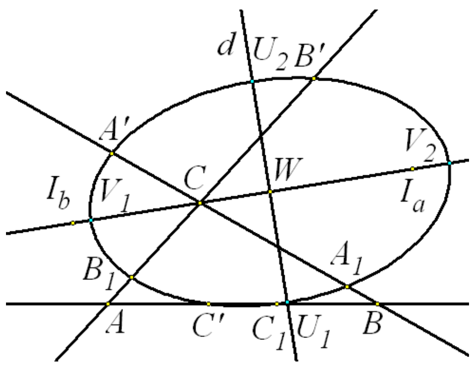

Fig. 11

It is easy to realize that if $\sphericalangle ACB = 90°$, then the points A_1, B_1, C_1, A', B' and C' are co-cyclic and the center of the circle on which they lie coincides with S_c. Consequently, the points A_1, B_1, C_1, A', B' and C' are co-cyclic iff $\sphericalangle ACB = 90°$. Now the question is how the points A_1, B_1, C_1, A', B' and C' are positioned in the general case. The expectations are that the 6 points lie on a second-order curve. Experiments by DSP confirm them. Theoretical grounding exists too: it is well-known that the lines AA_1, BB_1 and CC_1 are concurrent at a point N, the so called Nagel point for ΔABC. Additionally, since $AB' = BA' = p$, $AC' = CA' = p - a$ and

$BC' = CB' = p - b$, then $\frac{\overline{BA'}}{\overline{CA'}}.\frac{\overline{CB'}}{\overline{AB'}}.\frac{\overline{AC'}}{\overline{BC'}} = -1$. The Ceva theorem implies that the lines AA', BB' and CC' are concurrent at a point N'. According to a classical theorem if AA_1, BB_1, CC_1 and AA', BB', CC' are two triplets cevians of ΔABC, then A_1, B_1, C_1, A', B' and C' lie on a second-order curve. Thus, we have the following:

Theorem 6. *The points A_1, B_1, C_1, A', B' and C' lie on a second-order curve k for arbitrary ΔABC* (Fig. 10, Fig. 11).

The curve k has the following interesting properties:

Property 1. *The center W of the curve k lies on the line I_aI_b and if k is not degenerated, then I_aI_b and the perpendicular line d to it and through W are axes of k* (Fig. 10, Fig. 11).

Property 2. *The points A_1, B_1, C_1, A', B' and C' lie on two parallel lines iff $\sphericalangle ABC = 90°$ or $\sphericalangle BAC = 90°$.*

Property 3. *If one of the angles $\sphericalangle ABC$ or $\sphericalangle BAC$ is obtuse, then k is a hyperbola.*

Property 4. *If ΔABC is acute, then k is an ellipse* (Fig. 10).

Property 5. *There exist triangles for which $\sphericalangle ACB$ is obtuse and k is an ellipse. Also, there exist triangles for which $\sphericalangle ACB$ is obtuse and k is a hyperbola* (Fig. 10, Fig. 11).

Property 6. *If k is a hyperbola, then I_aI_b is its focal axis.*

Property 7. *If ΔABC is acute, then d is the focal axis of k, while if ΔABC is obtuse, then I_aI_b is the focal axis of k.*

Further details are included in the paper: Grozdev, S., V. Nenkov (2013 – [4]). A special kind of second-order curves, generated by the Nagel point, *Mathematics Plus*, 3, pp. 44 – 53, ISSN 0861 8321.

II.2. A problem for orthocenter and a collinearity. The second problem from the 52-nd IMO’2013 paper is the following:

Problem 5. *Let ABC be an acute-angled triangle with orthocenter H, and W be a point on the side BC, lying strictly between B and C. The points M and N are the feet of the altitudes from B and C, respectively. Denote by ω_1 the circumcircle of ΔBWN, and let X be the point on ω_1 such that WX is a diameter of ω_1. Analogously, denote by ω_2 the circumcircle* *of ΔCWM, and let Y be the point on ω_2 such that WX is a diameter of ω_2. Prove that X, Y and H are collinear.* (Fig. 12).

Firstly, note that if W, M and N are arbitrary points on the sides BC, CA and AB of ΔABC, respectively, while ω_1, ω_2 and ω_3 are the circumcircles of the triangles BWN, CWM and AMN, respectively, then these circles meet at a point T (Miquel theorem). Additionally, $\sphericalangle ATN = \sphericalangle AMN$.

