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Locus of the centroid of the equilateral triangle inscribed in an ellipse

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Abstract. An ellipse is the locus of the centroid of the equilateral triangle inscribed in an ellipse. The proof features some capabilities of the computer algebra system Maxima and the dynamic mathematics software GeoGebra.

Keywords. triangle geometry, locus, ellipse, conics, computer-discovered mathematics, Euclidean geometry, wxMaxima, GeoGebra.

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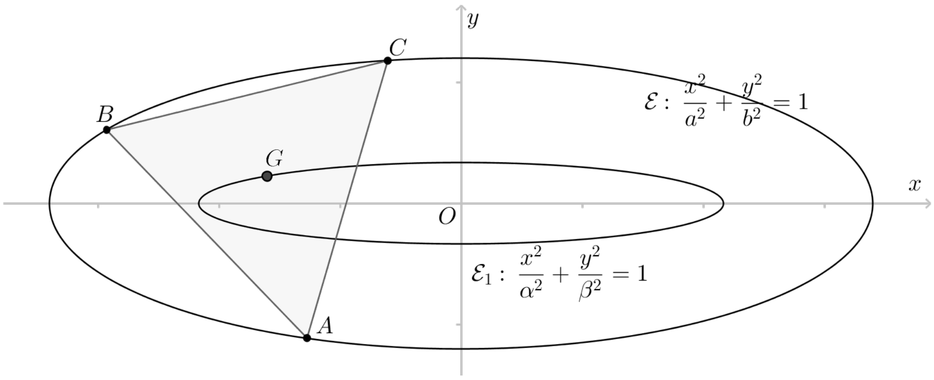
1. Introduction

It is well known the locus of the midpoint of the chord of an ellipse that is parallel to a given straight line. Moving a step ahead, one can calculate the locus of the centroid of the triangle that belongs to a family of triangles inscribed in an ellipse. It turns out that in many cases this locus is a very complicated set of points. Probably the simplest set is an ellipse which is obtained in the case of the family of equilateral triangles.

It seems that such a locus remains unexplored in the theory. The problem of describing this locus is a difficult one even in the case of equilateral triangles. Below, in the presented proof, there are some key computations which are rather complicated to be done by hand and they can be done in a reasonable period of time only using a computer algebra system.



The main theorem is Theorem 2.1 and it states that it is an ellipse which represents the locus $\mathcal{E}_1$ of the centroid of the equilateral triangle inscribed in a given ellipse $\mathcal{E}$. Every ellipse can serve as a such a locus $\mathcal{E}_1$ and for every $\mathcal{E}_1$ there exist only one ellipse $\mathcal{E}$ that has $\mathcal{E}_1$ as the corresponding locus.

The proof is, in fact, a script file that is executable in Maxima[10]. It seems reasonable to represent all comments as a mathematical argument for the validity of the theorem and to structure the proof as a series of lemmas. Thus, hopefully, the quality of the work has been improved.

The methods of Cartesian geometry are applied to reformulate the problem in terms of equations on coordinates of the points under consideration. Having noticed some symmetry of equations under permutations of points, it is quite natural to be aimed at some equations of elementary symmetric polynomials on three variables instead. Nevertheless, there are some quite complicated formulas. Note that the formulas do not represent in an obvious way geometrical properties of the ellipses or triangles under consideration and it will be a surprise to obtain shorter and smarter proof (for example, a proof that is based on the methods of the synthetic geometry).

We have found the locus of the centroid of equilateral triangles inscribed in a hyperbola and in a parabola, too. These two conics have been treated in the same way, but there are differences. In the case of inscribing triangles in a hyperbola, the locus can be a hyperbola (it is the same one when it is an equilateral hyperbola) or a pair of parallel straight lines, perpendicular to the major axis. If the triangles are inscribed in a parabola then the corresponding locus is a parabola, too. The latter case is less difficult than the former. In the current work, we will not state these results.

In general, there are theorems about some properties of triangles inscribed in ellipses or when they are placed in a some other way. For example, Marden's theorem [1, 2] is about foci of an ellipse that is tangential to the sides of a triangle at midpoints. Another example of a special relationship is the Steiner's ellipse [3, 4, 5] and an inscribed triangle with the maximal area. A list of remarkable triangle conics is compiled by E. Weinstein at mathworld.wolfram.com [6]. The Feuerbach's Conic Theorem [8] is a very impressive one. The theory about the geometry of triangles, conics and other types of lines that are in some special relation between each other, such a theory is in fact quite branched, difficult and interesting, especially when the computer algebra systems tend to become more and more advanced (see [7, 9]).

2. Main theorem

Theorem 2.1. *Suppose that in the Cartesian coordinate system Oxy ellipses $\mathcal{E}$ and $\mathcal{E}_1$ have their equations in the canonical form*

$$\mathcal{E} : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \mathcal{E}_1 : \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$$

where $a > b > 0$ and

$$\alpha = \frac{a(a^2 - b^2)}{3b^2 + a^2}, \quad \beta = \frac{b(a^2 - b^2)}{b^2 + 3a^2}. \tag{1}$$

If A, B and C are three different points such that the set of points $\{A, B, C\} \subset \mathcal{E}$ and the triangle $\triangle ABC$ is equilateral then the centroid G of $\triangle ABC$ belongs to $\mathcal{E}_1$.

If $\mathcal{E}_1$ is given with $\alpha > \beta > 0$ then then there exists an unique pair of real numbers a and b that complain with (1). Futhermore, for any point G from $\mathcal{E}_1$ there exist three points A, B and C such that $\{A, B, C\} \subset \mathcal{E}$, the triangle $\triangle ABC$ is equilateral and its centroid is G.

3. Proof of the main theorem

Here is the first command in Maxima.

```
(%i1) kill(all)$ reset()$
```

Claim 1. There exist equilateral triangles with vertices that are three different points on $\mathcal{E}$.

