International Journal of Computer Discovered Mathematics (IJCDM)
ISSN 2367-7775 ©IJCDM
Volume 6, 2021, pp.1-5
Received 17.12.2020. Published on-line 26.01.2021
web: http://www.journal-1.eu/

Location of Some Kimberling Centers Respect To Orthocentroidal Circle

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Abstract. By using barycentric coordinates and the computer program "Mathematica", we give theorems on the locations of some Kimberling centers respect to orthocentroidal circle.

Keywords. Orthocentroidal Circle, Kimberling Center, Euclidean Geometry, Barycentric Coordinates, Inequalities,

Mathematics Subject Classification (2010). 51-04, 68T01, 68T99.

1. Introduction

The orthocentroidal circle $\mathcal{S}_{GH}$ has diameter GH, where G is the centroid and H is the orthocenter of triangle ABC [1]. The orthocentroidal disk of the circle on diameter GH is the interior disc punctured at its center. We use the notation $\mathcal{D}_{GH}$ for the orthocentroidal disk.

The incenter X_1 lies in $\mathcal{D}_{GH}$. Since IGN_a are collinear and $IG : GN_a = 1 : 2$ it follows that Nagel's point X_8 is outside the disk. The symmedian point X_6 lies in the disc $\mathcal{D}_{GH}$. One Brocard point lies in $\mathcal{D}_{GH}$ and the other lies outside $\mathcal{S}_{GH}$, or they both lie simultaneously on $\mathcal{S}_{GH}$ (which happens if and only if the reference triangle is isosceles). X_7, Gergonne's point lies in the orthocentroidal disk $\mathcal{S}_{GH}$. First Fermat's point X_{13} ranges freely over the orthocentroidal disk $\mathcal{D}_{GH}$ and the second Fermat point X_{14} ranges freely over the region external to $\mathcal{S}_{GH}$. The Feuerbach point $Fe = X_{11}$ is always outside the orthocentroidal circle [2].

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2. Preliminaries

The Jerabek hyperbola is a circumconic that is the isogonal conjugate of the Euler line [3]. Since it is a circumconic passing through the orthocenter, it is a rectangular hyperbola and has center on the nine-point circle. The Jerabek center is Kimberling center X_{125} [4].

The Kiepert hyperbola is a hyperbola and triangle conic that is related to the solution of Lemoine's problem and its generalization to isosceles triangles constructed on the sides of a given triangle. Kiepert center is Kimberling center X_{115} [4].

Spieker center is Kimberling center X_{10}. It's the incenter of medial triangle [4].

Crosspoint of incenter and centroid is Kimberling center X_{37} [4].

Brocard midpoint center is Kimberling center X_{39} [4].

Crosspoint of incenter and symmedian point is Kimberling center X_{42} [4].

Given two points $P = (u_1 : v_1 : w_1)$ and $Q = (u_2 : v_2 : w_2)$ in normalized barycentric coordinates. Denote $x = u_1 - u_2$, $y = v_1 - v_2$ and $z = w_1 - w_2$. Then the square of the distance between P and Q is as follows [5]:

$$|PQ|^2 = -a^2yz - b^2zx - c^2xy$$

3. Theorems

Theorem 3.1. *X_{10}, Spieker center lies outside $\mathcal{S}_{GH}$.*

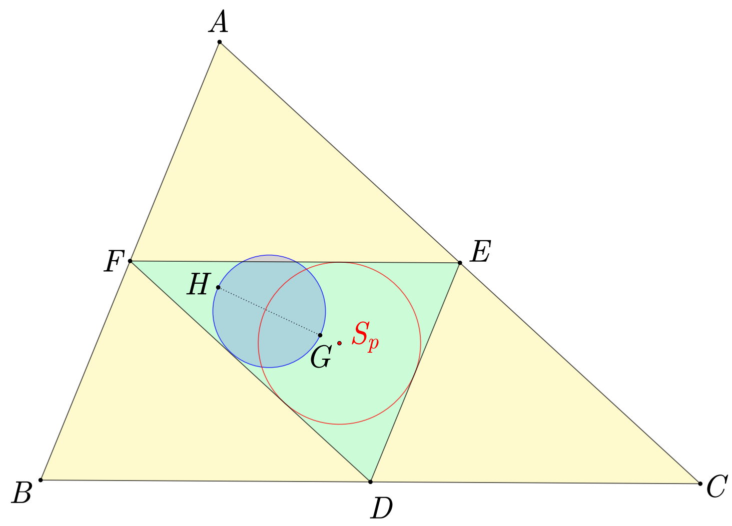

FIGURE 1.

