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SEVERAL PROPERTIES OF THE INSCRIBED CONIC SECTIONS AND A METHOD FOR PROOFS WITH COMPLEX NUMBERS

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Abstract: Several properties of conic sections are considered that are discovered by the software program GSP (The Geometer's Sketchpad). Some of the assertions are generalizations of well-known facts about circles. A general method is elaborated which could be applied to similar problems.

Key words: triangle, inscribe conic section, parabola, ellipse, hyperbola, complex number

I. Several properties of the diameters of conic sections which are inscribed in triangles. We consider the following properties.

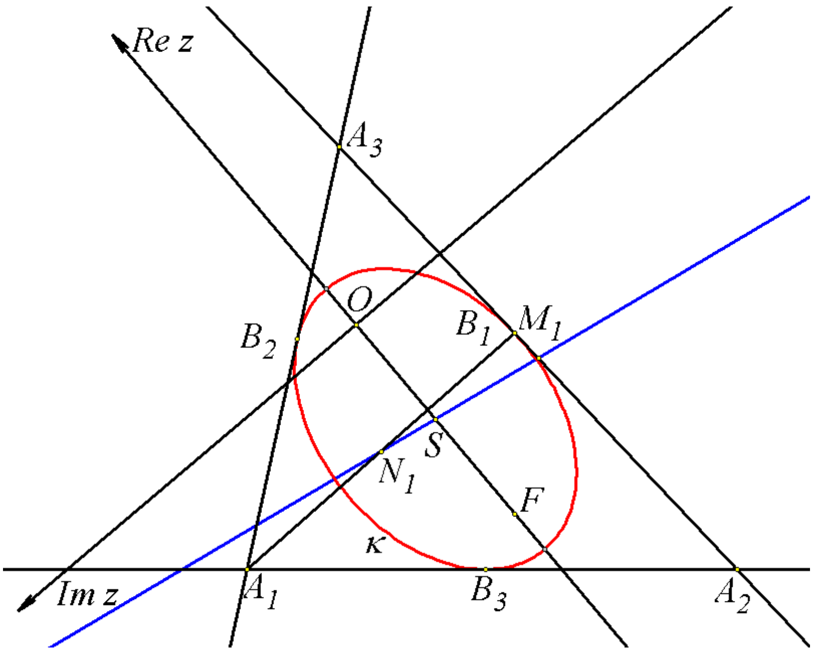

Figure 1

Theorem 1. *If the conic section* κ *is inscribed in* $\Delta A_1A_2A_3$ *and is tangent to the line* A_2A_3 *at the point* B_1, *while* M_1 *and* N_1 *are the midpoints of the segments* A_2A_3 *and* A_1B_1, *then:*

1) the line M_1N_1 *passes through the center of* κ*, when* κ *is ellipse or hyperbola* (Fig. 1, 2)*;*

2) the line M_1N_1 *is parallel to the axis of* κ*, when* κ *is parabola* (Fig. 3).

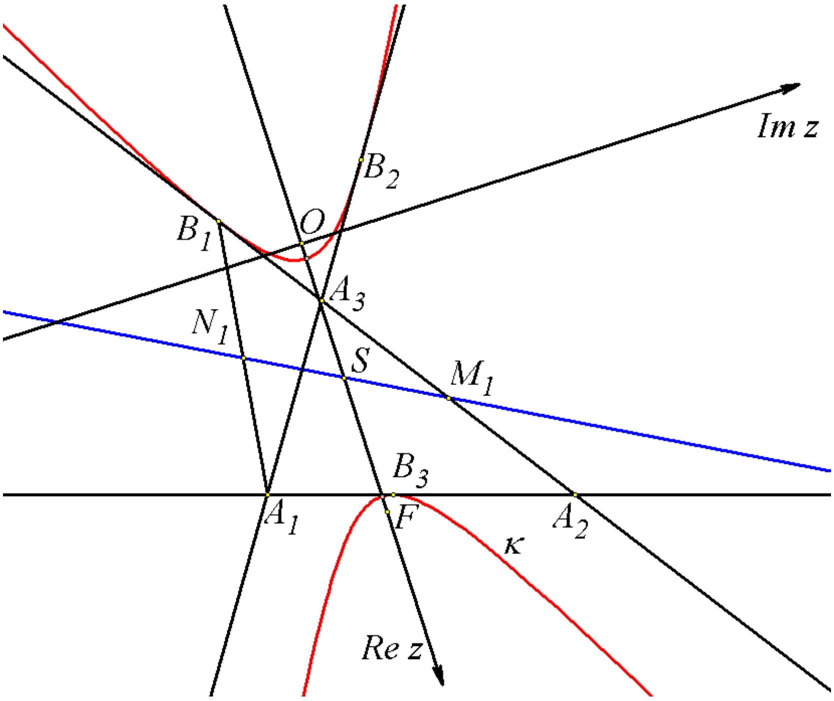

Figure 2

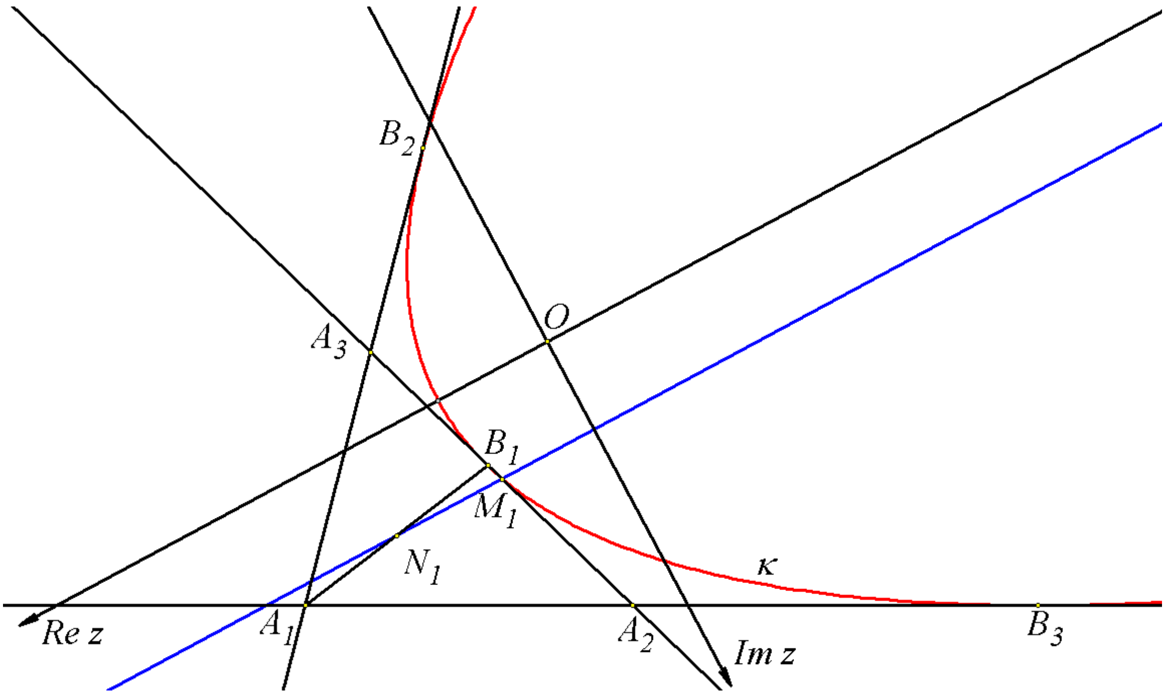

Figure 3

Theorem 2. *Let the conic section* κ *be tangent at the points* B_1 *u* B_3 *to the lines* A_2A_3 *and* A_1A_2*, which determine the triangle* $A_1A_2A_3$*. If the line through the mid-points of the segments* A_2A_3 *and* A_3A_1*, intersects the line* B_1B_3 *in point* U*, then*

1) the line A_1S *passes through* U*, when* κ *is ellipse or hyperbola with center* S*,* (Fig. 4, 5)*;*

2) the line A_1U *is parallel to the axis of* κ*, when* κ *is parabola* (Fig. 6).

