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Four Proofs of the Generalization of the Simson Line

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Abstract. We introduce four proofs of the generalization of the Simson line.

Keywords. Simson line, collinear, orthopole, circumcenter

1. INTRODUCTION

In 2014, I found a nice result in plane geometry, the result is a generalization of the Simson line theorem. The result was published in [1]. There are some proofs for this theorem which were published in [2]-[6].

Theorem 1.1. ([1]). *Let ABC be a triangle, line ℓ pass through the circumcenter O; point P lie on the circumcircle. Let AP, BP, CP meet ℓ at AP, BP, CP, respectively. Denote $A0$, $B0$, $C0$ the projections of AP, BP, CP onto BC, CA, AB, respectively. Then A_0, B_0, C_0 are collinear. Moreover, the new line passes through the midpoint of OH, where H the orthocenter of ABC (Figure 1).*

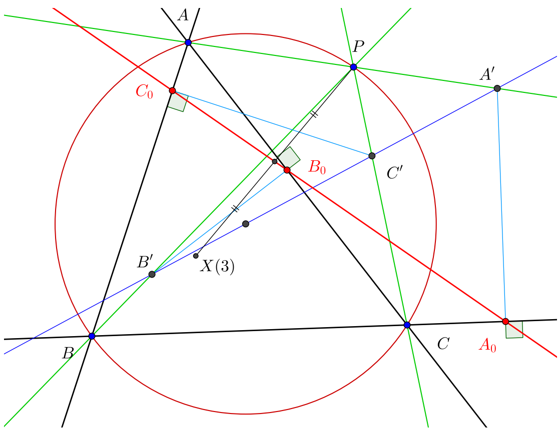

FIGURE 1.

Remark. *If ℓ passes through P, the line coincides with the Simson.*

In this paper we introduce 4 another proofs of the Theorem 1.1 in next section.

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2. SOME NEW PROOFS OF THEOREM 1.1

Proof 1 by Telv Cohl -[7]:

Let $X = \ell \cap AC$ and M be the midpoint of AC (See Figure 3).

Let Y, Z be the projection of P, B on AC, respectively.

Let H_A, H_B, H_C, H_P be the projection of A, B, C, P on ℓ, respectively.

Let A', B', C', P' be the orthopole of ℓ WRT $\triangle BCP, \triangle CAP, \triangle ABP, \triangle ABC$, respectively.

Let R be the Poncelet point of $\{A, B, C, P\}$ (It's well-known that R is the midpoint of P and the orthocenter of $\triangle ABC$).

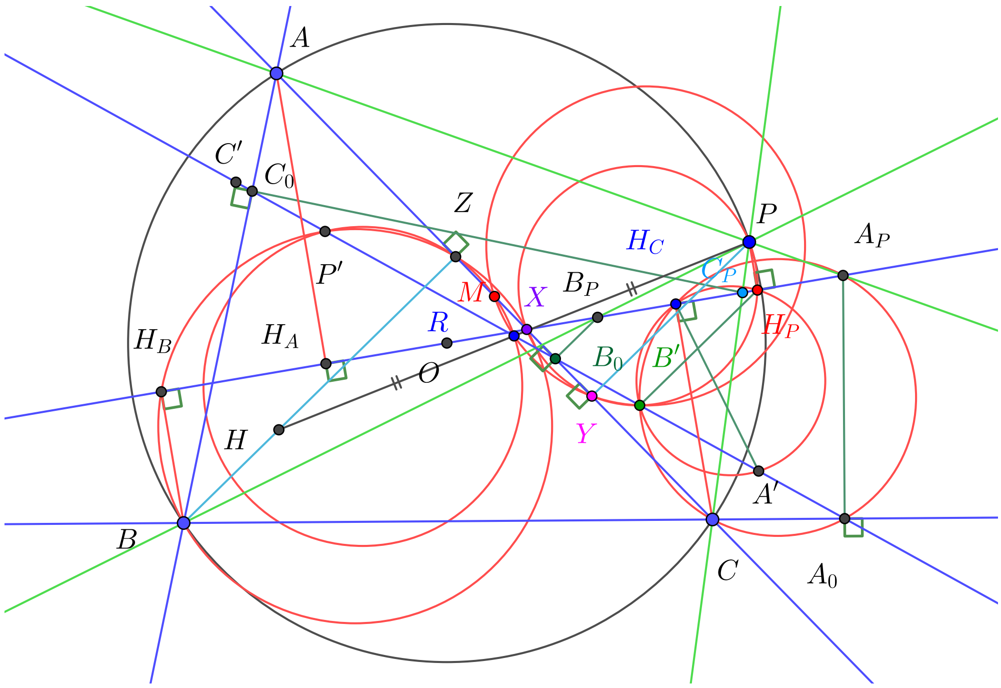

FIGURE 2.