Indeed,

- Let $A_0(a, 0)$, $B_0\left(\frac{a(a^2-3b^2)}{3b^2+a^2}, \frac{2\sqrt{3}ab^2}{3b^2+a^2}\right)$, $C_0\left(\frac{a(a^2-3b^2)}{3b^2+a^2}, -\frac{2\sqrt{3}ab^2}{3b^2+a^2}\right)$. Then these points are all different, they all belong to $\mathcal{E}$, the triangle $\triangle A_0B_0C_0$ is equilateral and has its centroid on $\mathcal{E}_1$. The triangle is equilateral because of

```
(%i2) block([xB0,yB0,AB,BC,CA],
       xB0:a*(a^2-3*b^2)/(3*b^2+a^2),
       yB0:2*sqrt(3)*a*b^2/(3*b^2+a^2),
```

```
A0:[a,0],  B0:[xB0,yB0],  C0:[xB0,-yB0],
AB:B0-A0,  BC:C0-B0,  CA:A0-C0,
ratsimp([AB.AB-BC.BC, CA.CA-BC.BC]) );
```

(%o2) $[0,0]$

i.e. $\overrightarrow{A_0B_0}^2-\overrightarrow{B_0C_0}^2=0$, $\overrightarrow{C_0A_0}^2-\overrightarrow{B_0C_0}^2=0$. The centroid G_0 of the triangle $\triangle A_0B_0C_0$ belongs to the ellipse $\mathcal{E}_1$ since its coordinates satisfy the equation of $\mathcal{E}_1$.

```
(%i3) block([G0,α,β], G0:1/3*(A0+B0+C0),
      α:a*(a^2-b^2)/(3*b^2+a^2),
      β:b*(a^2-b^2)/(b^2+3*a^2),
      ratsimp(G0[1]^2/α^2+G0[2]^2/β^2-1) );
```

(%o3) 0

- Let $A_1(-a,0)$, $B_1\left(-\frac{a(a^2-3b^2)}{3b^2+a^2},\ \frac{2\sqrt{3}ab^2}{3b^2+a^2}\right)$, $C_1\left(-\frac{a(a^2-3b^2)}{3b^2+a^2},\ -\frac{2\sqrt{3}ab^2}{3b^2+a^2}\right)$. Then these points are all different, they all belong to $\mathcal{E}$, the triangle $\triangle A_1B_1C_1$ is equilateral and has its centroid on $\mathcal{E}_1$. Note, $\triangle A_1B_1C_1$ and $\triangle A_0B_0C_0$ are symmetrical with respect to the y-axis and as a consequence of the previous item it follows that the triangle $\triangle A_1B_1C_1$ has the stated properties.
- It is possible to prove that for every point $A\in\mathcal{E}$ there exist two different points B and C such that each one of them lies on $\mathcal{E}$ and the triangle $\triangle ABC$ is equilateral. This could be proved if the every step of following algorithm is analyzed and proved. Now, there is no such proof and the details are omitted. It is not relevant for the proof of Theorem 2.1. Nevertheless, the algorithm is used for producing a dynamic figure [11] in GeoGebra and the steps are the following.

(Step 1.) Let the ellipse $\mathcal{E}'=\rho_A^{+60^o}\mathcal{E}$, where $\rho_A^{+60^o}$ is the rotation to $+60^o$ around the point A.
(Step 2.) Let $\mathcal{P}=\mathcal{E}'\cap\mathcal{E}$. The set $\mathcal{P}$ contains at least two different points, one of which is A.
(Step 3.) Let the point $B\in\mathcal{P}\setminus\{A\}$.
(Step 4.) Let $C=\rho_A^{-60^o}B$.

Then, the triangle $\triangle ABC$ is equilateral and its centroid is on $\mathcal{E}_1$. The proof is omitted because this algorithm is not used in the proof of Theorem 2.1.

Thus, $\mathcal{F}\neq\emptyset$, where $\mathcal{F}$ is the family of all equilateral triangles with vertices that are three different points on $\mathcal{E}$.

```
(%i4) kill(A0,B0,C0)$
```

Claim 2. If the point $G\in\mathcal{E}_1$ and the points A, B and C are such that $\{A,B,C\}\subset\mathcal{E}$, $\triangle ABC$ is equilateral and its centroid is G then G, A, B and C are four different points.

Indeed, if any two of the points coincide then all four coincide and the two ellipses $\mathcal{E}$ and $\mathcal{E}_1$ have a common point which is impossible: the system of their equations

$$\left|\begin{array}{l} b^2x^2+a^2y^2=a^2b^2 \\ \beta^2x^2+\alpha^2y^2=\alpha^2\beta^2 \end{array}\right.$$

is equivalent to

$$\left|\begin{array}{l} x^2=\frac{a^4}{a^2-b^2} \\ y^2=\frac{b^4}{b^2-a^2} \end{array}\right.$$

which does not have any real solution since $\frac{b^4}{b^2-a^2}<0$.

Note that for every ellipse $\mathcal{E}_1$ (α and β are given real numbers such that $\alpha>\beta>0$) there exists an unique pair of real numbers a and b that complain with (1). Futhermore, for any point G from $\mathcal{E}_1$ there exist three points A, B and C such that $\{A,B,C\}\subset\mathcal{E}$, the triangle $\triangle ABC$ is equilateral and its centroid is G. This assertion will not be proved here because it is not used in the proof of Theorem 2.1. It could be drown from the following algorithm. First, a and b are calculated by solving the system (1). Second, let $\mathcal{E}''=\rho_G^{+120^o}\mathcal{E}$. Third, the set $\mathcal{E}''\cap\mathcal{E}$ contains four different points. Fourth, there are three of them which are vertices of a triangle from the family $\mathcal{F}$ with its centroid at G. This algorithm is used for producing a dynamic figure in GeoGebra, only.

Let $\widetilde{\mathcal{E}}$ stand for $\mathcal{E}\setminus\{(-a,0)\}$ and $\widetilde{\mathcal{E}}=\mathcal{E}_1\setminus\{(-\alpha,0)\}$.

The map $t \mapsto (x(t), y(t))$ is one-to-one from $(-\infty; +\infty)$ onto $\widetilde{\mathcal{E}}$, where

$$x(t) = a\frac{1-t^2}{1+t^2}, \quad y(t) = b\frac{2t}{1+t^2}.$$

The points $A(x_A, y_A)$, $B(x_B, y_B)$ and $C(x_C, y_C)$ are three different points on $\widetilde{\mathcal{E}}$ iff there exist three different real numbers t_A, t_B and t_C such that

$$x_A = x(t_A), y_A = y(t_A), x_B = x(t_B), y_B = y(t_B), x_C = x(t_C), y_A = y(t_C). \tag{2}$$

Let $A(x_A, y_A)$, $B(x_B, y_B)$ and $C(x_C, y_C)$ be three different points on $\widetilde{\mathcal{E}}$ and let t_A, t_B and t_C be three real numbers such that (2) holds on.