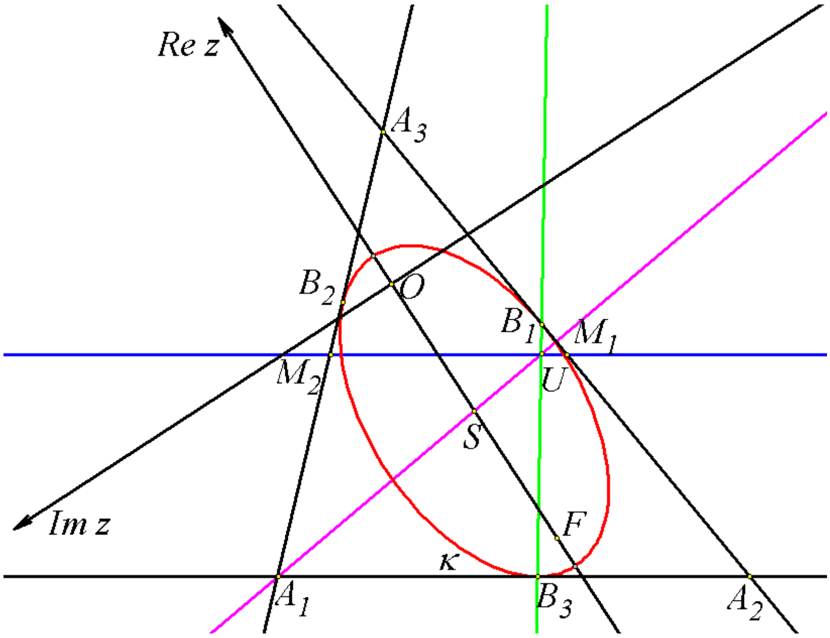

Figure 4

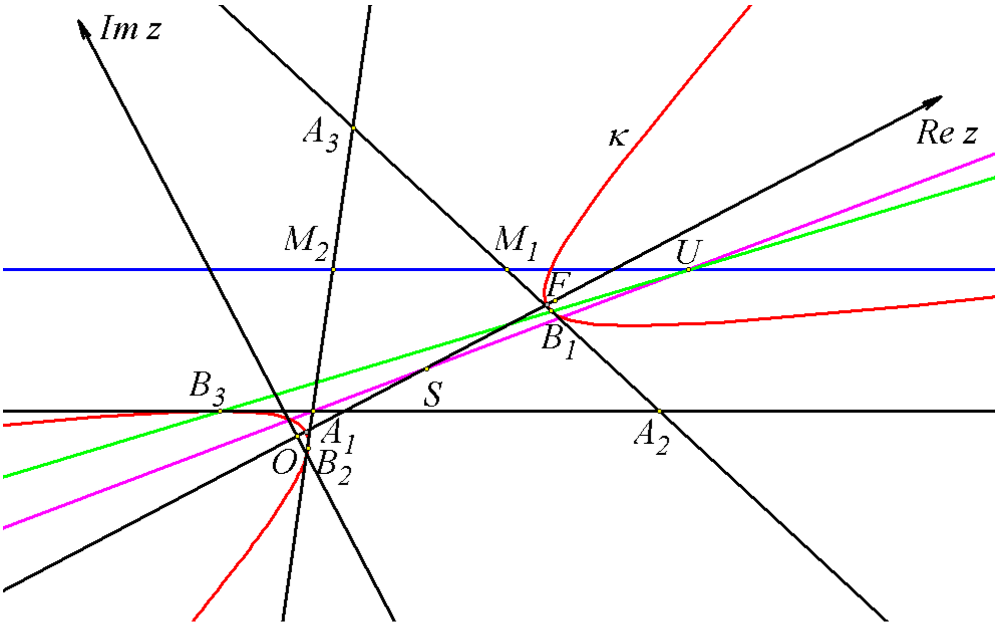

Figure 5

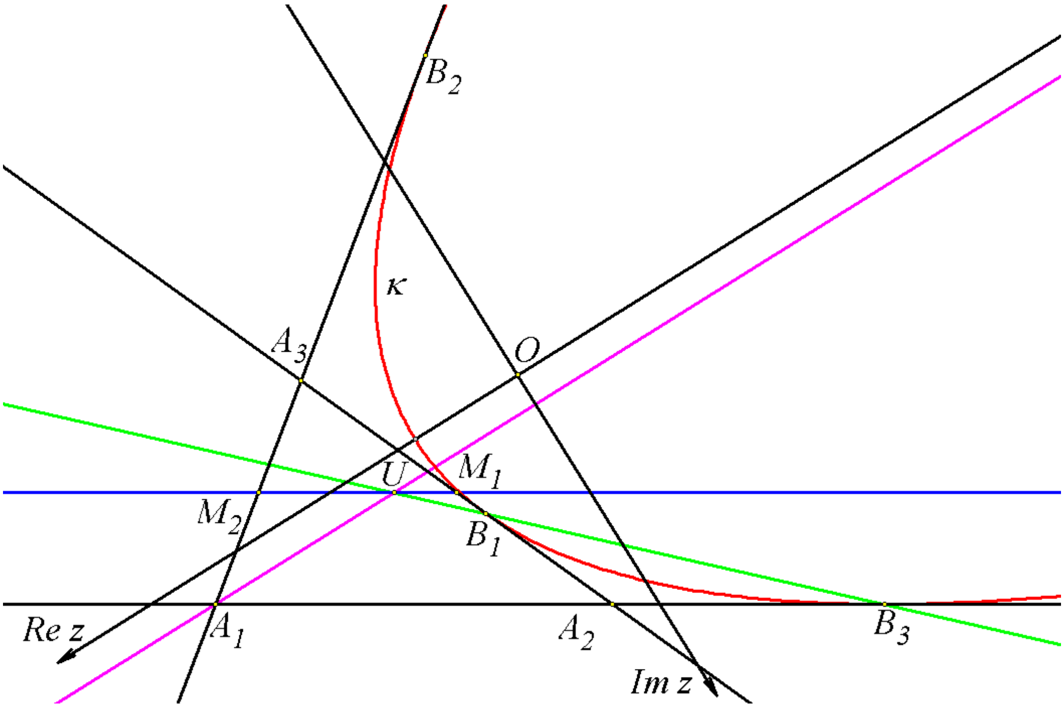

Figure 6

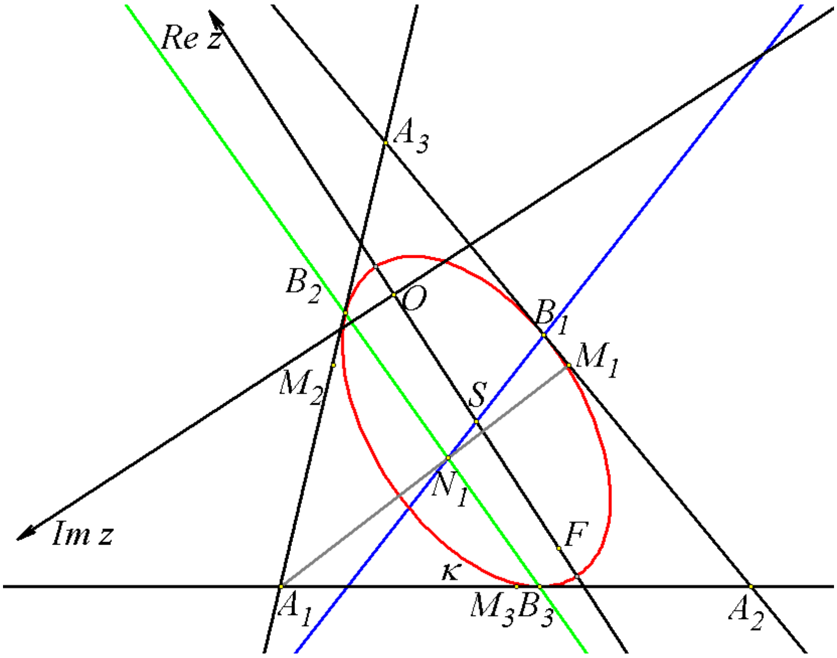

Figure 7

Theorem 3. *Let the conic section* κ *be tangent at the points* B_1, B_2 *and* B_3 *to the lines* A_2A_3, A_3A_1 *and* A_1A_2, *which determine the triangle* $A_1A_2A_3$. *If the medians through the vertices* A_1, A_2 *and* A_3 *intersect the lines* B_2B_3, B_3B_1 *and* B_1B_2 *in the points* N_1, N_2 *and* N_3 *respectively, then*

1) the lines B_1N_1, B_2N_2 *and* B_3N_3 *pass through the center of* κ, *when* κ *is ellipse or hyperbola* (Fig. 7, 8)*;*

2) the lines B_1N_1, B_2N_2 *and* B_3N_3 *are parallel to the axis of* κ, *when* κ *is parabola* (Fig. 9).

