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Odom's Triangle

Dedicated to the memory of Professor Deko Dekov

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Abstract. The late American artist and geometer George Phillips Odom Jr. is famous for his simple construction of the Golden Ratio with an equilateral triangle and its circumcircle. We use his construction to define *Odom's Triangle*. Using-dynamic geometry software as well as a computer algebra system and calculating with Cartesian coordinates, we find many more appearances of the Golden Ratio.

Keywords. Equilateral triangle, Golden Ratio.

Mathematics Subject Classification (2020). 51-03, 51M04, 51M15, 51N20.

1. Introduction

Constructions of segments divided in the Golden Ratio $\varphi = \frac{1+\sqrt{5}}{2}$ have long been in the interest of geometers world wide. A well known example is created by the late American artist and geometer George Phillips Odom Jr. and his simple construction of the Golden Ratio from an equilateral triangle, the line through two of its side midpoints and its circumcircle [6]. Inspired by this construction author of this paper Tran published a similar construction [9] using a square isosceles triangle in stead of an equilateral triangle. Similar variations were subsequently studied by Dao [1], Paunić and Yiu [7], Dao, Ngo and Yiu [2], Pietsch [8] and Tran [11], [12], [13].

We start our study with an equilateral triangle and Odom's construction. We will use Cartesian coordinates, with $A(\frac{1}{2}, \frac{\sqrt{3}}{2})$, $B(0,0)$ and $C(1,0)$ as the base

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equilateral triangle. The center of $\triangle ABC$ has coordinates $O_{ABC}(\frac{1}{2}, \frac{\sqrt{3}}{6})$ and its circumcircle ω has equation

$$\omega : \left(x - \frac{1}{2}\right)^2 + \left(y - \frac{\sqrt{3}}{6}\right)^2 = \frac{1}{3}. \tag{1}$$

The medial triangle is given by $D(\frac{1}{2}, 0)$, $E(\frac{3}{4}, \frac{\sqrt{3}}{4})$ and $F(\frac{1}{4}, \frac{\sqrt{3}}{4})$. Let G be the intersection of EF and ω such that F is between E and G. EG is the segment for which Odom proposed his famous statement that $\frac{EF}{FG} = \varphi$ [6].

Indeed we find coordinates $G\left(\frac{2-\sqrt{5}}{4}, \frac{\sqrt{3}}{4}\right)$ and it is easily verified that $\frac{EF}{FG} = \varphi$.

In correspondence among the authors Tran suggested to use the endpoints of Odom's segment and a vertex of the equilateral triangle to form a new triangle, $\triangle BEG$. Quickly that appeared to be fruitful. Using dynamic geometry software as well as a computer algebra system we found that this triangle yields the Golden ratio in several ways. In honour of his beautiful result we propose *Odom's triangle* as name for $\triangle BEG$. In this paper we present our findings, for which we use basic constructions as well as some triangle centers, as is done with the Golden Triangle in [3] and [10].

Most calculations that go with our findings are straightforward, more so for a computer algebra system, and will be omitted. For triangle centers we use barycentric coordinates and triangle functions. A triangle center X_n with respect to $\triangle A_1B_1C_1$ is defined by its triangle function $f(a, b, c)$ (where a, b, and c are the triangle sidelengths). Details can be found in [4] (although mainly using trilinear coordinates) and [5]. The Cartesian coordinates of this center X_n can be calculated from the Cartesian coordinates of B, E, and G by

$$X_n = \frac{f(b, e, g) \cdot B + f(e, g, b) \cdot E + f(g, b, e) \cdot G}{f(b, e, g) + f(e, g, b) + f(g, b, e)}, \tag{2}$$

where

$$b = GE = \frac{\sqrt{5} + 1}{4}, \; g = BE = \frac{\sqrt{3}}{2}, \; e = BG = \frac{\sqrt{2}(\sqrt{5} - 1)}{4}. \tag{3}$$

2. Odom's triangle and its circumcircle

The circumcenter of Odom's triangle has coordinates $O\left(\frac{5-\sqrt{5}}{8}, \frac{\sqrt{15}-\sqrt{3}}{8}\right)$. The circumcircle Ω of Odom's triangle hence has equation

$$\Omega : \left(x - \frac{5 - \sqrt{5}}{8}\right)^2 + \left(y - \frac{\sqrt{15} - \sqrt{3}}{8}\right)^2 = \frac{3 - \sqrt{5}}{4}. \tag{4}$$