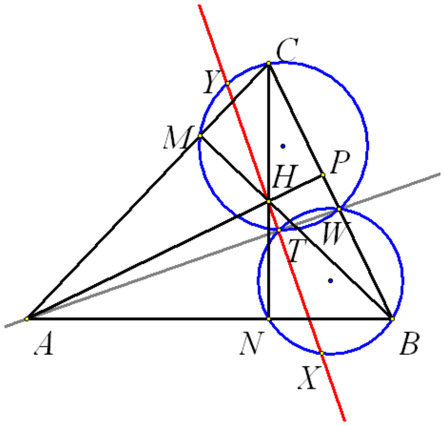

Fig. 12

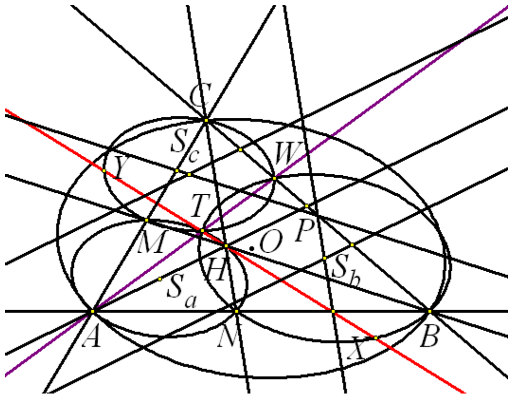

Fig. 13

Take now the points M and N as in Problem 5. Then, A, H, M and N are co-cyclic. Consequently, ω_3 passes through the orthocenter H, i. e. the points A, M, H, T and N are co-cyclic. On the other hand, the quadrilateral $BCMN$ is cyclic and it follows that $\sphericalangle CMN = 180° - \sphericalangle ABC$. We deduce that $\sphericalangle AMN = 180° - \sphericalangle CMN = 180° - (180° - \sphericalangle ABC) =$ $= \sphericalangle ABC$. Thus, $\sphericalangle ATN = \sphericalangle AMN = \sphericalangle ABC$. But $\sphericalangle NTW = 180° - \sphericalangle ABC$ and therefore $\sphericalangle ATN + \sphericalangle NTW = 180°$. This means that the points A, T and W are collinear. The secant property applied to ω_1 and A gives $AW.AT = AB.AN$. If $AH \cap BC = P$, then $BPHN$ is cyclic. Applying the secant property to the circumcircle of $BPHN$ and the point A, we get $AP.AH = AB.AN$. The last two equalities induce that $AW.AT = AP.AH$ and consequently the quadrilateral $HTWP$ is cyclic. Since $\sphericalangle HPW = 90°$, then $\sphericalangle HTW = 90°$. If HT meets ω_1 for a second time at X', then the segment WX' is seen from T under 90^0. Hence, WX' is diameter of ω_1. It follows that $X' \equiv X$. We deduce that the points H, T and X are collinear.

In a similar way we deduce that the points H, T and Y are collinear. It follows that the points H, T, X are Y collinear and this ends the solution.

Note that the point X is obtained as a symmetric one to W with regard to the center of ω_1, while the center of ω_1 is obtained as an intersection point of the perpendicular bisectors of the segments BW and BN, which on their turn are parallel to the altitudes AH and CH, respectively. To generalize, let H be an arbitrary point in the plane of ΔABC (Fig. 13). Let

$AH \cap BC = P$, $BH \cap CA = M$, $CH \cap AB = N$ and let W be an arbitrary point on the line BC. Draw lines through the midpoints of BW and BN, which are parallel to AH and CH, respectively. The intersection point of these lines is denoted by S_b, while the point, which is symmetric to W with respect to S_b, is denoted by X. An analogous construction leads to a point Y: draw lines through the midpoints of CW and CM, which are parallel to AH and BH, respectively; denote the intersection point of these lines by S_c, while the point, which is symmetric to W with respect to S_c, is denoted by Y. The construction leads to the following:

Theorem 7. *The points X, Y and H are collinear* (Fig. 13).