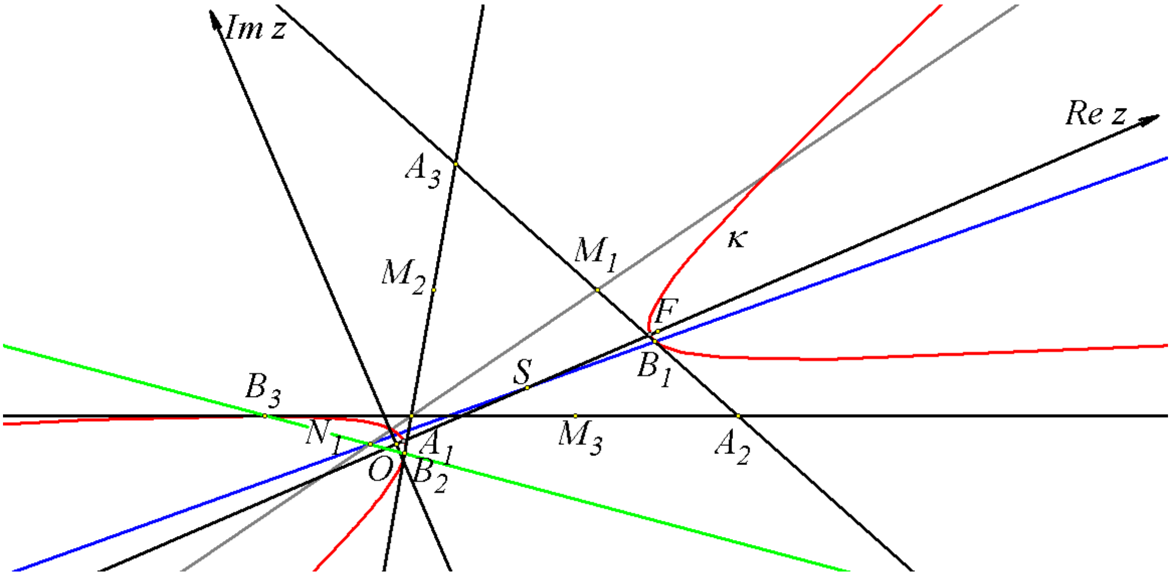

Figure 8

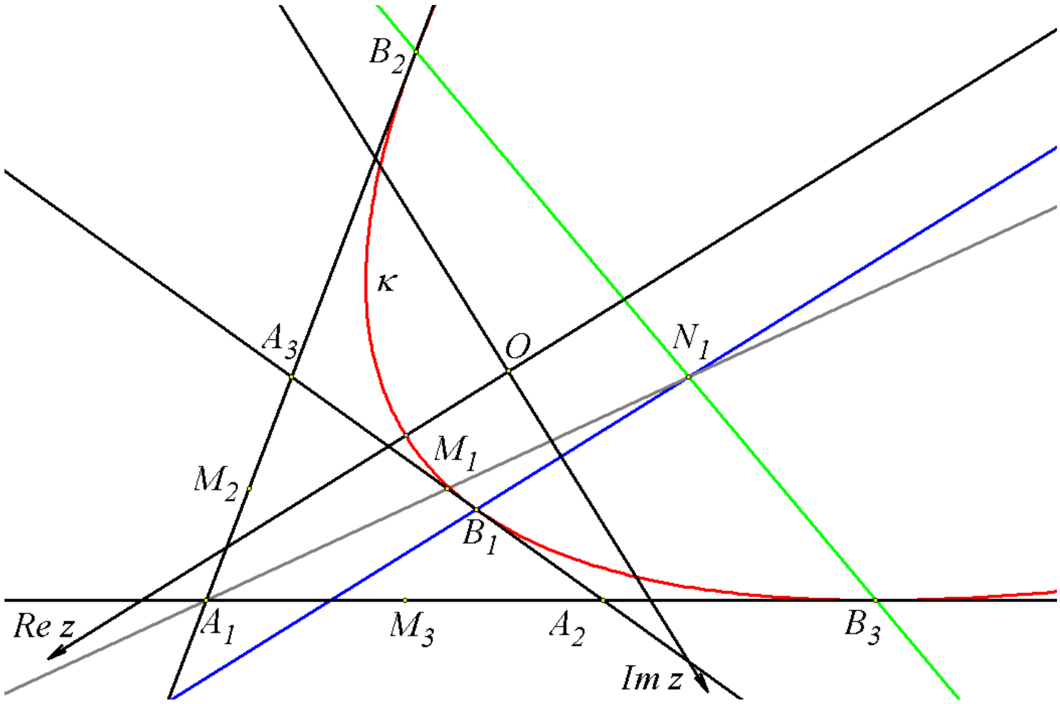

Figure 9

II. Two metric equalities which are determined by inscribed conic sections. The following metric relations are considered:

Theorem 4. *Given are triangle* $A_1A_2A_2$ *with area* S *and conic section* κ, *which is tangent to the lines* A_2A_3, A_3A_1 *and* A_1A_2 *at the points* B_1, B_2 *and* B_3, *respectively. If the areas of the triangles* $B_2B_3A_1$, $B_3B_1A_2$, $B_1B_2A_3$ *and* $B_1B_2B_3$ *are* S_1, S_2, S_3 *and* σ, *respectively, then*

$$\sigma = 2\sqrt{\frac{S_1 S_2 S_3}{S}}.$$

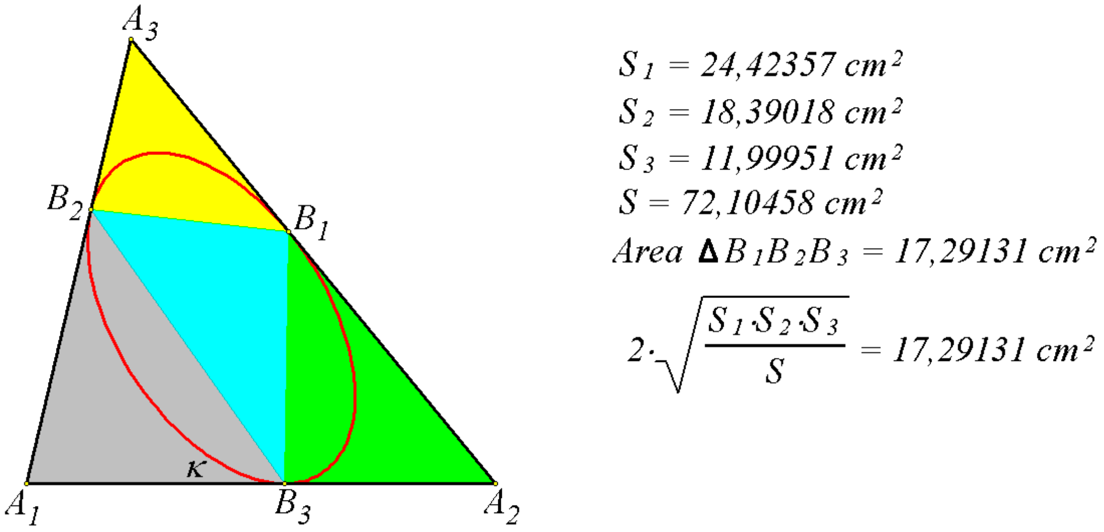

Figure 10

Theorem 5. *If $A_1A_2A_3\dots A_n$ $(n \ge 3)$ is the polygon obtained by the intersection of arbitrary lines A_1A_2, A_2A_3, …, $A_{n-1}A_n$ and A_nA_1, which are tangent to a given conic section κ at the points B_1, B_2, …, B_{n-1} and B_n, respectively, then*

$$\frac{\overline{A_1B_1}}{\overline{A_2B_1}}\cdot\frac{\overline{A_2B_2}}{\overline{A_3B_2}}\cdots\frac{\overline{A_{n-1}B_{n-1}}}{\overline{A_nB_{n-1}}}\cdot\frac{\overline{A_nB_n}}{\overline{A_1B_n}} = (-1)^n.$$

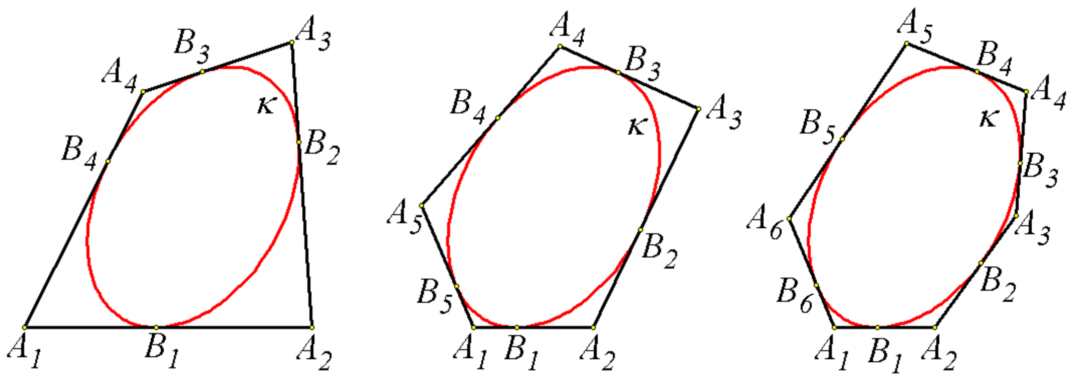

Figure 11

III. A linear property of the centers of the conic sections that are inscribed in a quadrilateral. The line g_0 passing through the mid-points of the diagonals of an arbitrary quadrilateral $ABCD$, which is not a parallelogram, is called to be *Gauss line for* $ABCD$. The next proposal is connected with it:

Theorem 6. *The centers of all inscribed conic sections in an arbitrary quadrilateral $ABCD$, which is not a parallelogram, lie on Gauss line for $ABCD$* (Fig. 12).