Note also that

$$\frac{DO}{OF} = \varphi. \tag{5}$$

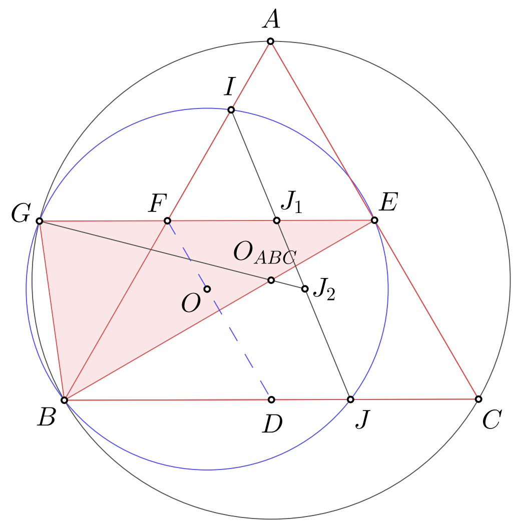

FIGURE 1. Odom's triangle and its circumcircle

Apart from B circle Ω intersects BC in J and AB in I. J and I are found by rotations or O through $\pm\frac{\pi}{3}$ about E, so that from (5) we conclude (see figure 1)

$$\frac{FI}{IA} = \frac{CJ}{JD} = \varphi. \tag{6}$$

This is supported by coordinates $J(\frac{5-\sqrt{5}}{4}, 0)$ and $I(\frac{1+\sqrt{5}}{8}, \frac{\sqrt{15}+\sqrt{3}}{8})$.

Finally we note that G, O, and C are collinear, so that it is obvious from (5) and the similarity $\triangle DOC \sim \triangle FOG$ that

$$\frac{CO}{OG} = \varphi. \tag{7}$$

O divides the chord of ω through O and B in parts congruent to those it divides GC in. So here again we find the Golden Ratio.

Remark. *Let BO meet EF in Q, so $Q(\frac{\sqrt{5}}{4}, \frac{\sqrt{3}}{4})$. Then $\frac{FQ}{QE} = \varphi$, and equivalently $FQ = FG = FI$.*

Remark. *It is easily seen that $\triangle BEG \cong \triangle EBI \cong \triangle EBH$.*

Now we consider chord IJ of Ω. It intersects EF in J_1 and GO_{ABC} in J_2, with

$$\begin{aligned} J_1 \quad & \left(\tfrac{11-4\sqrt{5}}{4}, \tfrac{\sqrt{3}}{4}\right), \\ J_2 \quad & \left(\tfrac{7\sqrt{5}-11}{8}, \tfrac{\sqrt{3}(\sqrt{5}-1)}{8}\right). \end{aligned}$$

