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Inequalities Involving Gergonne and Nagel Cevians

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Abstract. A Gergonne cevian is the cevian through the Gergonne point of a triangle. A Nagel cevian is the cevian through the Nagel point of a triangle. We present some new inequalities involving the lengths of the Gergonne and Nagel cevians of a triangle. Mathematica was used to both discover and prove some of these results.

Keywords. triangle geometry, Gergonne cevian, Nagel cevian, inequalities, Mathematica, computer-discovered mathematics.

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1. Introduction

Let s denote the semiperimeter of a triangle with sides of lengths a, b, and c. Let r and R denote the inradius and circumradius of the triangle. Let m_a, h_a, and w_a denote the length of the median to side a, the altitude to side a, and the angle bisector of angle A, respectively. Let the corresponding lengths to sides b and c be named similarly.

Inequalities involving these lengths abound in the literature. See [2] and [12] for a large number of such inequalities. Typical inequalities (taken from these books) are shown below.

$$\frac{3}{2}s < \sum m_a < 2s$$

$$\sum h_a \leq s\sqrt{3}$$

$$9r \leq \sum h_a \leq \sum w_a \leq \sum m_a \leq \frac{9}{2}R \tag{1}$$

$$\sum h_a^2 \leq \sum w_a^2 \leq s^2 \leq \sum m_a^2 \tag{2}$$

In this paper, we find similar inequalities involving the Gergonne and Nagel cevians of a triangle. (A *cevian* of a triangle is the line segment from a vertex to a point on the opposite side. A cevian through the Gergonne point of a triangle is called a *Gergonne cevian* and a cevian through the Nagel point of a triangle is called a *Nagel cevian.*)

The lengths of the Gergonne and Nagel cevians to side a of a triangle will be called g_a and n_a, respectively. The corresponding lengths to sides b and c are named similarly.

2. The Results

Theorem 1. *The lengths of the cevians to side a of $\triangle ABC$ satisfy the following chain of inequalities.*

$$2r < h_a \leq g_a \leq w_a \leq m_a \leq n_a < s \tag{3}$$

Proof. The first inequality comes from [12, p. 14]. The next four inequalities come from [13]. The last inequality comes from the fact that a Nagel cevian splits the perimeter of a triangle into two equal parts. See [11, pp. 1–14]. If AN is the Nagel cevian to side a of $\triangle ABC$ as shown in Figure 1, then $AC + CN = s$ and so $AN < s$ by the triangle inequality in $\triangle ANC$. □

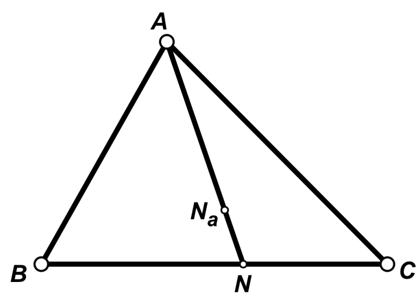

Figure 1. A Nagel cevian

In this paper, we will use the summation symbol, $\sum$, to represent a cyclic sum under the mapping $a \to b \to c$. Using this notation, we have the following immediate corollary.

Corollary 2. *The cyclic sums satisfy the following chain of inequalities.*

$$\sum h_a \leq \sum g_a \leq \sum w_a \leq \sum m_a \leq \sum n_a < 3s$$

Corollary 2 adds g_a and n_a into the inequality chain (1).

Since $0 < x < y$ implies that $x^2 < y^2$, we immediately get the following from equation (3).

Corollary 3. *The lengths of the cevians to side a of $\triangle ABC$ satisfy the following two chains of inequalities.*

$$4r^2 < h_a^2 \le g_a^2 \le w_a^2 \le m_a^2 \le n_a^2 < s^2 \tag{4}$$

and

$$12r^2 < \sum h_a^2 \le \sum g_a^2 \le \sum w_a^2 \le \sum m_a^2 \le \sum n_a^2 < 3s^2 \tag{5}$$

We note that equality holds for an equilateral triangle and that the first inequality in (5) is not the best possible. An improvement on the first inequality is

$$27r^2 \le \sum h_a^2. \tag{6}$$

This result comes from [12, p. 201].

Before improving these inequalities, let us note the formulas for the lengths of various cevians of a triangle. These formulas are easily obtained using Stewart's Theorem [1, p. 152].