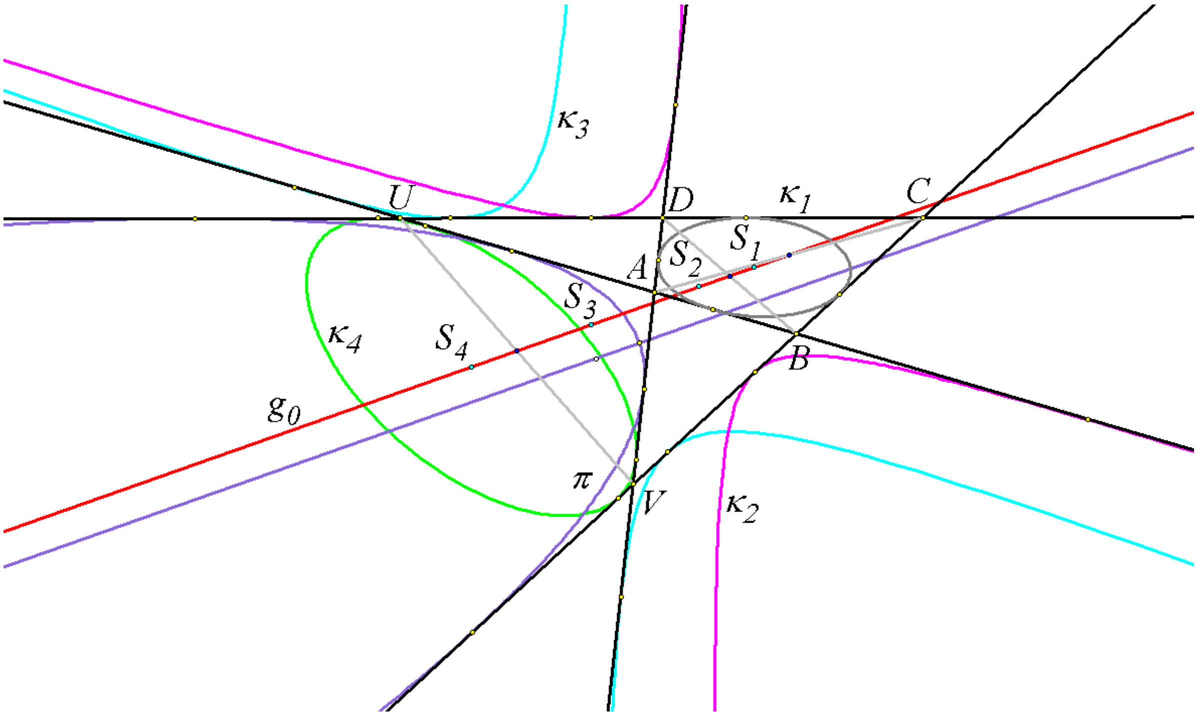

Figure 12

The sequel is dedicated to a method with complex numbers by means of which the formulated proposals will be proved. The method could be useful in problem solving connected with inscribed conic sections.

IV. Parametric equations of some curves in the complex plane. Let in the complex plane with respect to a coordinate system with center O be given the unit circle Γ with equation $|u|=1$ and an arbitrary point Z with affix z. If $Z\neq O$ and $\arg z=\varphi$, then only one point U with affix u exists on Γ with $\arg u=\varphi$. Consequently, we may write down for each point Z in the complex plane, which is different from the coordinate origin O, that

(1) $$z=|z|u\,,\ |u|=1\,.$$

The equality (1) remains true for $z=0$, i.e. $Z=O$. Thus, each complex number z may be represented in the form (1), and this means that each point in the complex plane has affix represented by (1). From (1) we get $z^2=|z|^2u^2=z\bar{z}u^2$. It follows that

(2) $$\bar{z}=z\bar{u}^2,\ |u|=1\,.$$

Using (1) and (2), we will show how to present the points from some curves in relation with a parameter, which is changing on the unit circle Γ.

IV.1. Equation of a line not passing through the coordinate origin. Each line l in the complex plane has equation of the type $A_0z+\bar{A}_0\bar{z}=C$, where C is a real number. If the line l does not pass through the coordinate origin O, then $C\neq 0$. Accounting for $\bar{C}=C$, the last equality may be written in the following way $\frac{A_0}{C}z+\frac{\bar{A}_0}{\bar{C}}\bar{z}=1$. We take $\alpha=\frac{A_0}{C}$ in this equality thus obtaining that each line l, which does not pass through the coordinate origin, has equation of the type

$$(3)\qquad l:\alpha z+\bar{\alpha}\bar{z}=1,\ \alpha\neq 0.$$

Substituting (2) in (3) and taking $t=\dfrac{\bar{\alpha}\bar{u}^2}{\alpha}$, we conclude, that the points on the line l verify

$$(4)\qquad z=\frac{1}{\alpha(1+t)},\ |t|=1.$$

IV.2. Equation of circle. Since the circle is a set of points in the plane, which are at distance R from a point Ω, then if the affix of Ω is ω, we have $|z-\omega|=R$ for each point Z on the circle. It follows from the last equality that $(z-\omega)(\bar{z}-\bar{\omega})=R^2$, and after simplification we get the equation

$$(5)\qquad z\bar{z}-\bar{\omega}z-\omega\bar{z}=R^2-\omega\bar{\omega}.$$

IV.3. Equation of conic section. Let O be focus of a given conic section κ, and d be directrice of κ corresponding to O. A polar coordinate system is chosen in the following way: the pole coincides with focus O, while the polar axis is with the same direction of the ray that passes through O, perpendicularly to d and directed to the line d. It is well-known that with respect to the introduced coordinate system κ has the following polar equation

$$(6)\qquad \rho=\frac{p}{1+e\cos\varphi},$$

where p is the focal parameter of κ, and e is its eccentricity.

If (ρ,φ) are the polar coordinates of point Z, accounting for (1) we obtain $z=\rho u$, $\cos\varphi=\dfrac{u+\bar{u}}{2}$ and $|u|=1$. From the last equalities and (6) we get $z=\dfrac{2p}{e\bar{u}^2+2\bar{u}+e}$. Substitute $t=\bar{u}$ in the last equality to deduce the equation of the conic section κ

$$(7)\qquad z=\frac{2p}{et^2+2t+e},\ |t|=1.$$

Note that in case of hyperbola the polar equation (6) describes only the branch, which lies in one and the same semi-plane with the focus O with respect to d. Consequently the equation (7) concerns this branch of the hyperbola only. It will be proved in the sequel that both branches of the hyperbola are described by the equation (7).

Further, the fact that the point Z_0 from the curve κ is obtained when the value of the parameter is $t=t_0$, will be notified in the following way: $Z_0(t_0)$.

IV.4. Equation of a tangent to a conic section. Under the above assumptions we will determine the equation of the tangent to κ at the point Z_0 with affix z_0. Since the tangent is a line that does not pass through the focus O, it is described by the equations $z=\dfrac{1}{\alpha(1+t')}$, $|t'|=1$ or $\alpha z+\bar{\alpha}\bar{z}=1,\ \alpha\neq 0$ (they follow from (4) and (3)). Substituting (7) in the last equation we obtain

$$(e-2p\bar{\alpha})t^2+2t+e-2p\alpha=0.$$

The tangent has a unique common point with κ, consequently the last equation has only one root t_0, which is $t_0=-\dfrac{1}{e-2p\bar{\alpha}}$. From here $\alpha=\dfrac{e+t_0}{2p}$. Thus, the equation of the tangent is $z=\dfrac{2p}{(e+t_0)(1+t')}$. Let $t'=\beta t$, $|\beta|=1$. For β we will use that $z_0=\dfrac{2p}{(e+t_0)(1+t')}=\dfrac{2p}{et_0^2+2t_0+e}$. From here $\beta=\dfrac{et_0+1}{t_0+e}$. Finally, the equation of the tangent at the point $Z_0(t_0)$ is

$$z=\frac{2p}{et_0t+t_0+t+e},\ |t|=1. \tag{8}$$

V. Inscribed conic sections in triangle. A conic section κ is inscribed in a triangle Δ if the lines that determine the triangle are tangent to κ.