We will use barycentric coordinated with respect to ΔABC with $A(1,0,0)$, $B(0,1,0)$, $C(0,0,1)$, $W(0,w,1-w)$, $H(\lambda,\mu,\nu)$ $(\lambda+\mu+\nu=1)$. It follows immediately, that $P\left(0,\frac{\mu}{1-\lambda},\frac{\nu}{1-\lambda}\right)$, $M\left(\frac{\lambda}{1-\mu},0,\frac{\nu}{1-\mu}\right)$, $N\left(\frac{\lambda}{1-\nu},\frac{\mu}{1-\nu},0\right)$. The lines l_1 and l_2 through the midpoints of BW and BN, which are parallel to AH and CH, respectively, are represented by the following parametric equations:

$$l_1:\begin{cases} x=\dfrac{\lambda}{2(1-\nu)}+\lambda t_1, \\ y=\dfrac{1+\mu-\nu}{2(1-\nu)}+\mu t_1, \\ z=(\nu-1)t_1, \end{cases} \qquad l_2:\begin{cases} x=(\lambda-1)t_2, \\ y=\dfrac{1+w}{2}+\mu t_2, \\ z=\dfrac{1-w}{2}+\nu t_2. \end{cases}$$

From here we find the coordinates of $S_b=l_1\cap l_2$:

$$S_b\left(\frac{\lambda(1-\lambda)w}{2\mu},\frac{1+(1-\lambda)w}{2},\frac{\mu-(1-\nu)(1-\lambda)w}{2\mu}\right).$$

Analogously:

$$S_c\left(\frac{\lambda(1-\lambda)(1-w)}{2\nu},\frac{\nu-(1-\lambda)(1-\mu)(1-w)}{2\nu},\frac{1+(1-\lambda)(1-w)}{2}\right).$$

The coordinates of the points X and Y are as it follows:

$$X\left(\frac{\lambda(1-\lambda)w}{\mu},1-\lambda w,-\frac{\nu\lambda w}{\mu}\right),\ Y\left(\frac{\lambda(1-\lambda)(1-w)}{\nu},-\frac{\lambda\mu(1-w)}{\nu},1-\lambda(1-w)\right).$$

From here we obtain:

$$\overrightarrow{HX}\left(\frac{\lambda\left[-\mu+(1-\lambda)w\right]}{\mu},\frac{\mu(1-\mu-\lambda w)}{\mu},\frac{-\nu(\mu+\lambda w)}{\mu}\right),$$

$$\overrightarrow{HY}\left(-\frac{\lambda\left[-\mu+(1-\lambda)w\right]}{\nu},-\frac{\mu(1-\mu-\lambda w)}{\nu},-\frac{-\nu(\mu+\lambda w)}{\nu}\right).$$

Now, check the relation $\overrightarrow{HY}=-\frac{\mu}{\nu}\overrightarrow{HX}$, which proves Theorem 7.

The proof of this theorem leads also to:

Corollary. *The simple ratio* $\overline{XH}:\overline{YH}$*, in which the point* H *divides the segment* XY*, does not depend on the place of the point* W *on the sideline* BC.

For a generalization of the circles ω_1 and ω_2 from the initial problem, consider conics $\bar{k}(S_b)$ and $\bar{k}(S_c)$ with centers S_b and S_c, which are circumscribed to the triangles BWN and CWM, respectively. Differently from Problem 5, the curves $\bar{k}(S_b)$ and $\bar{k}(S_c)$ are not necessary in the proof of Theorem 7. At the same time these curves have the properties of the circles ω_1 and ω_2, which are mentioned in the problem solution. Add the circumconic $\bar{k}(S_a)$ of ΔAMN, whose center is the midpoint S_a of AH. The last conic generalizes the circle ω_3. We have the following:

Theorem 8. *The curves* $\bar{k}(S_a)$, $\bar{k}(S_b)$ *and* $\bar{k}(S_c)$ *are of one and the same type and they are concurrent at a point* T (Fig. 13).

The point T has the following properties:

Property 1. *The points* A, T *and* W *are collinear* (Fig. 13).

Property 2. *The simple ratio* $\overline{AT}:\overline{WT}$*, in which the point* T *divides the segment* AW*, does not depend on the place of the point* W *on the line* BC.

Property 3. *The point* T *lies on the line* XY (Fig. 13).

Further details are included in the paper: Grozdev, S., V. Nenkov (2013-[5]). A type of central conics through a point, Mathematics Plus, 4, ISSN 0861 8321.

III. Some interesting properties, found in the process of generalizing. To this kind of problems belongs one from the 51-st IMO'2012 paper.

