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# Another Generalization of the Simson line

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**Abstract.** We introuduce a generalization of the Dao's generalization of the Simson line and a proof by Fedor Petrov.

**Keywords.** Simson line, collinear, orthopole, circumcenter

**Mathematics Subject Classification (2010).** 51-04, 68T01, 68T99.

## 1. Introduction

In 2014, I published without proof a remarkable generalization of the Simson line theorem:

**Theorem 1.1.** ([1]). *Let $ABC$ be a triangle, line $\ell$ pass through the circumcenter $O$; point $P$ lie on the circumcircle. Let $AP$, $BP$, $CP$ meet $\ell$ at $A'$, $B'$, $C'$, respectively. Denote $A_0$, $B_0$, $C_0$ the projections of $A'$, $B'$, $C'$ onto $BC$, $CA$, $AB$, respectively. Then $A_0$, $B_0$, $C_0$ are collinear. Moreover, the new line passes through the midpoint of $OH$, where $H$ the orthocenter of $ABC$ (Figure 1).*

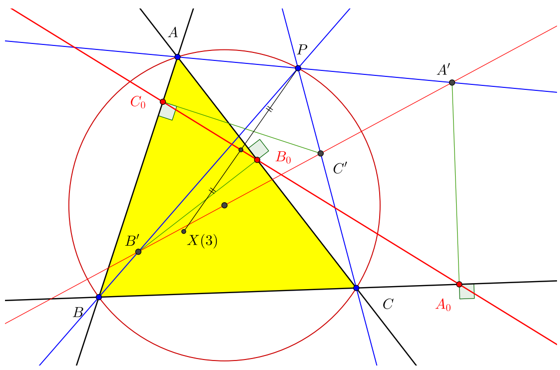


Figure 1.

**Remark.** *If $\ell$ passes through $P$, the line coincides with the Simson.*

There are many proof, You can see in [2]-[8].

The author also discovered another generalization of the Simson line and many other theorems, You can see in [9]-[12].

In this paper we introduce a projective of Theorem 1.1 is as follows:

**Theorem 1.2.** *([13]). Let $ABC$ be a triangle, let $P$ be a point in the circumcircle, the circumcenter is $O$. Let $Q$ be the point in the plane. The circles $(APQ)$, $(BPQ)$, $(CPQ)$ meet $OQ$ again at $A'$, $B'$, $C'$ respectively. Let $A_1$, $B_1$, $C_1$ be the projections of $A'$, $B'$, $C'$ onto $BC$, $CA$, $AB$ respectively. Then $A_1$, $B_1$, $C_1$ are collinear, and the new line through a fixed point on the Nine points circle when $Q$ be moved on the given line (or $P$ be moved in the circumcircle).*

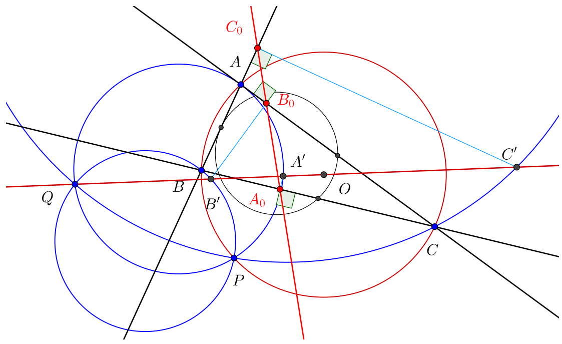


FIGURE 2.

**Remark.** *When Q in infinity, then the Theorem 1.2 become the Theorem 1.1*

In section 2, we introduce a proof of Theorem 1.2 by Fedor Petrov in [13].

## 2. PROOF THEOREM 1.2

Let $CC'$ meet a circle $\omega = (ABC)$ in a point $S \neq C'$. Then $\angle(CP, CS) = \angle(CP, CC') = \angle(QP, QC') = (QP, QO)$. Thus $AA'$, $BB'$ pass through the same point $S$. The following argument is not synthetic, but it explains what is this fixed point on an Euler circle and what is another point in which $A_1B_1C_1$ meets Euler circle. Thus it hopefully may help with a synthetic argument too.

Consider the complex coordinates for which $\omega = \{z : |z| = 1\}$, $A, B, C$ correspond to complex numbers $a, b, c$, $OQ$ to a real line, $S$ to $s$. Then $C'$ corresponds to $c' = (c+s)/(1+cs)$ (this is a formula for central projection from $\omega$ to a real line from the point $s$, as may be checked for three points $1, -1, -s$). Next, a projection of $z$ to a line between $a, b$ is $(z - \bar{z}ab + a + b)/2$, as may be checked for points $a, b, 0$. So, $C_1$ corresponds to $c_1 = (c'(1-ab) + a + b)/2$. Denote $c_2 = 2c_1 - (a+b+c)$. Note that $z \to 2z - (a+b+c)$ is a homothety which sends Euler circle of $ABC$ to $\omega$. Thus for points $a_2, b_2, c_2$ we should prove that they are collinear and the line passes through a point on $\omega$ not depending on $s$. We get $c_2 = c'(1-ab) - c$ and I claim that $c_2$ lies on a line between $s$ and $-abc$. Indeed, the direction between $s$ and $-abc$ is a direction of $s + abc$. The direction between $s$ and $c_2$ is a direction of $c_2 - s = -c'(ab + cs)$, that is, direction of $ab + cs$, but the ratio of $s + abc$ and $ab + cs$ is indeed real.

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