Proof. Let $J = X_{381}$ be midpoint of G and H. J has first barycentric coordinate: $(a^4 - 2(b^2 - c^2)^2 + a^2(b^2 + c^2) ::)$. X_{10} has barycentric coordinates: $(b+c : c+a : a+b)$. We want to show that $JX_{10} > JG$. Using the squared distances $JX_{10}^2 - JG^2$ is equal to:

$$\frac{1}{12}\left(a^2 - ab - ac + b^2 - bc + c^2\right)$$

This expression is positive immediately by rearrangement inequality, X_{10}, Spieker center lies outside $\mathcal{S}_{GH}$. ☐

Theorem 3.2. *X_{37}, crosspoint of incenter and centroid lies inside $\mathcal{D}_{GH}$.*

Proof. Let $J = X_{381}$ be midpoint of G and H. X_{37} has barycentric coordinates: $(a(b+c) : b(c+a) : c(a+b))$. The squared distance $JX_{37}^2 - JG^2$ is equal to:

$$\frac{abc\left(-3\left(a^3+b^3+c^3\right)+(a+b+c)(ab+bc+ca)\right)}{12(a(b+c)+bc)^2}$$

This denominator is positive. The numerator is negative by Chebyshev's inequality. Since sequences (a, b, c) and (a^2, b^2, c^2) are same ordered by Chebyshev's inequality:

$$3(a^3+b^3+c^3) \geq (a+b+c)(a^2+b^2+c^2)$$

So the expression $JX_{37}^2 - JG^2$ is negative. This proves X_{37}, lies inside $\mathcal{D}_{GH}$. ☐

Theorem 3.3. *X_{39}, Brocard Midpoint lies inside $\mathcal{D}_{GH}$.*

Proof. Let $J = X_{381}$ be midpoint of G and H. X_{39} has barycentric coordinates: $(a^2(b^2+c^2) : b^2(c^2+a^2) : c^2(a^2+b^2))$. Using the squared distances $JX_{39}^2 - JG^2$ is equal to:

$$\frac{a^2b^2c^2\left(-a^4+a^2b^2+a^2c^2-b^4+b^2c^2-c^4\right)}{12(a(b+c)+bc)^2}$$

This denominator is positive. The numerator is negative by Rearrangement Inequality. Since So the expression $JX_{39}^2 - JG^2$ is negative. This proves X_{39}, lies inside $\mathcal{D}_{GH}$. ☐

Theorem 3.4. *X_{42}, crosspoint of incenter and symmedian point lies inside $\mathcal{D}_{GH}$.*

Proof. Let $J = X_{381}$ be midpoint of G and H. X_{51} has first barycentric coordinate: $(a^2(b+c) : b^2(c+a) : c^2(a+b))$. Using the squared distances $JX_{42}^2 - JG^2$ is equal to:

$$\frac{-(ab+bc+ca)\left[\sum_{cyc}\left(ab(a^4+b^4)+3a^2b^2c^2\right]-\left[\sum_{cyc}2a^3b^3+abc\sum_{cyc}a^3\right]}{3\left(a^2(b+c)+a\left(b^2+c^2\right)+bc(b+c)\right)^2}$$

The denominator is positive. Its sufficent to show:

$$\left[\sum_{cyc}\left(ab(a^4+b^4\right)+3a^2b^2c^2\right]-\left[\sum_{cyc}2a^3b^3+abc\sum_{cyc}a^3\right]$$

is positive. Denote $a+b+c=p, ab+bc+ca=q, abc=r$. In terms of p, q, r it is written as:

$$q(p^2-2q)^2-4q^3+16pqr-2rp^3-9r^2 \geq 0$$

Note that if we fix p and q then LHS is a concave function of variable r. So its sufficent to replace r by its extremal values. We have two cases:
$a = x \geq 0$ and $b = c = 1$, the second case is $c = 0$. In the first case inequality becomes:

$$x^2(x-1)^2(2x+3) \geq 0$$

whic is obvious. In the second case we need to prove:

$$ab(a^4+b^4) \geq 2a^3b^3$$

whic is obvious by $AM-GM$ in eqauality. Using the squared distances $JX_{42}^2 - JG^2$ is negative. This proves X_{42}, lies inside $\mathcal{D}_{GH}$. ☐

Theorem 3.5. *X_{51}, centroid of orthic triangle lies inside $\mathcal{D}_{GH}$.*