Let the lines τ_1,τ_2 and τ_3 be tangent to the conic section κ with focus O at the points $B_1(t_1)$, $B_2(t_2)$ and $B_3(t_3)$, respectively. If κ is a hyperbola, we will suppose temporary that τ_1, τ_2 and τ_3 are tangent to the branch of κ, which contains the focus O. According to (8) the equations of the tangents are the following

$$\tau_j:\quad z=\frac{2p}{et_jt+t_j+t+e}.$$

(From now on we suppose that the equalities $|t_1|=|t_2|=|t_3|=|t|=1$ are going without saying and we will not mention them). If $\tau_2\cap\tau_3=A_1$, $\tau_3\cap\tau_1=A_2$ and $\tau_1\cap\tau_2=A_3$, we have that the affixes a_1, a_2 and a_3 of the points A_1, A_2 and A_3 are:

$$a_1=\frac{2p}{et_2t_3+t_2+t_3+e},\ a_2=\frac{2p}{et_3t_1+t_3+t_1+e},\ a_3=\frac{2p}{et_1t_2+t_1+t_2+e}. \tag{9}$$

Let k be the circumcircle of $\Delta A_1A_2A_3$. If Ω is the center of k and R is its radius, then the equation of k is expressed by (5). Replacing (9) in (5) we obtain the following linear system of equations with respect to ω, $\bar{\omega}$ and $R^2-|\omega|^2$:

$$\left|\begin{array}{l}\dfrac{4p^2t_1t_2}{(et_1t_2+t_1+t_2+e)^2}-\dfrac{2p\bar{\omega}}{et_1t_2+t_1+t_2+e}-\dfrac{2pt_1t_2\omega}{et_1t_2+t_1+t_2+e}=R^2-|\omega|^2,\\ \dfrac{4p^2t_2t_3}{(et_2t_3+t_2+t_3+e)^2}-\dfrac{2p\bar{\omega}}{et_2t_3+t_2+t_3+e}-\dfrac{2pt_2t_3\omega}{et_2t_3+t_2+t_3+e}=R^2-|\omega|^2,\\ \dfrac{4p^2t_3t_1}{(et_3t_1+t_3+t_1+e)^2}-\dfrac{2p\bar{\omega}}{et_3t_1+t_3+t_1+e}-\dfrac{2pt_3t_1\omega}{et_3t_1+t_3+t_1+e}=R^2-|\omega|^2.\end{array}\right.$$

From here we get the equalities:

$$\begin{aligned}\omega&=\frac{2p\left[e(e^2-2)t_1t_2t_3+t_1t_2+t_2t_3+t_3t_1+e(t_1+t_2+t_3)+e^2\right]}{(et_1t_2+t_1+t_2+e)(et_2t_3+t_2+t_3+e)(et_3t_1+t_3+t_1+e)},\\ \bar{\omega}&=\frac{2pt_1t_2t_3\left[e^2t_1t_2t_3+e(t_1t_2+t_2t_3+t_3t_1)+t_1+t_2+t_3+e(e^2-2)\right]}{(et_1t_2+t_1+t_2+e)(et_2t_3+t_2+t_3+e)(et_3t_1+t_3+t_1+e)},\end{aligned} \tag{10}$$

$$(11) \qquad R^2 - |\omega|^2 = \frac{4p^2\left(1-e^2\right)t_1t_2t_3}{\left(et_1t_2+t_1+t_2+e\right)\left(et_2t_3+t_2+t_3+e\right)\left(et_3t_1+t_3+t_1+e\right)}.$$

Let M_1, M_2 and M_3 be the feet of the perpendiculars from the focus O of κ, to the lines τ_1, τ_2 and τ_3, respectively. Consequently $\overrightarrow{OM_1} \perp \overrightarrow{A_2A_3}$, $\overrightarrow{OM_2} \perp \overrightarrow{A_3A_1}$ and $\overrightarrow{OM_3} \perp \overrightarrow{A_1A_2}$. From the condition of perpendicularity of vectors and the equalities (9) we obtain that the affixes m_1, m_2 and m_3 of the points M_1, M_2 and M_3, respectively are expressed by the equalities

$$(12) \qquad m_j = \frac{p}{e+t_j} \quad (j=1,2,3).$$

V.1. Relations between the inscribed parabolas in a triangle and its circumcircle. Taking into account that a conic section is a parabola exactly when $e=1$, it follows directly from (11) the following

Theorem 7. The conic section κ *inscribed in the triangle* Δ *is a parabola iff the focus of* κ *lies on the circumcircle* k *of* Δ (Fig. 13).

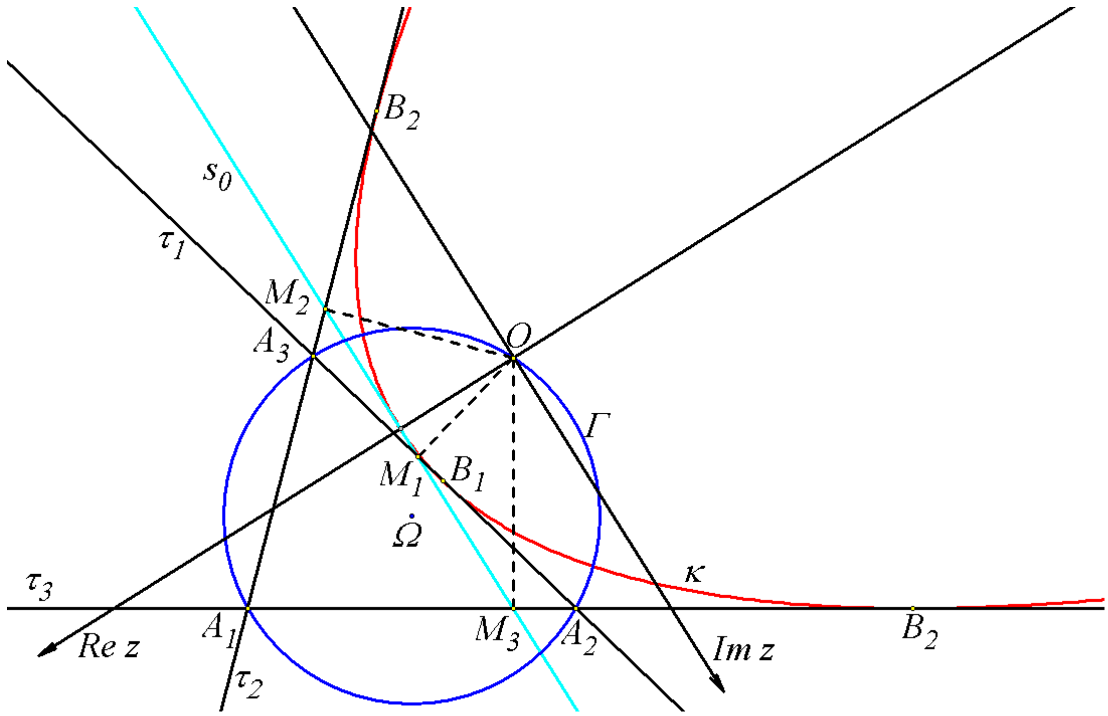

Figure 13

With respect to the coordinate system under consideration the vertex of the parabola κ has affix $\frac{p}{2}$ (Fig. 13). Since the tangent s_0 at the vertex is perpendicular to the axis of the parabola, then its equation is of the form

$$(13) \qquad s_0: \quad z+\overline{z}=p.$$

Replacing $e=1$ in (12), we see that the numbers m_1, m_2 and m_3 satisfy (13). The following theorem is true.

Theorem 8. *If the parabola κ is inscribed in the triangle Δ, then the Simson line s_0 of the focus of κ with respect to Δ is tangent to κ at its vertex* (Fig. 13).

V.2. Relations between the conic sections inscribed in a triangle and the pedal circles of their foci. Let P be a point in the plane of the triangle Δ, which is not on its circumcircle and the lines which determine Δ. If P_1, P_2 and P_3 are the orthogonal projections of P on the lines that determine Δ, then the circle passing through those points is called to be *pedal circle of the point P*.