From this we see

$$\frac{IJ_1}{J_1J_2} = \frac{JJ_2}{J_2J_1} = \varphi. \tag{8}$$

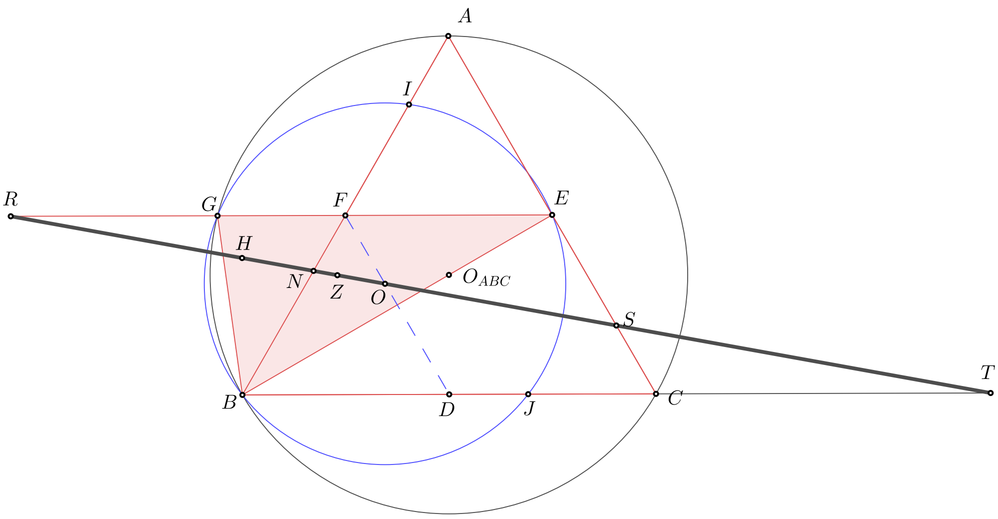

FIGURE 2. The Euler line of Odom's triangle

3. THE EULER LINE OF ODOM'S TRIANGLE

The centroid of Odom's triangle is given by $Z(\frac{5-\sqrt{5}}{12}, \frac{\sqrt{3}}{6})$. From this we see that the Euler line $e = OZ$ is given by

$$e : y = \frac{2\sqrt{15} - 5\sqrt{3}}{5}x + \frac{3\sqrt{3} - \sqrt{15}}{4}. \tag{9}$$

The orthocenter H thus has coordinates $H(0, \frac{3\sqrt{3}-\sqrt{15}}{4})$. The point of intersection of e with

- EF is $R(-\frac{\sqrt{5}}{4}, \frac{\sqrt{3}}{4})$;
- AC is $S(\frac{\sqrt{5}+5}{8}, \frac{3\sqrt{3}-\sqrt{15}}{8})$;
- BC is $T(\frac{5+\sqrt{5}}{4}, 0)$.

Note that S is the midpoint of HT.

We now have 7 occurences of the Golden ratio

$$\frac{RG}{GF} = \frac{ES}{SC} = \frac{SH}{HR} = \frac{RO}{OS} = \frac{TO}{OR} = \frac{TC}{CD} = \frac{TJ}{JB} = \varphi. \tag{10}$$

See figure 2.

Remark. *Note that the Nine-Point Center $N(\frac{5-\sqrt{5}}{16}, \frac{5\sqrt{3}-\sqrt{15}}{16})$ of Odom's triangle lies on AB.*

4. GOLDEN CYCLIC HEXAGON

Let GZ meet BC in U, GH meet FZ in V, and DK and FH meet in W. So

- $U(\frac{1+\sqrt{5}}{4}, 0)$,
- $V(\frac{3-\sqrt{5}}{4}, 0)$, hence V lies on BC as well, and
- $W(\frac{1-\sqrt{5}}{8}, \frac{\sqrt{15}-\sqrt{3}}{8})$.

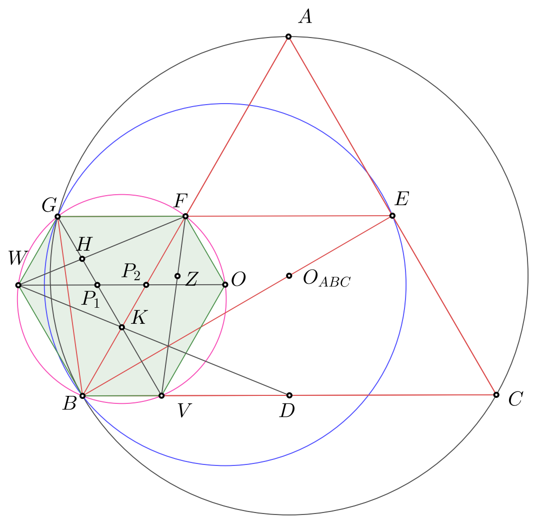

FIGURE 3. The Golden cyclic hexagon $BVOFGW$

We find that

$$\frac{DV}{VB} = \frac{DU}{UC} = \frac{TU}{UV} = \frac{DK}{KW} = \frac{FH}{HW} = \varphi. \tag{11}$$

Now we take a closer look to hexagon $BVOFGW$ and note a number of properties, see figure 3:

- $BVOFGW$ is cyclic and equiangular,
- each pair adjacent sides of $BVOFGW$ has φ as length ratio,
- diagonals BF, GV and WO bound an equilateral triangle $\triangle KP_1P_2$, where we have taken $P_1 \in GV$ and $P_2 \in BF$,
- the sidelengths of equilateral triangles $\triangle FGK$, $\triangle BP_2W$, and $\triangle VOP_1$ are φ times those of $\triangle BVK$, $\triangle OFP_2$, and $\triangle GWP_1$, which are φ times those of $\triangle KP_1P_2$.