Here, in the proof of the theorem, the following notations are used

$$\begin{aligned} S_x &= x_A + x_B + x_C, \quad S_y = y_A + y_B + y_C,\\ S_{1t} &= t_A + t_B + t_C,\\ S_{2t} &= t_A t_B + t_B t_C + t_C t_A,\\ S_{3t} &= t_A t_B t_C \end{aligned} \tag{3}$$

and

$$\begin{aligned} \lambda &= \frac{b^2 - 3a^2}{3(b^2 - a^2)}, \quad \mu = \frac{3b^2 - a^2}{3(b^2 - a^2)},\\ X &= -\tfrac{1}{3}S_x S_y, \quad Y = \lambda S_y, \quad Z = \mu S_x. \end{aligned} \tag{4}$$

Hence, λ and μ are such that $\lambda + \mu = \frac{4}{3}$, $\lambda > 1$. The values of X, Y and Z do not change when the points A, B and C are interchanged in any arbitrary way.

Let the point $G(x_G, y_G)$ be the centroid of the triangle $\triangle ABC$. Hence,

$$3x_G = S_x, \quad 3y_G = S_y$$

and

$$X = -3x_G y_G, \quad Y = 3\lambda y_G, \quad Z = 3\mu x_G.$$

The following notations are used in the script file `xA` stands for x_A and y_A, x_B, y_B, x_C, y_C, t_A, t_B, t_C are denoted in a similar way; `S1X` stands for S_x; `S1Y` $\longleftrightarrow S_y$; `S1T` $\longleftrightarrow S_{1t}$; `S2T` $\longleftrightarrow S_{2t}$; `S3T` $\longleftrightarrow S_{3t}$.

```
(%i5) load(sym)$
(%i6) x(t):=a*(1-t^2)/(1+t^2)$
(%i7) y(t):=b*2*t/(1+t^2)$
(%i8) tData:[xA=x(tA),yA=y(tA),xB=x(tB),yB=y(tB),xC=x(tC),yC=y(tC)]$
```

Lemma 3.1. *Suppose $A(x_A, y_A)$, $B(x_B, y_B)$ and $C(x_C, y_C)$ are three different points that belong to $\mathcal{E}$. The triangle $\triangle ABC$ is equilateral iff*

$$\left| \begin{aligned} &(x_B + x_C - 2x_A)a^2(y_B + y_C) - b^2(x_B + x_C)(y_B + y_C - 2y_A) = 0\\ &(x_A + x_C - 2x_B)a^2(y_A + y_C) - b^2(x_A + x_C)(y_A + y_C - 2y_B) = 0\\ &(x_B + x_A - 2x_C)a^2(y_B + y_A) - b^2(x_B + x_A)(y_B + y_A - 2y_C) = 0. \end{aligned} \right. \tag{5}$$

Proof of Lemma 3.1. The coordinates of the points are such that

$$b^2x_A^2 + a^2y_A^2 = a^2b^2 \tag{6}$$

$$b^2x_B^2 + a^2y_B^2 = a^2b^2 \tag{7}$$

$$b^2x_C^2 + a^2y_C^2 = a^2b^2 \tag{8}$$

The proof of the lemma is divided in two parts.

Part 1. Suppose $\triangle ABC$ is equilateral. It has to be proved that (5) is satisfied. Thus, as it is supposed,

$$|AB| = |AC|, \qquad |AB| = |BC|, \qquad |BC| = |AC|.$$

It is enough to show the proof of the first equation of (5) since there are similar arguments that prove the other two equations. So, the claim is to verify the truth of the first equation of (5).

It follows from $|AB| = |AC|$, (7) and (8) that

$$\left| \begin{array}{l} (x_B - x_A)^2 + (y_B - y_A)^2 - (x_C - x_A)^2 - (y_C - y_A)^2 = 0 \\ b^2(x_B^2 - x_C^2) + a^2(y_B^2 - y_C^2) = 0 \end{array} \right.$$

and hence,

$$\text{(9)} \qquad \left| \begin{array}{l} (x_B - x_C)(x_B + x_C - 2x_A) + (y_B - y_C)(y_B + y_C - 2y_A) = 0 \\ b^2(x_B - x_C)(x_B + x_C) + a^2(y_B - y_C)(y_B + y_C) = 0. \end{array} \right.$$

Now, it is necessary to consider the following two cases separately: $x_B \neq x_C$ and $x_B = x_C$.

Case $x_B \neq x_C$. The first equation of (9) is multiplied by $a^2(y_B + y_C)$, the second equation of (9) is multiplied by $(y_B + y_C - 2y_A)$. Next, the latter is subtracted from the former equation to obtain

$$(x_B - x_C)(x_B + x_C - 2x_A)a^2(y_B + y_C) - b^2(x_B - x_C)(x_B + x_C)(y_B + y_C - 2y_A) = 0$$

and the last one implies the first equation of (5).

Case $x_B = x_C$. In the current case, it follows that $y_B = -y_C$ and $y_C \neq 0$ because if the opposite is true i.e. either $y_B = y_C$ or $y_C = 0$ then the points B and C coincide which contradicts to the assumption that A, B and C are three different points. Now, it follows from the first equation of (9) that $(-2y_C)(-2y_A) = 0$ and hence, $y_A = 0$. Thus, $y_A = 0$, $y_B = -y_C$ and a simple check shows that the first equation of (5) is satisfied.

Therefore, the first equation of (5) follows from $|AB| = |AC|$, (7) and (8).

The other equations of (5) are deduced in a similar way.

So, the proof of Part 1 is over.

Part 2. Suppose the coordinates of the points satisfy (5). It has to be proved that $\triangle ABC$ is equilateral.

Accordingly to the assumption of the lemma, A, B and C are three different points on $\mathcal{E}$. Hence, the point $O(0,0)$ does not belong to two sides of the triangle $\triangle ABC$, at least.