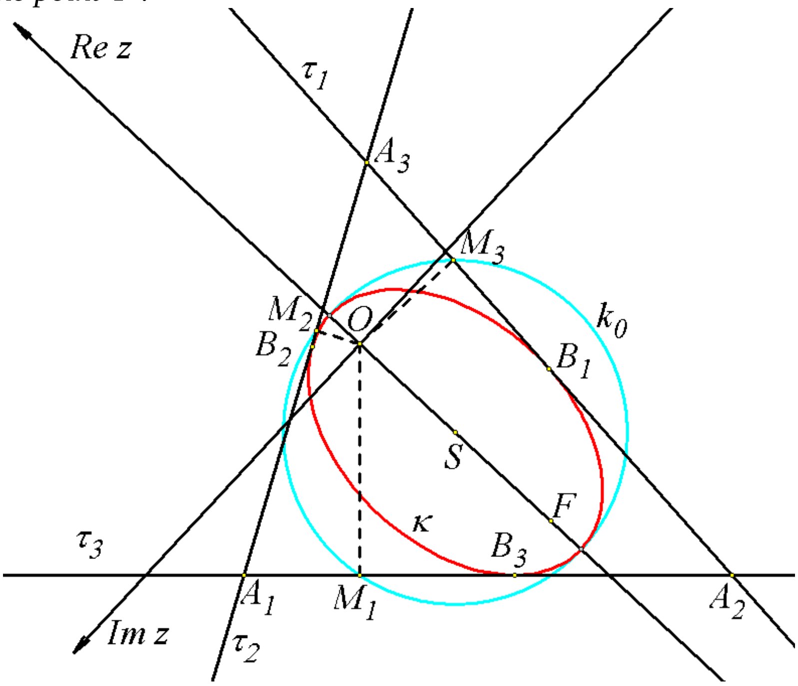

Figure 14

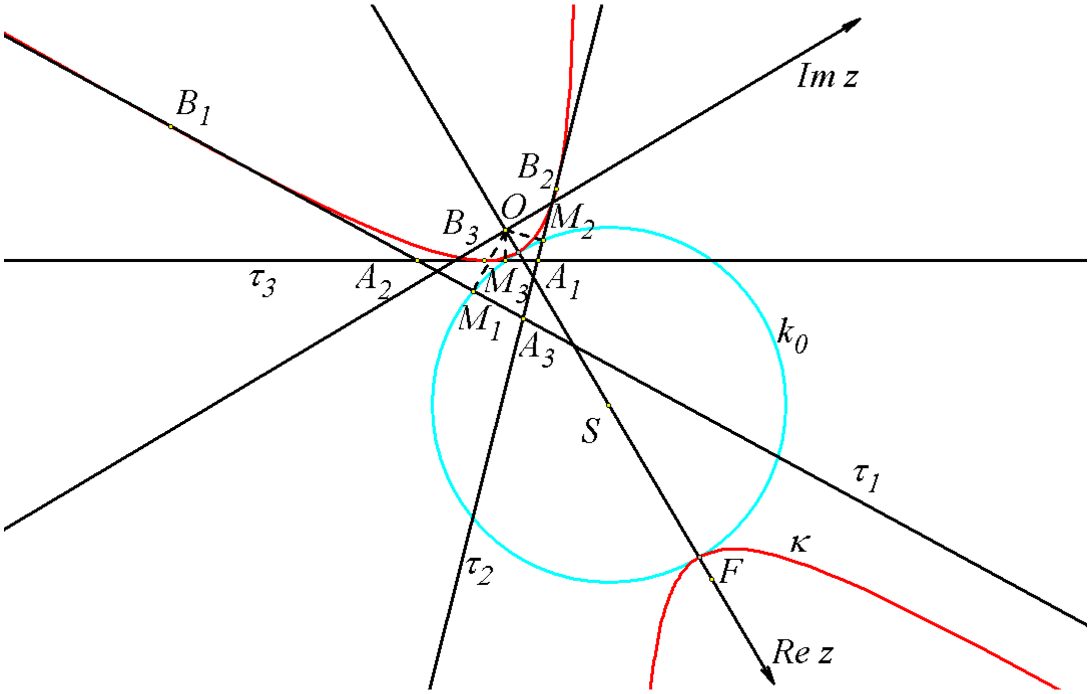

Figure 15

Let κ be ellipse or hyperbola with focus O and inscribed in $\Delta A_1A_2A_3$ (Fig. 14, 15, 16). Let S be the center and R_0 be the radius of the pedal circle k_0 of the focus O with respect to $\Delta A_1A_2A_3$. From (5) and (12) we obtain the following equation

$$(14)\qquad k_0: \quad z\bar{z} - \frac{pe}{e^2-1}z - \frac{pe}{e^2-1}\bar{z} = \frac{p^2}{1-e^2}.$$

Note from (14) that the following equalities are satisfied for the affix s of S and the radius R_0 of k_0

$$(15)\qquad s = \frac{pe}{e^2-1},$$

$$(16)\qquad R_0 = \frac{p}{\left|e^2-1\right|}.$$

It is seen from (15) that the center S of k_0 lies on the real axis of the coordinate system. Additionally, the system consisted of the equations (14) and (7) has only two solutions which are real and could be obtained when $t'=1$ and $t''=-1$. This shows that the pedal circle k_0 and κ are tangent (because $t'=1$ and $t''=-1$ are double roots of the system). The corresponding tangent points have affixes $z' = \dfrac{p}{e+1}$ and $z'' = \dfrac{p}{e-1}$. They are the affixes of the common points of κ and the real axis, i.e. of the vertices of κ on the real axis. Consequently, S is the center of κ and the other focus F of κ is symmetric to O with respect to S. This means that the points O and F are isogonally conjugated with respect to $\Delta A_1A_2A_3$.

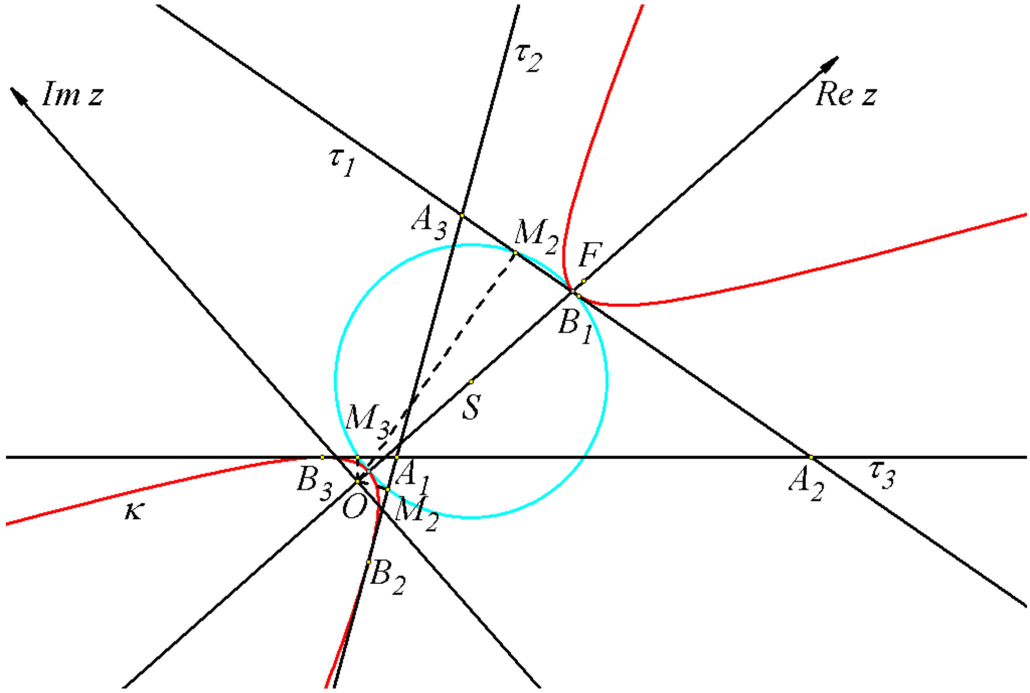

Figure 16

Now, let κ be hyperbola and $Z_0(t_0)$ be a point from its branch for which the equality (7) is satisfied. Then, from (15) we obtain that the point Z_0', which is symmetric to $Z_0(t_0)$ with respect to the center S of κ, has affix $z_0' = \dfrac{2p(et_0+1)^2}{(e^2-1)(et_0^2+2t_0+e)} = \dfrac{2p}{et_0'^2+2t_0'+e}$, where $t_0' = -\dfrac{t_0+e}{et_0+1}$. Since $|t_0'|=1$, the last equality shows that the point $Z_0'(t_0')$ lies on the other branch of the hyperbola and satisfies (7) when $t=t_0'$. Thus, we establish that the equation (7) describes both branches of the hyperbola. This means that in the case of hyperbola it is not necessary to consider different situations depending on the position of the hyperbola under consideration.

Accounting for the last conclusion and the results obtained before we reach the following

Theorem 9. *If a conic section κ inscribed in a triangle Δ is ellipse or hyperbola, then its foci are isogonally conjugated with respect to Δ, and their common pedal circle is tangent to κ at a point which lies on its focal axis* (Fig. 14, 15, 16).

VI. Proofs of theorems 1-6.

VI.1. Proof of Theorem 1. Using the considerations for the affix of the point B_1 we have $b_1 = \dfrac{2p}{et_1^2+2t_1+e}$, while the equalities (9) are satisfied for the affixes of the points A_1, A_2 and A_3. The affixes of M_1 and N_1 are respectively

$$m_1 = \frac{a_2+a_3}{2} = \frac{p(et_1t_2+et_1t_3+2t_1+t_2+t_3+2e)}{(et_1t_2+t_1+t_2+e)(et_3t_1+t_3+t_1+e)},$$

$$n_1 = \frac{a_1 + b_1}{2} = \frac{p\left(et_1^2 + et_2t_3 + 2t_1 + t_2 + t_3 + 2e\right)}{\left(et_1^2 + 2t_1 + e\right)\left(et_2t_3 + t_2 + t_3 + e\right)}.$$

If κ is ellipse or hyperbola with center S, it follows from the last two equalities and (15) that

$$\frac{n_1 - s}{m_1 - s} = \frac{\left(et_1t_2 + t_1 + t_2 + e\right)\left(et_3t_1 + t_3 + t_1 + e\right)}{\left(et_1^2 + 2t_1 + e\right)\left(et_2t_3 + t_2 + t_3 + e\right)} = \frac{\overline{n}_1 - s}{\overline{m}_1 - s},$$

which means that the points M_1, N_1 and S are collinear.

If κ is parabola, then $e = 1$ and $n_1 - m_1 = \dfrac{p\left(t_1 - t_2\right)\left(t_1 - t_3\right)}{\left(1 + t_1\right)^2\left(1 + t_2\right)\left(1 + t_3\right)} = \overline{n}_1 - \overline{m}_1$. This equality shows that the line M_1N_1 is parallel to the axis of the parabola. Thus, Theorem 1 is proved.