This is exactly the *Golden cyclic hexagon* described by De Villiers in [14].

5. ODOM'S TRIANGLE AND THE FERMAT POINTS

Let $\triangle B_1E_1G_1$ be the triangle of apices of equilateral triangles pointed oudwardly on the sides of Odom's triangle. It is well known that BB_1, EE_1 and GG_1 concur in the first Fermat point F_1. Coordinates are $B_1(\frac{5-\sqrt{5}}{8}, \frac{\sqrt{3}(3+\sqrt{5})}{8})$, and $E_1(-\frac{1+\sqrt{5}}{8}, \frac{\sqrt{3}(3-\sqrt{5})}{8})$, $G_1(\frac{3}{4}, -\frac{\sqrt{3}}{4})$, so that

$$F_1\left(\frac{5(8-3\sqrt{5})}{76}, \frac{\sqrt{3}(\sqrt{5}+10)}{76}\right).$$

Let K_1 and K_3 be the second intersections of BB_1 and GG_1 with ω, and K_2 and K_4 the second intersection of EE_1 and BB_1 with Ω. See figure 4. Coordinates

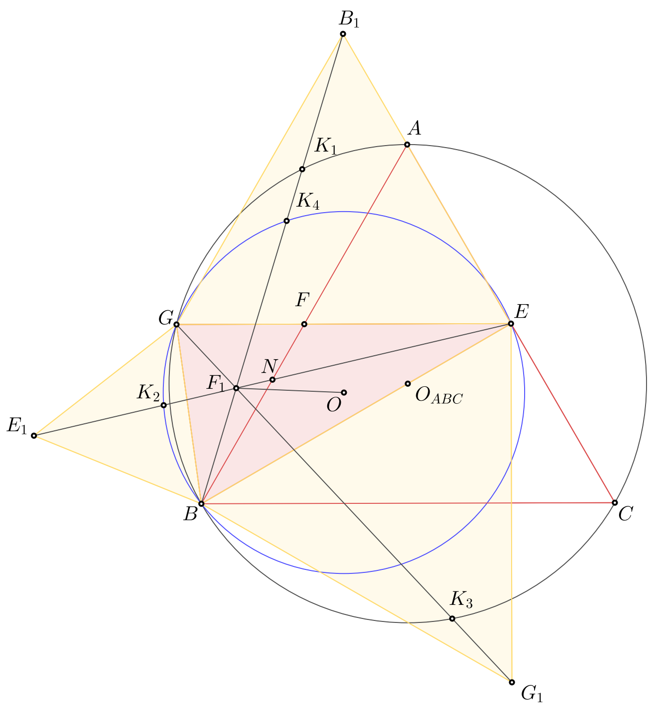

FIGURE 4. The first Fermat point

are

$$K_1 \quad \left(\tfrac{25-7\sqrt{5}}{38}, \tfrac{\sqrt{3}(3\sqrt{5}+11)}{38}\right),$$
$$K_2 \quad \left(\tfrac{145-71\sqrt{5}}{152}, \tfrac{5\sqrt{3}(5\sqrt{5}-7)}{152}\right),$$
$$K_3 \quad \left(\tfrac{55-4\sqrt{5}}{76}, -\tfrac{\sqrt{3}(10+\sqrt{5})}{76}\right),$$
$$K_4 \quad \left(\tfrac{65-22\sqrt{5}}{76}, \tfrac{\sqrt{3}(21+4\sqrt{5})}{76}\right).$$