Proof. Let $J = X_{381}$ be midpoint of G and H. X_{51} has first barycentric coordinate: $(a^2((b^2-c^2)^2 - a^2(b^2+c^2)::)$. Using the squared distances $JX_{51}^2 - JG^2$ is equal to:

$$\frac{a^8 - a^6b^2 - a^6c^2 + a^4b^2c^2 - a^2b^6 + a^2b^4c^2 + a^2b^2c^4 - a^2c^6 + b^8 - b^6c^2 - b^2c^6 + c^8}{36a^2b^2c^2}$$

The denominator is positive. Substituting $x = a^2, y = b^2, z = c^2$ we get $-x^4 - y^4 - z^4 - xyz(x+y+z) + (x^3y + xy^3 + y^3z + yz^3 + z^3x + zx^3)$ It's negative since $x^4 + y^4 + z^4 \geq xyz(x+y+z)$ gives;

$$2(x^4 + y^4 + z^4) \geq \sum_{sym} x^3y$$

Applying $AM - GM$ to RHS:

$$2(x^4 + y^4 + z^4) \geq 2(x^2y^2 + y^2z^2 + z^2x^2)$$

$$x^4 + y^4 + z^4 \geq x^2y^2 + y^2z^2 + z^2x^2$$

which is true by rearrangement ineqaulity. $JX_{51}^2 - JG^2$ is negative. This proves X_{51}, lies inside $\mathcal{D}_{GH}$. □

Theorem 3.6. *X_{115}, center of Kiepert hyperbola lies outside $\mathcal{S}_{GH}$.*

Proof. Let $J = X_{381}$ be midpoint of G and H. X_{115} has barycentric coordinates: $((b^2-c^2)^2 : (c^2-a^2)^2 : (a^2-b^2)^2)$. Using the squared distances $JX_{115}^2 - JG^2$ is equal to:

$$\frac{a^6 - a^4b^2 - a^4c^2 - a^2b^4 + 3a^2b^2c^2 - a^2c^4 + b^6 - b^4c^2 - b^2c^4 + c^6}{12\left(a^4 - a^2b^2 - a^2c^2 + b^4 - b^2c^2 + c^4\right)}$$

Numerator is positive by the inequality used in Theorem 2.1. Substituting $x = a^2, y = b^2, z = c^2$ denominator is positive by rearrangement inequality. So this expression is positive. Hence $JX_{115} > JG$. So X_{115}, center of Kiepert hyperbola lies outside $\mathcal{S}_{GH}$. □

Theorem 3.7. *X_{125}, center of Jerabek hyperbola lies outside $\mathcal{S}_{GH}$.*

Proof. Let $J = X_{381}$ be midpoint of G and H. X_{125} has first barycentric coordinates: $((b^2-c^2)(a^2(b^2-c^2) - b^4 + c^4)::)$. We want to show that $JX_{125} > JG$. Using the squared distance $JX_{125}^2 - JG^2$ is equal to:

$$\frac{a^8 - a^6b^2 - a^6c^2 + a^4b^2c^2 - a^2b^6 + a^2b^4c^2 + a^2b^2c^4 - a^2c^6 + b^8 - b^6c^2 - b^2c^6 + c^8}{12(a^6 - a^4b^2 - a^4c^2 - a^2b^4 + 3a^2b^2c^2 - a^2c^4 + b^6 - b^4c^2 - b^2c^4 + c^6)}$$

It's sufficent to show that this expression is positive. Substituting $x = a^2, y = b^2, z = c^2$ in the denominator gives this quantity is positive for all real a, b, c except $a = b = c$. This follows from the well known inequality for non-negative x, y and z that:

$$x^3 + y^3 + z^3 + 3xyz \geq \sum_{sym} x^2y$$

Again substituting $x = a^2, y = b^2, z = c^2$ in the numerator we get $x^4 + y^4 + z^4 + xyz(x + y + z) - (x^3y + xy^3 + y^3z + yz^3 + z^3x + zx^3)$. It's positive since $x^4 + y^4 + z^4 \geq xyz(x + y + z)$ gives;

$$2(x^4 + y^4 + z^4) \geq \sum_{sym} x^3y$$

Applying $AM - GM$ to RHS:

$$2(x^4 + y^4 + z^4) \geq 2(x^2y^2 + y^2z^2 + z^2x^2)$$
$$x^4 + y^4 + z^4 \geq x^2y^2 + y^2z^2 + z^2x^2$$

which is true by rearrangement inequality. □

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