So, without loss of generality, let $O(0,0)$ does not belong to AC and BC. The conclusion $\triangle ABC$ is equilateral is to be drawn from the first and the second equation of (5). It follows $|AB| = |AC|$ from the first equation of (5) and in a similar way, $|AB| = |BC|$ follows from the second one. To end the proof it is enough to show the proof of the former, only. The rest of the proof is structured in two cases and the second case is branched into two sub-cases.

Case $y_B + y_C \neq 0$. First equation of (5) is multiplied by $x_B - x_C$ and then $(x_B - x_C)b^2(x_B + x_C)$ is substituted with $-a^2(y_B - y_C)(y_B + y_C)$ to obtain

$$(x_B - x_C)(x_B + x_C - 2x_A)a^2(y_B + y_C) + a^2(y_B - y_C)(y_B + y_C)(y_B + y_C - 2y_A) = 0.$$

Hence,

$$\text{(10)} \qquad (x_B - x_C)(x_B + x_C - 2x_A) + (y_B - y_C)(y_B + y_C - 2y_A) = 0.$$

Case $y_B + y_C = 0$. In this case, $x_B = x_C$ and moreover, $y_C \neq 0$. Now, it follows from (5) that

$$\text{(11)} \qquad \left| \begin{array}{l} -b^2(2x_C)(-2y_A) = 0 \\ (x_A - x_C)a^2(y_A + y_C) - b^2(x_A + x_C)(y_A + 3y_C) = 0 \\ (x_A - x_C)a^2(-y_C + y_A) - b^2(x_C + x_A)(y_A - 3y_C) = 0. \end{array} \right.$$

Sub-case $x_C \neq 0$. It follows from (11) that $y_A = 0$ and hence, (10) is satisfied.

Sub-case $x_C = 0$. Now, $x_B = 0$, $y_B = \pm b$, $y_C = \mp b$. Moreover, $x_A \neq 0$ (it is impossible $x_A = 0$ since A, B and C are three different points on $\mathcal{E}$). In this sub-case, it follows from (11) that

$$\left| \begin{array}{l} a^2(y_A + y_C) - b^2(y_A + 3y_C) = 0 \\ a^2(-y_C + y_A) - b^2(y_A - 3y_C) = 0 \end{array} \right.$$

and hence,

$$\left| \begin{array}{l} (a^2 - b^2)y_A + (a^2 - 3b^2)y_C = 0 \\ (a^2 - b^2)y_A - (a^2 - 3b^2)y_C = 0. \end{array} \right.$$

This system has a solution if $a^2 = 3b^2$, only. So, here, $a^2 = 3b^2$ and the solution is $y_A = 0$. Therefore, (10) is satisfied.

Thus, in both cases, (10) follows the first equation of (5). Note, that (10) is equivalent to $|AB|^2 = |AC|^2$.

In a similar way, $|AB|^2 = |BC|^2$ follows from the second equation of (5).

Therefore, $\triangle ABC$ is equilateral. $\square$

Lemma 3.2. *Suppose $A(x_A, y_A)$, $B(x_B, y_B)$ and $C(x_C, y_C)$ are three different points on $\mathcal{E}$ and X, Y, Z are calculated accordingly to (3) and (4). The triangle $\triangle ABC$ is equilateral iff X, Y and Z satisfy the system*

$$\left| \begin{array}{l} X + Yx_A + Zy_A = x_A y_A \\ X + Yx_B + Zy_B = x_B y_B \\ X + Yx_C + Zy_C = x_C y_C. \end{array} \right. \tag{12}$$

Proof of Lemma 3.2. The first equation of (5) is equivalent to the first equation of (12). Indeed, it follows from $x_B + x_C = S_x - x_A$, $y_B + y_C = S_y - y_A$ and he first equation of (5) that

$$\begin{aligned} & (S_x - 3x_A)a^2(S_y - y_A) - b^2(S_x - x_A)(S_y - 3y_A) = 0 \\ \Longrightarrow\ & (a^2 - b^2)S_x S_y + (-3a^2 + b^2)S_y x_A + (-a^2 + 3b^2)S_x y_A + 3(a^2 - b^2)x_A y_A = 0 \\ \Longrightarrow\ & -\frac{1}{3}S_x S_y + \frac{b^2 - 3a^2}{3(b^2 - a^2)}S_y x_A + \frac{3b^2 - a^2}{3(b^2 - a^2)}S_x y_A = x_A y_A. \end{aligned}$$

The last one is exactly the first equation of (12).

The two other equations of (5) are treated in a similar way and details are omitted. $\square$

Remark. *The system (12) has unique solution. Indeed, the determinant*

$$D = \begin{vmatrix} 1 & x_A & y_A \\ 1 & x_B & y_B \\ 1 & x_C & y_C \end{vmatrix} = \pm 2(\text{ “area of } \triangle ABC\text{” }) \neq 0,$$

because it is not possible for three different points on $\mathcal{E}$ to be collinear. Hence, the solution is given by the Cramer's rule and it is used bellow in the proof of the theorem.

Remark. *Suppose A, B, C are three different points on $\mathcal{E} \setminus \{(-a, 0)\}$. The solution of (12) does not depend on the permutation of the three points A, B, C. Accordingly to the Cramer's rule, each one of X, Y and Z is expressed as a fraction of determinants which, of course, change the sign when two of points are interchanged. By now, X, Y and Z are expressed with formulas that include λ, β and x_A, y_A, x_B, y_B, x_C, y_C. All these six coordinates are substituted accordingly to (2). Thus, the numerators and the denominators in the Cramer's rule are expressed in terms of t_A, t_B, t_C. Both have the common factor $(t_B - t_A)(t_C - t_A)(t_C - t_B)$. This factor is not equal to 0, it cancels out and the remaining polynomials in both, the numerators and the denominators, are symmetric polynomials. Next, this symmetric polynomials are rewritten in terms of elementary symmetric polynomials S_{1t}, S_{2t} and S_{3t}. As a result, X, Y and Z depend on S_{1t}, S_{2t} and S_{3t}. Later, this is transformed to obtain S_{1t} and S_{3t} in terms of S_{2t}.*