We have

$$\frac{F_1B}{F_1K_2} = \frac{F_1E}{F_1K_4} = \frac{F_1K_4}{F_1O} = \varphi, \tag{12}$$

and

$$\frac{F_1K_1}{F_1G} = \frac{F_1K_3}{F_1B} = \varphi^2. \tag{13}$$

Remark. *Note that the Nine Point Center N lies on EE_1.*

Now consider $G_2(0, \frac{\sqrt{3}}{2})$ and $B_2(\frac{5-\sqrt{5}}{8}, \frac{\sqrt{3}(1-\sqrt{5})}{8})$ as the apices of equilateral triangles erected on BE and GE, pointing inwardly in $\triangle BEG$, and note that BOG is equilateral as well. It is well known that BB_2, EO and GG_2 concur in the second Fermat point

$$F_2 \left(\frac{\sqrt{5}(\sqrt{5}-3)}{16}, \frac{\sqrt{3}(3-\sqrt{5})}{16}\right).$$

Remark. *Note that F_2B_2 passes through R and that $B_2 \in \omega$.*

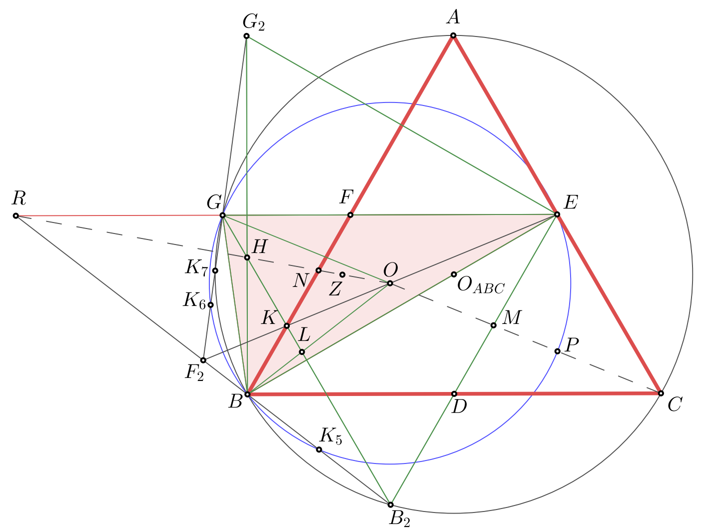

FIGURE 5. The second Fermat point

F_2E, B_2G and AB concur in $K(\frac{3-\sqrt{5}}{8}, \frac{\sqrt{3}(3-\sqrt{5})}{8})$. Let GC and EB_2 meet in $M(\frac{7-\sqrt{5}}{8}, \frac{\sqrt{3}(3-\sqrt{5})}{8}))$ and let OB and GB_2 meet in $L(\frac{\sqrt{5}(\sqrt{5}-2)}{4}, \frac{\sqrt{3}(\sqrt{5}-2)}{4})$. Then let $P(\frac{3}{4}, \frac{\sqrt{3}(\sqrt{5}-2)}{4})$ be the point where GC meets ω apart from G. Finally, note that the orthocenter H lies on BG_2 as well as GB_2. See figure 5. We find

$$\frac{FK}{KB} = \frac{EO}{OK} = \frac{G_2H}{HB} = \frac{B_2M}{ME} = \frac{GM}{MC} = \frac{OL}{LB} = \frac{OM}{MP} = \varphi. \tag{14}$$

Let K_5 and K_6 be the points where F_2B and F_2G meet Ω for a second time. Let K_7 be the point where F_2G meets ω for the second time. We have coordinates

$$\begin{array}{ll} K_5 & \left(\frac{5-\sqrt{5}}{16}, \frac{\sqrt{3}(1-\sqrt{5})}{16}\right), \\ K_6 & \left(\frac{3(2-\sqrt{5})}{8}, \frac{\sqrt{3}}{8}\right), \\ K_7 & \left(\frac{1-\sqrt{5}}{16}, \frac{\sqrt{3}(5-\sqrt{5})}{16}\right). \end{array}$$