From [8] we get A', B', C', P' lie on a line τ .

Since $\odot(A_P A_0 C)$ is the pedal circle of A_P WRT $\triangle PAC$, so from Fontene theorem we get $B' \in \odot(A_P A_0 C) \Longrightarrow A_0, A_P, C, H_C, B'$ are concyclic.

Since $H_C A' \perp PB, H_C B' \perp PA, H_P A' \perp BC, H_P B' \perp AC$, so $\angle B' H_C A' = \angle APB = \angle ACB = \angle B' H_P A' \Longrightarrow A', B', H_C, H_P$ are concyclic . From Reim theorem and $H_P A' \parallel A_P A_0$ we get $A_0 \in A'B' \equiv \tau$ (similar discussion for B_0, C_0) .

It's well-known that R lie on the 9-point circle of $\triangle ABC$, so P', M, Z, R are concyclic at the 9-point circle of $\triangle ABC$. Similarly B', M, Y, R are concyclic at the 9-point circle of $\triangle ABC$. Since B, H_B, P', X, Z are concyclic at the pedal circle of X WRT $\triangle ABC$, so $\angle AZP' = \angle XH_B P' = 90° - \angle(AC, \tau)$ (notice that $H_B P' \perp AC$) .

Similarly we can prove $\angle CYB' = 90° - \angle(AC, \tau) \Longrightarrow ZP' \parallel YB'$, so from Reim theorem we get P', R, B' are collinear . i.e. $R \in \tau \equiv \overline{A_0 B_0 C_0}$

Proof 2 by Luis Gonzalez-[9]:

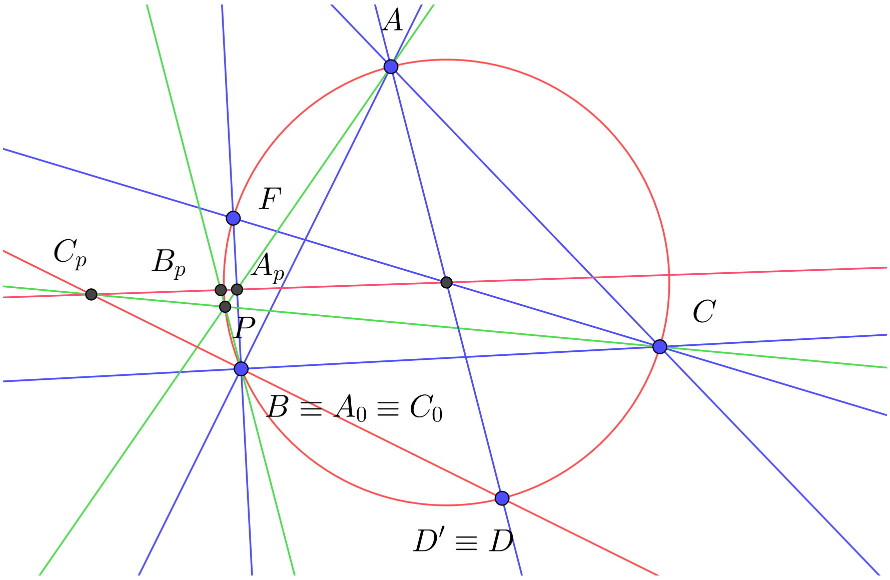

FIGURE 3.

Fix the line ℓ and animate P. The pencils PA, PB, PC are projective inducing a proyectivity on ℓ, i.e. the series A_p, B_p, C_p are projective $\Longrightarrow$ series A_0, B_0, C_0 are projective.

Let D, F be the antipodes of A, C on the circumcircle (O) and consider the case when $A_p \in BF$. If BC_p cuts (O) again at D', then by Pascal theorem for $APCFBD'$, it follows that $A_p, C_p, CF \cap AD'$ are collinear $\Longrightarrow D \equiv D' \Longrightarrow \angle C_pBA = 90^\circ \Longrightarrow B \equiv A_0 \equiv C_0 \Longrightarrow A_0 \mapsto C_0$ is a perspectivity $\Longrightarrow A_0C_0$ goes through a fixed point. When P coincides with $\{X, Y\} \equiv \ell \cap (O)$, then A_0C_0 becomes Simson lines of X, Y meeting at the orthopole T of $\ell \Longrightarrow T \in A_0C_0$ and similarly $T \in B_0C_0 \Longrightarrow A_0, B_0, C_0$ are collinear on a line τ passing through T

Let H be the orthocenter of $\triangle ABC$ and let X be the midpoint of HP lying on 9-point circle (N). It's known that $T \in (N)$ when $O \in \ell$. Now since $X \barwedge P \barwedge A_0$ with fixed points at $(N) \cap BC$, then it follows that $X \mapsto A_0$ is a stereographic projection of (N) onto $BC \Longrightarrow X \in TA_0 \equiv \tau$.