Lemma 3.3. *Suppose A, B, C are three different points on $\mathcal{E} \setminus \{(-a, 0)\}$. and X, Y, Z are calculated accordingly to (3) and (4). The triangle $\triangle ABC$ is equilateral iff S_{1t}, S_{2t} and S_{3t} satisfy the system*

$$\left| \begin{array}{l} 9S_{3t}^3 - 7S_{1t}S_{2t}S_{3t}^2 - 12S_{1t}S_{3t}^2 + 2S_{2t}^3 S_{3t} - 5S_{2t}^2 S_{3t} + 6S_{1t}^2 S_{2t} S_{3t} + 12 S_{2t} S_{3t} \\ \qquad + 5S_{1t}^2 S_{3t} - 9S_{3t} - S_{1t}S_{2t}^3 - 6S_{1t}S_{2t}^2 + S_{1t}^3 S_{2t} + 7S_{1t}S_{2t} - 2S_{1t}^3 = 0 \\ S_{2t}S_{3t}\lambda - 3S_{3t}\lambda + S_{1t}S_{2t}\lambda + S_{1t}\lambda - S_{2t}S_{3t} + 2S_{3t} - S_{1t} = 0 \\ 3S_{3t}^2\mu - 2S_{1t}S_{3t}\mu + S_{2t}^2\mu + 2S_{2t}\mu - S_{1t}^2\mu - 3\mu - S_{3t}^2 - S_{2t}^2 + S_{1t}^2 + 1 = 0 \\ (S_{1t} - S_{3t})^2 + (S_{2t} - 1)^2 > 0. \end{array} \right. \tag{13}$$

Proof of Lemma 3.3. By Lemma 3.2, the system (12) has to be solved and the solution for X, Y and Z is substituted in (4).

```
(%i10)  sys:[X+Y*xA+Z*yA=xA*yA,
            X+Y*xB+Z*yB=xB*yB,
            X+Y*xC+Z*yC=xC*yC]$
(%i11)  determinant(submatrix(augcoefmatrix(sys,[X,Y,Z]),4));
```

(%o11) $-xA(yC-yB)+xByC-xCyB+(xC-xB)yA$

```
(%i12)  D:%,tData,factor;
```

(D) $\frac{4ab(tB-tA)(tC-tA)(tC-tB)}{(tA^2+1)(tB^2+1)(tC^2+1)}$

```
(%i13) solve(sys,[X,Y,Z])$
(%i14) [X,Y,Z],%$
(%i15) %,tData,ratsimp$
(%i16) block([n,d,r],r:[],
           for s in % do(
            contract(num(s),[tA,tB,tC]),
            n:elem([3,S1T,S2T,S3T],%%,[tA,tB,tC]),
            contract(denom(s),[tA,tB,tC]),
            d:elem([3,S1T,S2T,S3T],%%,[tA,tB,tC]),
            r:append(r,[factor(n)/factor(d)])), r );
```

(%o16) $[\frac{-2ab(S2T\,S3T-S1T)}{S3T^2-2S1T\,S3T+S2T^2-2S2T+S1T^2+1}, \frac{-2b(S2T\,S3T-2S3T+S1T)}{S3T^2-2S1T\,S3T+S2T^2-2S2T+S1T^2+1}, \frac{a(S3T^2+S2T^2-S1T^2-1)}{S3T^2-2S1T\,S3T+S2T^2-2S2T+S1T^2+1}]$

```
(%i17) solXYZ:%$
(%i18) [-1/3*(xA+xB+xC)*(yA+yB+yC),λ*(yA+yB+yC),μ*(xA+xB+xC)],tData,ratsimp$
(%i19) ''(%i16)$
(%i20) %-solXYZ,ratsimp$
(%i21) map('num,%)$
(%i22) [%[1]/(-2*a*b),%[2]/(2*b),%[3]/(-a)],factor;
       /* The result is in the left hand side */
       /* of the first three equations of (13). */
```

It follows from $(S_{1t}-S_{3t})^2+(S_{2t}-1)^2=(tA^2+1)(tB^2+1)(tC^2+1)>0$ the inequality of (13). □

Remark. *The equations of* (13) *are obtained by the corresponding equations of the system*

$$\left|\begin{array}{l} \frac{-1}{3}S_xS_y = (\text{“}X \text{ from the solution of } (12)\text{”}) \\ \lambda S_y = (\text{“}Y \text{ from the solution of } (12)\text{”}) \\ \mu S_x = (\text{“}Z \text{ from the solution of } (12)\text{”}). \end{array}\right.$$

Lemma 3.4. *Suppose A, B, C are three different points on $\mathcal{E}\setminus\{(-a,0)\}$. The triangle $\triangle ABC$ is equilateral iff S_{1t}, S_{2t} and S_{3t} satisfy the system*

$$\left|\begin{array}{l} S_{1t}^2 = -\frac{(S_{2t}\lambda-3\lambda-S_{2t}+2)^2(3S_{2t}\lambda+9\lambda-S_{2t}-9)}{(2\lambda-1)(6S_{2t}\lambda^2+18\lambda^2-5S_{2t}\lambda-27\lambda-S_{2t}+11)} \\ S_{3t} = \frac{1-\lambda-\lambda S_{2t}}{(\lambda-1)S_{2t}+2-3\lambda}S_{1t} \\ -\frac{18\lambda^2-27\lambda+11}{6\lambda^2-5\lambda-1} < S_{2t} \le -\frac{9\lambda-9}{3\lambda-1}. \end{array}\right. \tag{14}$$

Proof of Lemma 3.4. By Lema 3.3, the second and the third equations of (13) allow us to express S_{1t} and S_{3t} in terms of S_{2t}. The results are presented as the first and the second equations of (14).

If S_{1t}, $S2t$ and S_{3t} satisfy (14) then the inequality of (13) is satisfied since $S_{2t}\le-\frac{9\lambda-9}{3\lambda-1}<0$ (which implies, in particular, that $S_{2t}-1<0$). Next, a simple check verifies the first equation of (13).

```
(%i23)  sys2:[%[2],%[3]]$
```

In order to obtain the second equation of (14) it is necessary to solve the second equation of (13).

```
(%i24)  ratcoef(sys2[1],S3T); /* see (15). */
```

The coefficient of S_{3t} in the second equation of (13) is

$$(S_{2t}-3)\lambda - S_{2t} + 2. \tag{15}$$

It is impossible S_{2t} to be such that (15) annihilates. Indeed, if $(S_{2t}-3)\lambda - S_{2t}+2 = 0$ then $S_{2t} = \frac{3\lambda-2}{\lambda-1}$ and the second equation of (13) implies $S_{1t} = 0$ and furthermore, from the third equation of (13), it follows

```
(%i25) sys2[2],S1T=0,S2T=(3*λ-2)/(λ-1)$
(%i26) %,λμData,factor;   /* see the left hand side of (16) */
```

$$\frac{4S_{3t}^2 b^6 + 2b^6 + 7a^2b^4 + 4a^4b^2 + 3a^6}{2b^4(b^2-a^2)} = 0 \tag{16}$$

which is impossible.