Problem 6. *Given is a right-angled triangle* ABC $(\sphericalangle ACB=90^\circ)$*, for which* $C_1\in AB$ *is the foot of the altitude from the vertex* C*. Let* P *be an interior point of the segment* CC_1*. The* *point K from the segment AP is such that $BK = BC$, while the point L from the segment BP is such that $AL = AC$. If M is the intersection point of AL and BK, prove that $MK = ML$.*

We will enlarge this geometric configuration considering an arbitrary ΔABC. Let $A_1 \in BC$, $B_1 \in CA$ and $C_1 \in AB$ be the feet of the altitudes of ΔABC from the vertices A, B and C, respectively, while P be an arbitrary point on the line CC_1. If k_a is a circle with center A and radius AB_1, while k_b is a circle with center B and radius BA_1, then denote by K_1 and K_2 the intersection points of k_b with the line AP (in case they exist), denote by L_1 and L_2 the intersection points of k_a with the line BP (in case they exist). Also, consider the points $M_{11} = BK_1 \cap AL_1$, $M_{12} = BK_1 \cap AL_2$, $M_{21} = BK_2 \cap AL_1$ and $M_{22} = BK_2 \cap AL_2$.

Some properties are expected for the points M_{11}, M_{12}, M_{21} and M_{22} in relation with the position of P on CC_1. Let H be the orthocenter of ΔABC and H' be the symmetric one of H with regard to AB. After experiments by GSP we deduce the following:

Theorem 9. *If the point P runs the segment HH', then two of the points M_{11}, M_{12}, M_{21} and M_{22} describe an ellipse E, which passes through the vertex C and its foci are the vertices A and B. The other two points describe a hyperbola χ, which passes through the vertex C and its foci are the vertices A and B or the perpendicular bisector s of the side AB.*

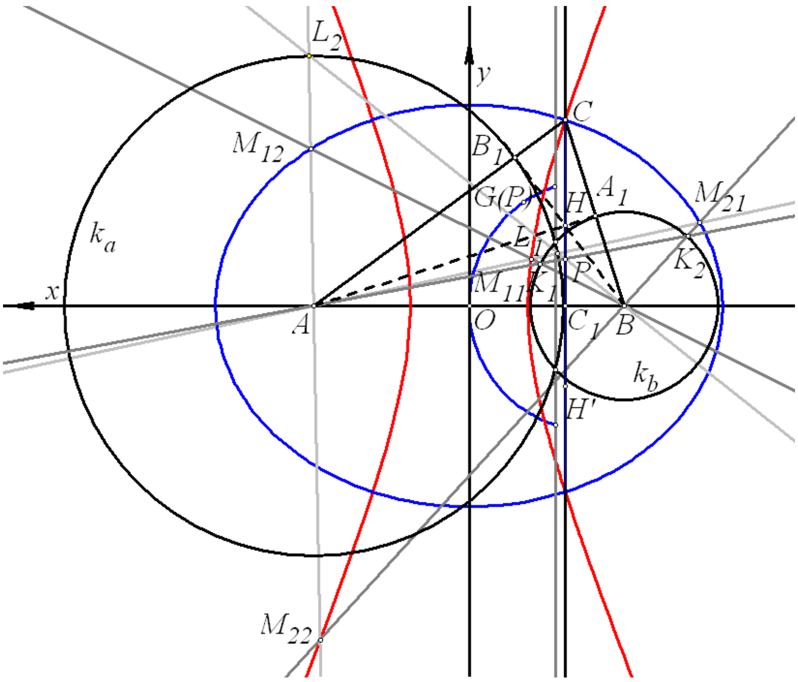

Fig. 14

In the case of Problem 6 $(\sphericalangle ACB = 90°)$ the following equalities are verified: $K_1M_{11} = L_1M_{11}$, $K_2M_{22} = L_2M_{22}$, $K_1M_{12} = L_2M_{12}$ and $K_2M_{21} = L_1M_{21}$. In the general case an equality of ratios appears:

Theorem 10. *The points K_i, L_i and M_{ij} $(i = 1, 2; j = 1, 2)$ are connected by the equality:*

$$\frac{K_1M_{11}}{L_1M_{11}} \cdot \frac{K_2M_{22}}{L_2M_{22}} = \frac{K_1M_{12}}{L_2M_{12}} \cdot \frac{K_2M_{21}}{L_1M_{21}}.$$

Depending on the generating point P, the points K_1, K_2, L_1 and L_2 have a gravity center $G(P)$. Experiments by GSP lead to the following:

Theorem 11. *When the point P runs the segment HH', the point $G(P)$ describes an arc from a circle or a segment, whose extremities lie on the radical axes of the circles k_a and k_b.*

Further details are included in the paper: Grozdev, S., V. Nenkov (2013-[6]). Three properties of the triangle, defined by its altitudes. *Mathematics and Informatics*, 1, pp. 176 – 183. ISSN 1310-2230

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