With the coordinates we see

$$\frac{GK_6}{K_6F_2} = \frac{FK_6}{K_6K_7} = \frac{F_2K_5}{K_5B_2} = \frac{K_5B}{BF_2} = \varphi, \tag{15}$$

where it is to be noted that $F_2B = F_2K_6$ and $F_2G = F_2K_5$.
The Golden Ratio is also found by

$$\frac{G_2F_2}{F_2O} = \frac{EF_2}{F_2B_2} = \varphi. \tag{16}$$

Remark. *The anticomplement of F_2 lies on ω.*

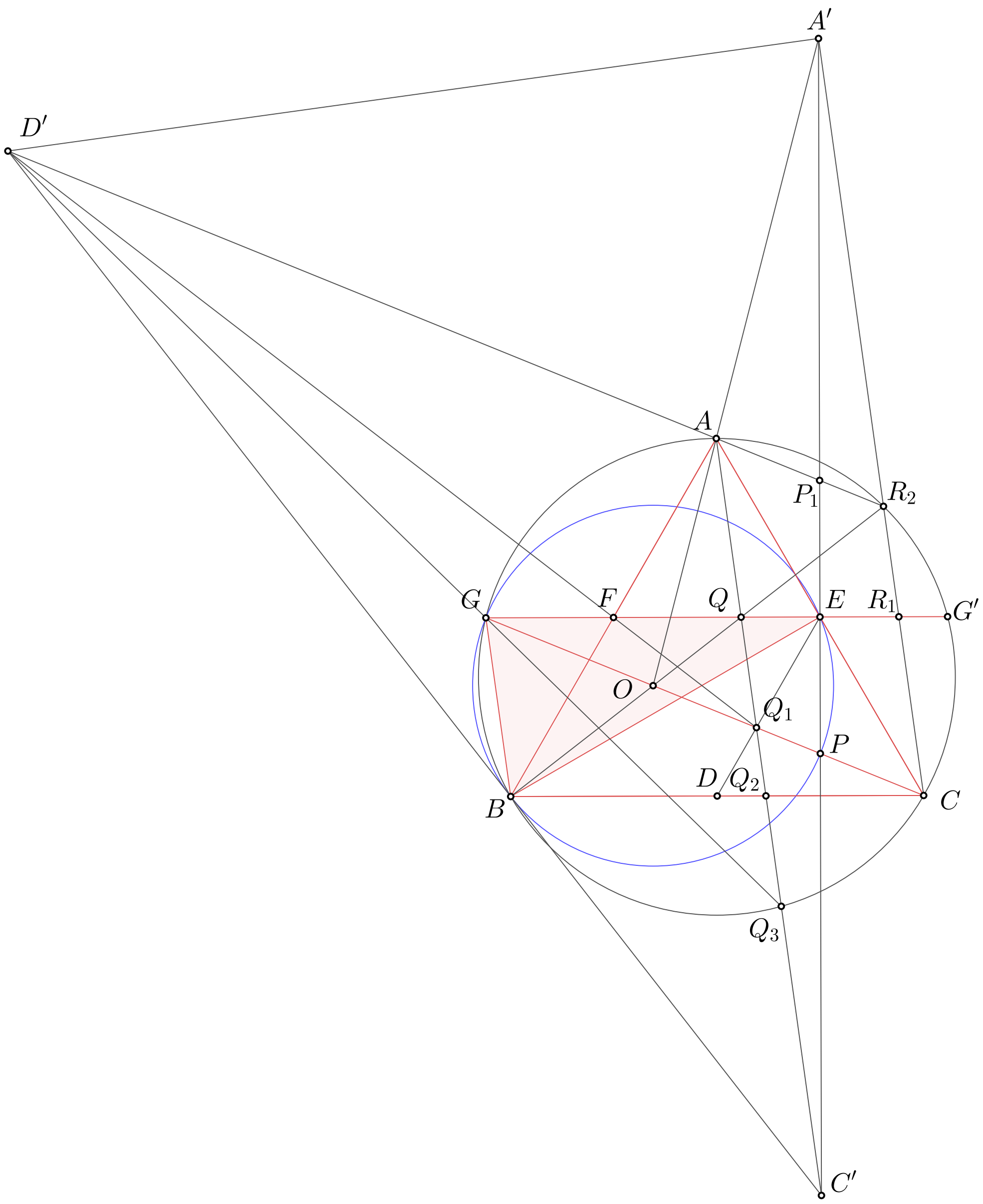

Figure 6. Isogonal conjugates of A, C and D

6. The isogonal conjugates of A, C, and D

Consider

$$A'\left(\frac{3}{4}, \frac{\sqrt{3}(\sqrt{5}+2)}{4}\right),$$

$$C'\left(\frac{3}{4}, -\frac{\sqrt{15}}{4}\right),$$

and

$$D'\left(-\frac{3(1+\sqrt{5}}{8}, \frac{\sqrt{3}(5+\sqrt{5}}{8}\right),$$

the isogonal conjugates of A, C and D respectively.