VI.2. Proof of Theorem 2. Under the assumed notations for the affixes of the points B_1 and B_3 we have respectively $b_1 = \dfrac{2p}{et_1^2 + 2t_1 + e}$ and $b_3 = \dfrac{2p}{et_3^2 + 2t_3 + e}$, while the equalities (9) are satisfied for the points A_1, A_2 and A_3. Let M_1 and M_2 be the midpoints of A_2A_3 and A_3A_1, respectively. We obtain for the equations of the lines B_1B_3 and M_1M_2 that

$$B_1B_3: \quad \left(2t_3t_1 + et_3 + et_1\right)z + \left(et_3 + et_1 + 2\right)\overline{z} = 2p\left(t_3 + t_1\right),$$

$$M_1M_2: \quad \left(t_3 + e\right)\left(et_1t_2 + t_1 + t_2 + e\right)z + \left(et_3 + 1\right)\left(et_1t_2 + t_1 + t_2 + e\right)\overline{z} =$$
$$= p\left(2et_1t_2t_3 + t_3^2 + t_1t_2 + t_2t_3 + t_3t_1 + 2et_3\right).$$

From those equations we find that the affix u of the common point U of B_1B_3 and M_1M_2 is expressed by the equality

$$(17) \qquad u = \frac{p\left(et_3^2 + et_1t_2 - et_2t_3 + et_3t_1 + 2t_3 + 2t_1 + 2e\right)}{\left(et_3^2 + 2t_3 + e\right)\left(et_1t_2 + t_1 + t_2 + e\right)}.$$

If κ is ellipse or hyperbola with center S, it follows from (15) and (17) that

$$\frac{a_1 - s}{u - s} = \frac{\left(et_3^2 + 2t_3 + e\right)\left(et_1t_2 + t_1 + t_2 + e\right)}{\left(et_2t_3 + t_2 + t_3 + e\right)\left(et_3t_1 + t_3 + t_1 + e\right)} = \frac{\overline{a}_1 - s}{\overline{u} - s},$$

which means that the points A_1, U and S are colinear.

If κ is parabola, when $e = 1$ we obtain from (17) that

$$u - a_1 = \frac{p\left(t_1 - t_3\right)\left(t_2 - t_3\right)}{\left(1 + t_1\right)\left(1 + t_2\right)\left(1 + t_3\right)^2} = \overline{u} - \overline{a}_1,$$

which means that the line A_1U is parallel to the axis of κ. Thus, Theorem 2 is proved.

VI.3. Proof of Theorem 3. Under the assumed notations for the affixes of the points B_1, B_2 and B_3 we have respectively $b_1 = \dfrac{2p}{et_1^2 + 2t_1 + e}$, $b_2 = \dfrac{2p}{et_2^2 + 2t_2 + e}$ and $b_3 = \dfrac{2p}{et_3^2 + 2t_3 + e}$. The equalities (9) are satisfied for the affixes of the points A_1, A_2 and A_3, while the affixes of the midpoints M_1, M_2 and M_3 of A_2A_3, A_3A_1 and A_1A_2, respectively are expressed by the equalities $m_1 = \dfrac{a_2+a_3}{2}$, $m_2 = \dfrac{a_3+a_1}{2}$ and $m_3 = \dfrac{a_1+a_2}{2}$.

Let κ be ellipse or hyperbola and $B_2B_3 \cap SB_1 = Q_1$. We find for the equations of the lines B_2B_3 and SB_1 that

$$B_2B_3: \quad \left(2t_2t_3 + et_2 + et_3\right)z + \left(et_2 + et_3 + 2\right)\overline{z} = 2p\left(t_2 + t_3\right),$$

$$SB_1: \quad \left[\left(e^2-2\right)t_1^2 - 2et_1 - e^2\right]z + \left(e^2t_1^2 + 2et_1 + 2 - e^2\right)\overline{z} = 2pe\left(t_1^2 - 1\right).$$

From those equations we obtain that the affix q_1 of the point Q_1 satisfies the equality

$$q = \frac{2p\left(et_1^2 + et_1t_2 + et_3t_1 + t_2 + t_3 + e\right)}{et_1t_2t_3\left(et_1+2\right) + et_1^2\left(t_2+t_3\right) + \left(2-e^2\right)\left(t_1^2 + t_2t_3\right) + 2e^2t_1\left(t_2+t_3\right) + \left(2t_1 + t_2 + t_3 + e\right)e}.$$

From here we get

$$\frac{a_1 - m_1}{a_1 - q_1} = \frac{\overline{a}_1 - \overline{m}_1}{\overline{a}_1 - \overline{q}_1} =$$

$$= \frac{et_1t_2t_3\left(et_1+2\right) + et_1^2\left(t_2+t_3\right) + \left(2-e^2\right)\left(t_1^2 + t_2t_3\right) + 2e^2t_1\left(t_2+t_3\right) + \left(2t_1 + t_2 + t_3 + e\right)e}{\left(et_1t_2 + t_1 + t_2 + e\right)\left(et_3t_1 + t_3 + t_1 + e\right)},$$

which means that the point Q_1 lies on the median A_1M_1. Consequently $Q_1 \equiv N_1$ and the line B_1N_1 passes through S.

Now, let κ be parabola and d_1 be the line passing through B_1, parallel to the axis of κ. Since $e=1$, the equations of the lines d_1 and B_2B_3 are the following

$$d_1: \quad z - \overline{z} = \frac{2p\left(1-t_1\right)}{1+t_1}, \quad B_2B_3: \quad \left(2t_2t_3 + t_2 + t_3\right)z + \left(t_2 + t_3 + 2\right)\overline{z} = 2p\left(t_2 + t_3\right).$$

If $d_1 \cap B_2B_3 = Q_1$, then $q_1 = \dfrac{2p\left(1 - t_1 + t_2 + t_3\right)}{\left(1+t_1\right)\left(1+t_2\right)\left(1+t_3\right)}$. It is easily seen from here that the equalities $\dfrac{a_1 - m_1}{a_1 - q_1} = \dfrac{1}{2} = \dfrac{\overline{a}_1 - \overline{m}_1}{\overline{a}_1 - \overline{q}_1}$ are satisfied. Consequently $Q_1 \equiv N_1$ and B_1N_1 is parallel to the axis of κ. Thus, Theorem 3 is proved.

VI.4. Proof of theorem 4. Denote the orientated areas of the triangles $B_2B_3A_1$, $B_3B_1A_2$, $B_1B_2A_3$, $A_1A_2A_3$ and $B_1B_2B_3$ by $\tilde{S}_1$, $\tilde{S}_2$, $\tilde{S}_3$, $\tilde{S}$ and $\tilde{\sigma}$, respectively. The orientated area of $A_1A_2A_3$ is determined in the following way

$$\tilde{S} = \frac{i}{4}\begin{vmatrix} a_1 & \overline{a}_1 & 1 \\ a_2 & \overline{a}_2 & 1 \\ a_3 & \overline{a}_3 & 1 \end{vmatrix} = \frac{ip^2\left(t_1-t_2\right)\left(t_2-t_3\right)\left(t_3-t_1\right)}{\left(et_1t_2 + t_1 + t_2 + e\right)\left(et_2t_3 + t_2 + t_3 + e\right)\left(et_3t_1 + t_3 + t_1 + e\right)}.$$

Analogously, we find

$$\tilde{\sigma} = -\frac{2ip^2\left(t_1-t_2\right)\left(t_2-t_3\right)\left(t_3-t_1\right)}{\left(et_1^2 + 2t_1 + e\right)\left(et_2^2 + 2t_2 + e\right)\left(et_3^2 + 2t_3 + e\right)},$$

$$\tilde{S}_1 = -\frac{ip^2\left(t_2-t_3\right)^3}{\left(et_2^2 + 2t_2 + e\right)\left(et_3^2 + 2t_3 + e\right)\left(et_2t_3 + t_2 + t_3 + e\right)},$$

$$\tilde{S}_2 = -\frac{ip^2\left(t_3-t_1\right)^3}{\left(et_3^2+2t_3+e\right)\left(et_1^2+2t_1+e\right)\left(et_3t_1+t_3+t_1+e\right)},$$

$$\tilde{S}_3 = -\frac{ip^2\left(t_1-t_2\right)^3}{\left(et_1^2+2t_1+e\right)\left(et_2^2+2t_2+e\right)\left(et_1t_2+t_1+t_2+e\right)}.$$

It follows easily from here that $\tilde{\sigma}^2\tilde{S} = -4\tilde{S}_1\tilde{S}_2\tilde{S}_3$. Consequently $\sigma = 2\sqrt{\frac{S_1S_2S_3}{S}}$. Thus, Theorem 4 is proved.