So, (15) is not zero and it is possible to solve the second equation of (13) with respect to S_{3t}

```
(%i27) solve(sys2[1],S3T),factor;
       /* The result is the second equation of (14). */
```

It is important to note that $S_{2t} - 1 \neq 0$. Indeed, if $S_{2t} - 1 = 0$ then it follows by the second equation of (14) that $S_{3t} = S_{1t}$ which is impossible because of the inequality of (13).

```
(%i28) sys2[2],%,μ=4/3-λ,ratsimp$
(%i29) factor(num(%))$
(%i30) νEq:ν=factor(%/(-(S2T-1))); /* see (17). */
```

Thus, it follows from the third equation of (13), the second equation of (14) and $\mu = \frac{4}{3} - \lambda$ that

$$(S_{2t}-1)\nu = 0$$

where

$$\begin{aligned}\nu = {} & 3S_{2t}^3\lambda^3 - 9S_{2t}^2\lambda^3 + 12S_{1t}^2S_{2t}\lambda^3 - 27S_{2t}\lambda^3 + 36S_{1t}^2\lambda^3 + 81\lambda^3 - 7S_{2t}^3\lambda^2 \\ & + 9S_{2t}^2\lambda^2 - 16S_{1t}^2S_{2t}\lambda^2 + 99S_{2t}\lambda^2 - 72S_{1t}^2\lambda^2 - 189\lambda^2 + 5S_{2t}^3\lambda + 5S_{2t}^2\lambda + 3S_{1t}^2S_{2t}\lambda \\ & - 102S_{2t}\lambda + 49S_{1t}^2\lambda + 144\lambda - S_{2t}^3 - 5S_{2t}^2 + S_{1t}^2S_{2t} + 32S_{2t} - 11S_{1t}^2 - 36.\end{aligned} \tag{17}$$

Therefore, in order to obtain the first equation of (14), it is necessary to solve $\nu = 0$ for S_{1t}^2.

The equation $\nu = 0$ is in fact an equation of the form $\rho S_{1t}^2 + \sigma = 0$ where the coefficients ρ and σ does not contain S_{1t}.

```
(%i31) ρ=factor(ratcoeff(ν,S1T,2)),νEq; /* see (18). */
```

Hence, the coefficient

$$\rho = (2\lambda-1)(6S_{2t}\lambda^2 + 18\lambda^2 - 5S_{2t}\lambda - 27\lambda - S_{2t} + 11). \tag{18}$$

It is impossible that $\rho = 0$. Indeed, otherwise,

```
(%i32) solve(ρ,S2T),%$
(%i33) νEq,%$
(%i34) ν,%,factor; /* The result is the left hand side of (19). */
```

$$\frac{324(2\lambda-1)^4(6\lambda-5)}{(\lambda-1)(6\lambda+1)^3} = 0, \tag{19}$$

which is impossible.

Thus, $\rho \neq 0$.

Hence, $S_{1t}^2 = -\frac{\sigma}{\rho}$ which is the first equation of (14).

```
(%i35) solve(ν,S1T^2),νEq,factor;
       /* The result is the first equation of (14). */
```

Now, the inequality of (14) follows from $S_{1t}^2 \geq 0$.

The first equation of (13) is satisfied

```
(%i36) %o22[1],%o27,factor$
(%i37) %,%o35,factor;
(%o37) 0
```

□

Lemma 3.5. *Suppose the complex numbers t_1, t_2 and t_3 are the roots of the equation*

$$t^3 - S_1 t^2 + S_2 t - S_3 = 0$$

where S_1, S_2 and S_3 are real numbers. Let us denote by λ the fraction $\frac{b^2-3a^2}{3(b^2-a^2)}$, where a and b are real numbers such that $a > b > 0$. If

$$\left| \begin{array}{l} S_1^2 = -\frac{(S_2\lambda-3\lambda-S_2+2)^2(3S_2\lambda+9\lambda-S_2-9)}{(2\lambda-1)(6S_2\lambda^2+18\lambda^2-5S_2\lambda-27\lambda-S_2+11)} \\ S_3 = \frac{1-\lambda-\lambda S_2}{(\lambda-1)S_2+2-3\lambda} S_1 \\ -\frac{18\lambda^2-27\lambda+11}{6\lambda^2-5\lambda-1} < S_2 \le -\frac{9\lambda-9}{3\lambda-1}, \end{array} \right. \tag{20}$$

then t_1, t_2 and t_3 are three different real numbers.

Proof of Lemma 3.5. Accordingly to the properties of roots of cubic equations, the discriminant

$$\Delta = (t_1 - t_2)^2 (t_2 - t_3)^2 (t_3 - t_1)^2$$

is positive iff the three roots t_1, t_2 and t_3 are three different real numbers. So, it it enough to prove that $\Delta > 0$.

```
(%i38) (t1-t2)^2*(t1-t3)^2*(t2-t3)^2,expand$
(%i39) Δ:expand(elem([3,S1,S2,S3],contract(%,[t1,t2,t3]),
                  [t1,t2,t3]))$
(%i40) [%o27,%o35],[S1T=S1,S2T=S2,S3T=S3]$
(%i41) ratvars(λ,S3,S2,S1)$
(%i42) Δ,%th(2),ratsimp$
(%i43) %,[%th(3),%th(3)^2],factor;    /* see (21). */
```

$$\Delta = \frac{(S_2-1)^2(\lambda-1)(3\lambda-1)(S_2^2\lambda + 6S_2\lambda + 9\lambda - S_2^2 + 2S_2 - 9)^2}{(2\lambda-1)^2(6S_2\lambda^2 + 18\lambda^2 - 5S_2\lambda - 27\lambda - S_2 + 11)^2}. \tag{21}$$

If follows by the inequalities of (20) that

- $S_2 \neq 1$ since $0 > -\frac{9\lambda-9}{3\lambda-1} \ge S_2$,
- $6S_2\lambda^2 + 18\lambda^2 - 5S_2\lambda - 27\lambda - S_2 + 11 > 0$ because of
$$-\frac{18\lambda^2 - 27\lambda + 11}{6\lambda^2 - 5\lambda - 1} = -\frac{18\lambda^2 - 27\lambda + 11}{(\lambda-1)(6\lambda+1)} < S_2.$$

Claim.

$$S_2^2\lambda + 6S_2\lambda + 9\lambda - S_2^2 + 2S_2 - 9 = (\lambda-1)S_2^2 + (6\lambda+2)S_2 + (9\lambda-9) \neq 0.$$

In fact, the left hand side has a negative value.