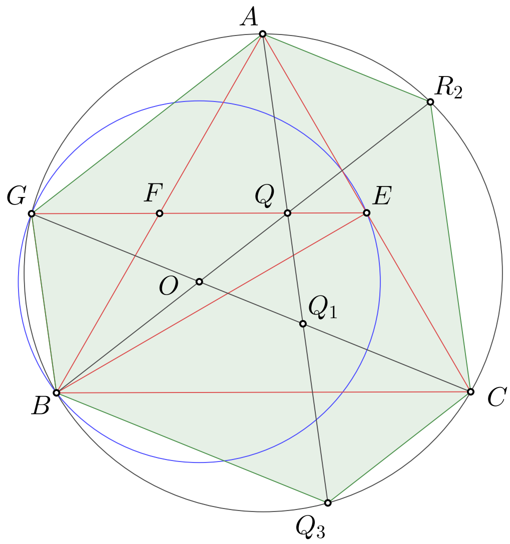

FIGURE 7. Golden cyclic hexagon $AGBQ_3CR_2A$

Note that $A'C' \perp BC$, that $AC' \perp A'D'$, that $P \in A'C'$ and that $\triangle ECC' \cong \triangle EAA'$. Let

- $Q_1(\frac{7-\sqrt{5}}{8}, \frac{\sqrt{3}(3-\sqrt{5})}{8}) = CG \cap AC'$,
- $Q_2(\frac{1}{\varphi}, 0) = BC \cap AC'$,
- $Q_3(\frac{\sqrt{5}+3}{8}, \frac{\sqrt{3}(1-\sqrt{5})}{8})$, the second intersection of AC' and ω,
- $R_1(\frac{6-\sqrt{5}}{4}, \frac{\sqrt{3}}{4}) = EG \cap A'C$,
- $R_2(\frac{\sqrt{5}+5}{8}, \sqrt{3}(1+\sqrt{5}))$ the second intersection of $A'C$ and ω,
- $P_1(\frac{3}{4}, \sqrt{3} - \frac{\sqrt{15}}{4}) = AR_2 \cap A'C'$,
- $G'(\frac{\sqrt{5}+2}{4}, \frac{\sqrt{3}}{4})$, the second intersection of EG and ω.

See figure 6.

From coordinates and figure it is easily seen that

$$\frac{BQ_2}{Q_2C} = \frac{QQ_2}{Q_2Q_3} = \frac{AP_1}{P_1R_2} = \frac{C'D'}{D'A'} = \varphi. \tag{17}$$

Points A, O, and A' are collinear, with ratio

$$\frac{AO}{OA'} = \varphi. \tag{18}$$

D' is remarkable as it lies on the lines BC', GQ_3, FQ_1, and AR_2. we find ratios

$$\frac{D'B}{BC'} = \frac{D'G}{G, Q_3} = \varphi, \tag{19}$$

and

$$\frac{D'A}{AR_2} = \frac{D'F}{FQ_1} = \varphi^3. \tag{20}$$

Also this configuration yields a second Golden cyclic hexagon $AGBQ_3CR_2A$ with central equilateral triangle OQQ_1. See figure 7.

Finally we see, noting that D, Q_1 and E are collinear,

$$\frac{ER_1}{R_1G'} = \frac{CR_1}{R_1R_2} = \frac{A'R_2}{R_2C} = \frac{EQ_1}{Q_1D} = \frac{GQ_1}{Q_1C} =$$

$$\frac{AQ_3}{Q_3C'} = \frac{BQ_2}{Q_2C} = \frac{A'P}{PC'} = \frac{C'P_1}{P_1A'} = \varphi. \quad (21)$$

7. Conclusion

Odom's Golden Ratio construction has given us Odom's Triangle, in which the Golden Ratio appears to be obiquitous. We are happy to realize that the modern use of computer power is helpful to reveal the classical beauty of Euclidean geometry.

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