VI.5. Proof of theorem 5. Consider a coordinate system as shown in Fig. 17. The following equalities are satisfied with respect to it

$$b_j = \frac{2p}{et_j^2+2t_j+e} \quad \left(j=1,2,\ldots,n\right),$$

$$a_1 = \frac{2p}{et_nt_1+t_n+t_1+e}, \; a_j = \frac{2p}{et_{j-1}t_j+t_{j-1}+t_j+e} \quad \left(j=2,3,\ldots,n\right).$$

From the ratio $\frac{\overline{A_kB_j}}{\overline{A_lB_j}} = \frac{a_k-b_j}{a_l-b_j}$ for three points A_k, A_l and B_j we obtain the following relations:

$$\frac{\overline{A_1B_1}}{\overline{A_2B_1}} = \frac{t_n-t_1}{t_2-t_1}.\frac{a_1}{a_2}, \; \frac{\overline{A_kB_k}}{\overline{A_{k+1}B_k}} = \frac{t_{k-1}-t_k}{t_{k+1}-t_k}.\frac{a_k}{a_{k+1}} \; \left(k=2,3,\ldots,n-1\right), \; \frac{\overline{A_nB_n}}{\overline{A_1B_n}} = \frac{t_{n-1}-t_n}{t_1-t_n}.\frac{a_n}{a_1}.$$

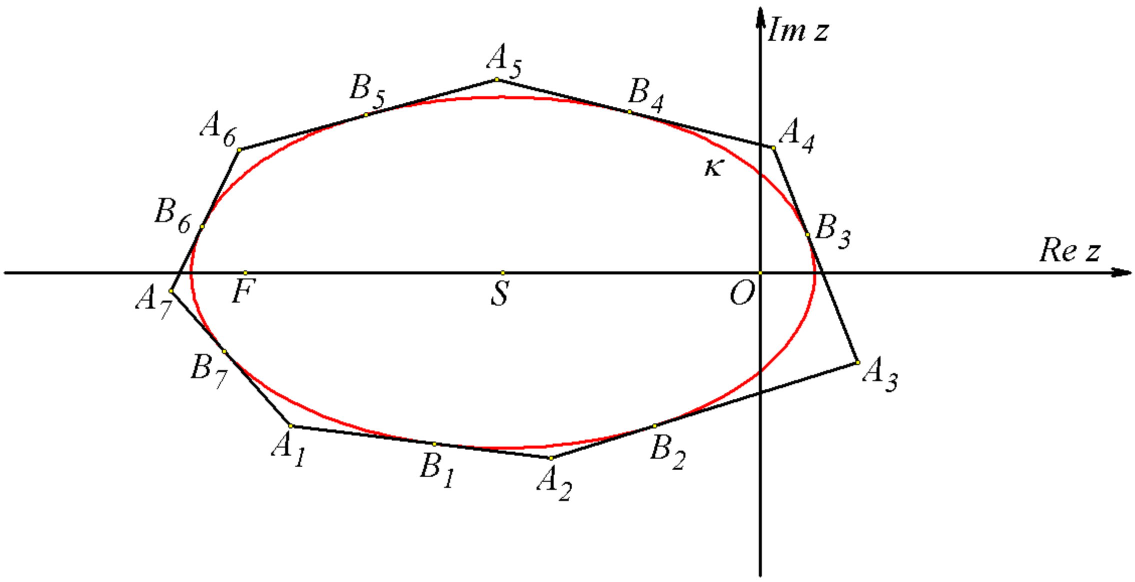

Figure 17

From we have $\frac{\overline{A_1B_1}}{\overline{A_2B_1}}.\frac{\overline{A_2B_2}}{\overline{A_3B_2}}\cdots\frac{\overline{A_{n-1}B_{n-1}}}{\overline{A_nB_{n-1}}}.\frac{\overline{A_nB_n}}{\overline{A_1B_n}} = \left(-1\right)^n$. Thus, Theorem 5 is proved.

VI.6. Proof of Theorem 6 and some consequences. We will use the notations in Fig. 18.

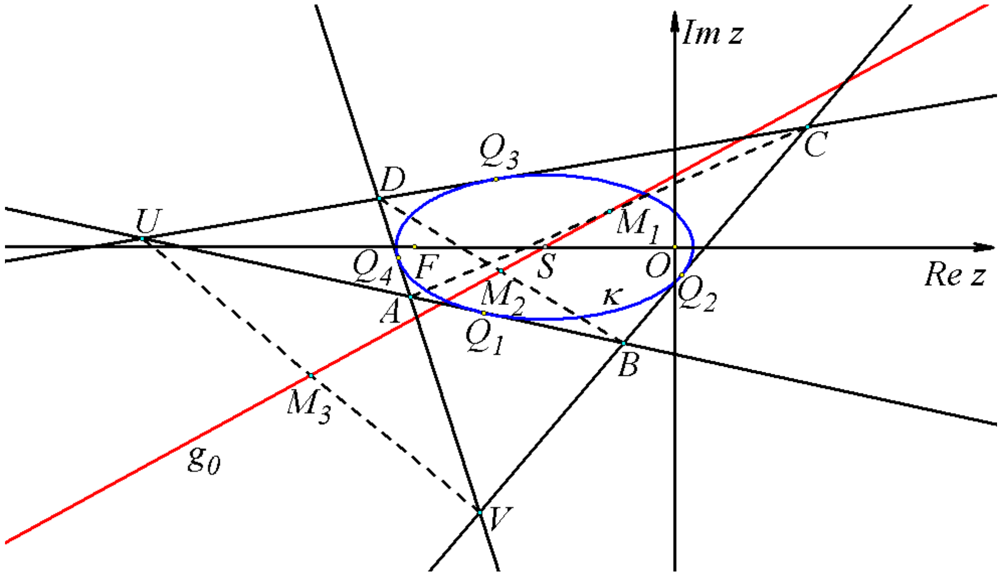

Figure 18

Let Q_1, Q_2, Q_3 and Q_4 be the tangent points of the conic section κ with the lines AB, BC, CD and DA respectively, while the equalities $q_j = \dfrac{2p}{et_j^2 + 2t_j + e}$ $(j = 1,2,3,4)$ are satisfied for their affixes. Consequently, the affixes of the points A, B, C, D, U and V are expressed by the equalities

$$a = \frac{2p}{et_4t_1 + t_4 + t_1 + e}, b = \frac{2p}{et_1t_2 + t_1 + t_2 + e}, c = \frac{2p}{et_2t_3 + t_2 + t_3 + e},$$

$$d = \frac{2p}{et_3t_4 + t_3 + t_4 + e}, u = \frac{2p}{et_1t_3 + t_1 + t_3 + e}, v = \frac{2p}{et_2t_4 + t_2 + t_4 + e}.$$

For the affixes of the points M_j $(j = 1,2,3)$ we have

$$m_1 = \frac{a+c}{2},\ m_2 = \frac{b+d}{2},\ m_3 = \frac{u+v}{2}$$

If the conic section κ is inscribed in $ABCD$ and if it is ellipse or hyperbola, then from the above equalities and (15) we obtain

$$\frac{m_1 - s}{m_2 - s} = \frac{(et_2t_3 + t_2 + t_3 + e)(et_4t_1 + t_4 + t_1 + e)}{(et_1t_2 + t_1 + t_2 + e)(et_3t_4 + t_3 + t_4 + e)} = \frac{\overline{m}_1 - s}{\overline{m}_2 - s},$$

$$\frac{m_1 - s}{m_3 - s} = \frac{(et_1t_3 + t_1 + t_3 + e)(et_2t_4 + t_2 + t_4 + e)}{(et_1t_2 + t_1 + t_2 + e)(et_3t_4 + t_3 + t_4 + e)} = \frac{\overline{m}_1 - s}{\overline{m}_3 - s}.$$

Consequently, the points M_1, M_2, M_3 and S are co-linear.

If κ is parabola, inscribed in $ABCD$, then $e = 1$ and we have the equalities

$$m_1 - m_2 = \frac{(t_1 - t_3)(t_2 - t_4)\,p}{(1+t_1)(1+t_2)(1+t_3)(1+t_4)} = \overline{m}_1 - \overline{m}_2,$$

$$m_1 - m_3 = \frac{(t_1 - t_4)(t_2 - t_3)\,p}{(1+t_1)(1+t_2)(1+t_3)(1+t_4)} = \overline{m}_1 - \overline{m}_3.$$

Consequently, the points M_1, M_2 and M_3 lie on a line, which is parallel to the axis of κ. Thus, Theorem 6 is proved.

Taking into account that the parabola could be considered as a conic section with infinite center – the infinity point of κ, the last result means that the center of the parabola κ is the infinity point of the line g_0. Each line has only one infinity point and we establish the following

Corollary 1. *Each quadrilateral without parallel sides has unique inscribed parabola* (Fig. 17).

Corollary 2. *If a given quadrilateral is circumscribed or escribed of circle, then the center of the circle is on the Gauss line for the quadrilateral.*

It is well known that the line defined by a vertex of a quadrilateral and the center of gravity of the triangle formed by the other three vertices passes through a constant point G of the quadrilateral which is called *med-center* of the quadrilateral. The affix of G is expressed by the equality $g = \frac{a+b+c+d}{4}$ It is easy to see that the point G is on the line g_0.

Accounting for the cases when $ABCD$ is a parallelogram, we obtain the following

Corollary 3. *The med-center of an arbitrary quadrilateral $ABCD$ is the center of a conic section which is inscribed in $ABCD$.*

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