In order to prove this assertion, it is necessary to estimate the values of the quadratic function

$$F(s) = (\lambda-1)s^2 + (6\lambda+2)s + (9\lambda-9)$$

when s is such that $s_1 \overset{def}{=} -\frac{18\lambda^2-27\lambda+11}{6\lambda^2-5\lambda-1} < s \le s_2 \overset{def}{=} -\frac{9\lambda-9}{3\lambda-1}$.

```
(%i44) [s1,s2]:[-(18*λ^2-27*λ+11)/(6*λ^2-5*λ-1),-(9*λ-9)/(3*λ-1)]$
(%i45) map(lambda([s],factor((λ-1)*s^2+(6*λ+2)*s+(9*λ-9))),[s1,s2]);
```

(%o45) $\left[\frac{-108(2\lambda-1)^3}{(\lambda-1)\ (6\lambda+1)^2}, \frac{-108(\lambda-1)(2\lambda-1)}{(3\lambda-1)^2}\right]$

So, the quadratic coefficient of $F(s)$ is $\lambda - 1 > 0$ and

- $F(s_1) = \frac{-108(2\lambda-1)^3}{(\lambda-1)(6\lambda+1)^2} < 0$,
- $F(s_2) = \frac{-108(\lambda-1)(2\lambda-1)}{(3\lambda-1)^2} < 0$.

Hence, both s_1 and s_2 are placed between the two roots of the equation $F(t) = 0$. Therefore, s is placed between these roots, too.

Consequently, $F(s) < 0$ and Claim is proved.

Hence, $\Delta > 0$. □

Lemma 3.6. *Suppose $A(x(t_A), y(t_A))$, $B(x(t_B), y(t_B))$ and $C(x(t_A), y(t_A))$ are three different points on $\mathcal{E} \setminus \{(-a, 0)\}$. If the triangle $\triangle ABC$ is equilateral then its centroid $G(x_G, y_G)$ is such that*

$$(22) \qquad \left| \begin{array}{l} x_G = -\frac{a(S_{2t}\lambda+3\lambda-2)}{S_{2t}-1}, \\ y_G = \frac{S_{1t}b(6S_{2t}\lambda^2+18\lambda^2-5S_{2t}\lambda-27\lambda-S_{2t}+11)}{3(S_{2t}-1)(S_{2t}\lambda-3\lambda-S_{2t}+2)} \end{array} \right.$$

Proof of Lemma 3.6.

```
(%i46)  [xG=(xA+xB+xC)/3,yG=(yA+yB+yC)/3],tData,ratsimp$
(%i47)  ''(%i16)$
(%i48)  %,%o27,factor$
(%i49)  %,%o35,factor; /*see the right hand side of xG and yG in (22).*/
```

□

Lemma 3.7. *Suppose A, B and C are three different points on $\mathcal{E}$. If the triangle $\triangle ABC$ is equilateral then its centroid G belongs to $\mathcal{E}_1$.*

Proof of Lemma 3.7. As it is pointed out earlier, see Claim 1, a simple check shows that the triangle $\triangle A_1B_1C_1$ is equilateral and has its centroid on $\mathcal{E}_1$.

So, let $A(x(t_A), y(t_A))$, $B(x(t_B), y(t_B))$ and $C(x(t_A), y(t_A))$ be three different points on $\mathcal{E} \setminus \{(-a, 0)\}$ such that the triangle $\triangle ABC$ is equilateral. Let the point $G(x_G, y_G)$ be the centroid of the triangle $\triangle ABC$. By Lemma 3.6, x_G and y_G are expressed in terms of S_{1t}, S_{2t} and S_{3t} in the system (22). Note that S_{1t}, S_{2t} and S_{3t} satisfy the system (14), by Lemma 3.4.

Claim. There are a unique pair of α and β such that $\alpha > \beta > 0$ and $\beta^2 x_G^2 + \alpha^2 y_G^2 - \alpha^2\beta^2 = 0$ for all S_{2t} which satisfy the inequality of (14). The pair is (1).

Indeed,

```
(%i50)  β^2*xG^2+α^2*yG^2-α^2*β^2,[xG=%[1],yG=%[2]],ratsimp$
(%i51)  %,%o27,ratsimp$
(%i52)  num(%)$
(%i53)  %,[%o35,%o35^2],factor$
(%i54)  num(%)$
(%i55)  part(%,2)$
(%i56)  makelist(coeff(%,S2T,i),i,2,0,-1)$
(%i57)  %,α^2=u,β^2=v$
(%i58)  solve(%,[u,v])$
(%i59)  %[1],[u=α^2,v=β^2],λμData,factor;   /* see (23). */
```

$$(23) \qquad \alpha^2 = \frac{a^2(a^2-b^2)^2}{(3b^2+a^2)^2}, \quad \beta^2 = \frac{b^2(a^2-b^2)^2}{(b^2+3a^2)^2}$$

Thus, the assertion of the claim is proved. □

Lemma 3.8. *Suppose α and β are real numbers. If $\alpha > \beta > 0$, then there exists a unique pair of real numbers a and b that comply with (1). Moreover, $a > b > 0$ and the ellipse $\mathcal{E} : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is such that every equilateral triangle $\triangle ABC$ with vertices three different points on $\mathcal{E}$ has its centroid on $\mathcal{E}_1$.*