Formulary. *The lengths of various cevians of a triangle can be found from the following formulas.*

$$h_a^2 = \frac{4K^2}{a^2}$$

$$m_a^2 = \frac{2b^2 + 2c^2 - a^2}{4}$$

$$w_a^2 = \frac{4bcs(s-a)}{(b+c)^2}$$

$$g_a^2 = \frac{(s-a)(as - (b-c)^2)}{a}$$

$$n_a^2 = \frac{c^2(s-b) + b^2(s-c) - a(s-b)(s-c)}{a}$$

where

$$K = \sqrt{s(s-a)(s-b)(s-c)}$$

We can now look for inequalities involving the squares of the lengths of the Gergonne and Nagel cevians.

Theorem 4. *The squares of the lengths of the Gergonne and Nagel cevians satisfy the following inequalities.*

$$\frac{1}{4}\sum n_a^2 < \sum g_a^2 \le \sum n_a^2$$

Proof. The second inequality comes from equation (5). The first inequality can be proven using Mathematica. The following code segment sets up the necessary variables. (The csum function creates a cyclic sum.)

```
assum = a>0 && b>0 && c>0 && a+b>c && b+c>a && c+a>b;
csum[expr_]:= expr+(expr/.{a->b,b->c,c->a})
              +(expr/.{a->c,b->a,c->b});
s = (a+b+c)/2;
ga = Sqrt[(s-a)(a*s-(b-c)^2)/a];
na = Sqrt[((s-b)c^2+(s-c)b^2-a(s-b)(s-c))/a];
```

The command Simplify[expr,assum] instructs Mathematica to simplify the specified expression using the specified assumptions.

The following Mathematica command then proves the inequality.

```
Simplify[(1/4)csum[na^2]<csum[ga^2], assum]
```

The result of executing this command produces

```
True
```

indicating that the inequality is true. □

If we had not known the constant 1/4 in this inequality, but merely suspected that there was some integer k such that $k\sum n_a^2 < \sum g_a^2$, we could use Mathematica to find the best possible value for k as follows. The Mathematica command Minimize[{expr,constraint},vars] instructs Mathematica to minimize the specified expression subject to the specified constraint involving the specified variables.

The Mathematica command

```
Minimize[{csum[ga^2]/csum[na^2], assum}, {a,b,c}]
```

produces the result

```
{1/4, {a->1, b->1/2, c->1/2}}
```

indicating that the minimum value is $\frac{1}{4}$ and that this minimum is attained when $a = 1$, $b = \frac{1}{2}$, and $c = \frac{1}{2}$. The Mathematica output also includes a note that there is no minimum in the specified region and that the returned result is on the boundary of that region. This provides an alternate proof to Theorem 4.

Using Mathematica in the same manner, we were able to prove the following results.

Theorem 5. *The lengths of the Gergonne and Nagel cevians satisfy the following inequalities.*

$$\frac{1}{2}w_a < g_a \leq w_a$$

$$m_a \leq n_a < 2m_a$$

Theorem 6. *The squares of the lengths of the Gergonne cevians satisfy the following inequalities.*

$$\frac{1}{2}s^2 < \sum g_a^2 \le s^2$$
$$\frac{1}{3}(a^2+b^2+c^2) < \sum g_a^2 \le \frac{3}{4}(a^2+b^2+c^2)$$
$$\frac{4}{9}\sum m_a^2 < \sum g_a^2 \le \sum m_a^2$$
$$27r^2 \le \sum g_a^2 \le \frac{27}{4}R^2$$
$$3\sqrt{3}K \le \sum g_a^2$$

Theorem 7. *The squares of the lengths of the Nagel cevians satisfy the following inequalities.*

$$s^2 \le \sum n_a^2 < 3s^2$$
$$\frac{3}{4}(a^2+b^2+c^2) \le \sum n_a^2 < \frac{3}{2}(a^2+b^2+c^2)$$
$$\sum m_a^2 \le \sum n_a^2 < 2\sum m_a^2$$
$$27r^2 \le \sum n_a^2 \le 12R^2$$
$$3\sqrt{3}K \le \sum n_a^2$$

Theorem 8. *The squares of the lengths of the Gergonne and Nagel cevians satisfy the following inequalities.*

$$\sum n_a^2 - \sum g_a^2 < 2s^2$$
$$\frac{3}{4}s^2 \le \sum n_a^2 - \frac{1}{4}\sum g_a^2 < \frac{11}{4}s^2$$

For additional inequalities involving Gergonne and Nagel cevians, see references [3, 4, 5, 6, 7, 8, 9, 10].

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