Proof of Lemma 3.8. The pair of a and b is the solution of the system (1). In order to solve this system, first it is necessary to express b in terms of a, α, β.

```
(%i60) solve(α*(3*b^2+a^2)-a*(a^2-b^2),b^2)$
(%i61) num(ratsimp(ev(β*(b^2+3*a^2)-b*(a^2-b^2),%)))
(%i62) solve(%,b);
```
(%o62) $[b = \frac{(2\alpha+a)\beta}{\alpha}]$

Next, b is substituted in the second equation of the system (1).

```
(%i63) ratvars(α,β,a)$
(%i64) (α*(3*b^2+a^2)-a*(a^2-b^2))*α^2,%th(2),ratsimp;
```
(%o64) $a^2(7\alpha\beta^2+\alpha^3)+a^3(\beta^2-\alpha^2)+12\alpha^3\beta^2+16a\alpha^2\beta^2$

So, a is a solution of the equation $f(t) = 0$, where

$$f(t) = t^3(\beta^2-\alpha^2)+t^2(7\alpha\beta^2+\alpha^3)+16t\alpha^2\beta^2+12\alpha^3\beta^2,$$

$t \in (-\infty; +\infty)$.

Claim. The discriminant of $f(t) = 0$ is negative.

Indeed,

```
(%i65) f(a):=''(%)$
(%i66) [c3,c2,c1,c0]:makelist(coeff(f(a),a,i),i,3,0,-1)$
(%i67) -27*c0^2*c3^2+18*c0*c1*c2*c3-4*c1^3*c3-4*c0*c2^3+c1^2*c2^2,factor;
       /* see (24). */
```

So, the discriminant is

$$-16\alpha^8\beta^2(3\beta^4+506\alpha^2\beta^2+3\alpha^4) \tag{24}$$

and it is negative, of course. Thus, Claim is proved.

Therefore, the equation $f(t) = 0$ has a unique real root. Let us denote this root by a. Moreover, $a > 0$ since the cubic coefficient $\beta^2-\alpha^2 < 0$ and the free term $16a\alpha^2\beta^2 > 0$ have different signs.

Hence, a and $b = \frac{(2\alpha+a)\beta}{\alpha}$ (note, $b > 0$) is the only real solution of (1).

Furthermore,

$$a > \frac{2\alpha\beta}{\alpha-\beta} > 0. \tag{25}$$

Indeed,

```
(%i68) f(t)*c3,t=2*α*β/(α-β),factor; /* see the left hand side of (26). */
```

$$f\left(\frac{2\alpha\beta}{\alpha-\beta}\right)(\beta^2-\alpha^2) = \frac{16\alpha^5\beta^2(\beta+\alpha)}{\beta-\alpha} \tag{26}$$

So, $f(\frac{2\alpha\beta}{\alpha-\beta})(\beta^2-\alpha^2) < 0$, i.e. the cubic coefficient $\beta^2-\alpha^2$ and the value $f(\frac{2\alpha\beta}{\alpha-\beta})$ have different signs and hence, there is a real root of $f(t) = 0$ (which is a since it is the only real root) that is greater that $\frac{2\alpha\beta}{\alpha-\beta}$. Thus, (25) is verified.

Now, $a > b$ follows from (25)—it is obvious and we omit the details. In order to end the proof, note that the remaining assertions of the lemma follow from Lemma 3.7. □

Lemma 3.9. *Suppose α and β are real numbers such that $\alpha > \beta > 0$. Let a and b be the pair of real numbers that comply with (1) and $\mathcal{E}_1 : \frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2} = 1$, $\mathcal{E} : \frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$. Then, for every point $G \in \mathcal{E}_1$ there exist three different points A, B and C on $\mathcal{E}$ such that triangle $\triangle ABC$ is equilateral and its centroid is G.*

Proof of Lemma 3.9. The pair a and b is well defined as it is proved in Lemma 3.8.

If $G(\alpha, 0)$ then the points are A_0, B_0 and C_0 as it is calculated in Claim 1.

If $G(-\alpha, 0)$ then the points are A_1, B_1 and C_1 as it is calculated in Claim 1.

Now, let $G(x_G, y_G) \in \mathcal{E}_1 \setminus \{(\alpha, 0), (-\alpha, 0)\}$. Note, $\alpha^2 > x_G^2$ and $y_G \neq 0$. Let

$$S_2 = -\frac{3a\lambda - x_G - 2a}{a\lambda + x_G}, \tag{27}$$

(the denominator $a\lambda + x_G > 0$ since $a > \alpha$, $\lambda > 1$, $x_G > -\alpha$).

Claim. S_2 satisfies the inequalities of (20). Indeed,

```
(%i69)  (S2-s1)*(S2-s2),S2=-(3*a*λ-xG-2*a)/(a*λ+xG),factor$
(%i70)  %,λμData,factor$
(%i71)  %/(α^2-xG^2),α=(a*(a^2-b^2))/(3*b^2+a^2),factor;
        /* see the right hand side of (28). */
```

$$\frac{(S_2 - s_1)(S_2 - s_2)}{\alpha^2 - x_G^2} = -\frac{9(a^2 - b^2)^2(b^2 + 3a^2)(3b^2 + a^2)^2}{a^2b^2(7a^2 - 3b^2)(-3b^2xG + 3a^2xG - ab^2 + 3a^3)^2}. \tag{28}$$

So, $(S_2 - s_1)(S_2 - s_2) < 0$ and Claim is proved.

S_1 and S_3 are calculated accordingly to (20). The sign of S_1 has to be chosen to be the same as the sign of y_G.

By Lemma 3.5, the roots of

$$t^3 - S_1t^2 + S_2t - S_3 = 0$$

are three different real numbers. Let name them t_A, t_B, t_C.

By Lemma 3.4, $A(x(t_A), y(t_A))$, $B(x(t_B), y(t_B))$ and $C(x(t_A), y(t_A))$ are three different points on $\mathcal{E}$ and the triangle $\triangle ABC$ is equilateral.

By Lemma 3.6, the abscissa of the centroid G' of $\triangle ABC$ is x_G.

G' is a point on $\mathcal{E}_1$, by Lemma 3.7. Its second coordinate has same the sign as S_1, by (22).

Hence, $G' = G$ and the proof is over. □

Theorem 2.1 follows from Lemmas 3.7, 3.8, 3.9.

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