Proof 3 by Tran Quang Huy-[10]:

If there exists a point W on the circumcircle of $\triangle ABC$, denote by d_W the Simson line of W WRT $\triangle ABC$

Now, let Q be the reflection of P WRT the line ℓ and let M be the midpoint of PH (H is the orthocenter of $\triangle ABC$)

We will prove that: $MA_0 \parallel d_Q$. Indeed:

T, D are the projections of P, H on BC $MP = MH \Rightarrow MD = MT$.

On the other hand:

$$(TM, TD) = (d_P, d_S) = \frac{1}{2}.\widehat{SP} = (AS, AP) = (AO, AP) = -(PO, PA) (\text{mod} \pi)$$

$\Rightarrow \triangle TMD \sim \triangle POA$ (1)

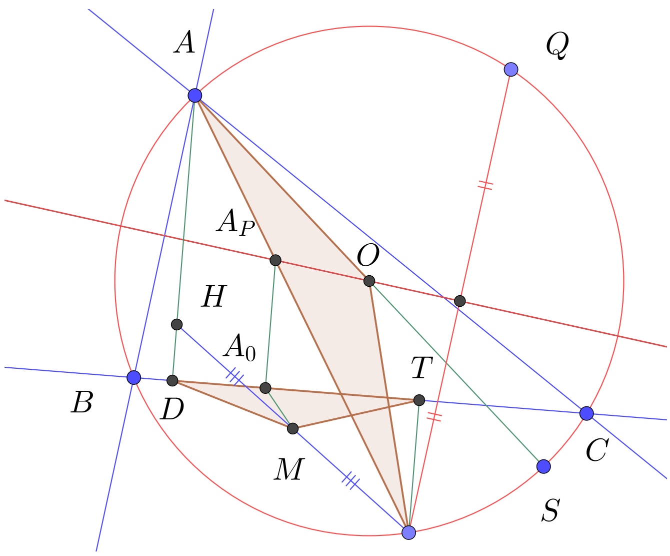

FIGURE 4.

We have:

$$\frac{A_0T}{A_0D} = \frac{A_PP}{A_PA}$$

because $AD \parallel A_PA_0 \parallel PT$) (2)

Combine (1), (2) $\Rightarrow \triangle TMA_0 \sim \triangle POA_P \Rightarrow (MT, MA_0) = (OA_P, OP) = \frac{1}{2}.\widehat{QP} = (d_P, d_Q)$ (mod π) $\Rightarrow \boxed{MA_0 \parallel d_Q}$

Similarly: $MB_0, MC_0 \parallel d_Q \Rightarrow \overline{M, A_0, B_0, C_0}$ or $\overline{A_0, B_0, C_0}$ bisects HP

Proof 4 by Dukejukem-[11]:

We can prove a stronger result as follows:

Let H be the orthocenter of $\triangle ABC$ and let Q be the isogonal conjugate of the infinite point on ℓ. Then A_0, B_0, C_0 are collinear on the H-midline of $\triangle HPQ$.

Let A^* be the antipode of A on the circumcircle (O) and let H^* be reflection of H in BC. Let D be the reflection of A in A_P and let A_1 be the reflection of H in A_0. Let R, S lie on (O) so that $QR \parallel BC$ and $QS \perp BC$.

By considering the homothety $\mathbf{H}(H, 2)$, it's enough to show that $A_1 \in PQ$.

It's well-known that $H^* \in (O)$ and A^*H^* is the image of BC under the homothety $\mathbf{H}(H, 2)$. Hence, $A_1 \in A^*H^*$.

Notice that ℓ is the A-midline of $\triangle AA^*D$, so $\ell \parallel A^*D$. Since $AR \parallel \ell$ by definition and A^*S is the image of AR under the homothety $\mathbf{H}(O, -1)$ it follows that $A^*S \parallel \ell$. Hence A^*, S, D are collinear.

From $AH \parallel A_PA_0$ we get $DA_1 \parallel AH \parallel A_PA_0 \parallel QS$. By the converse of Pascal's Theorem for PAH^*A^*SQ we deduce that $A_1 \in PQ$, as desired. □

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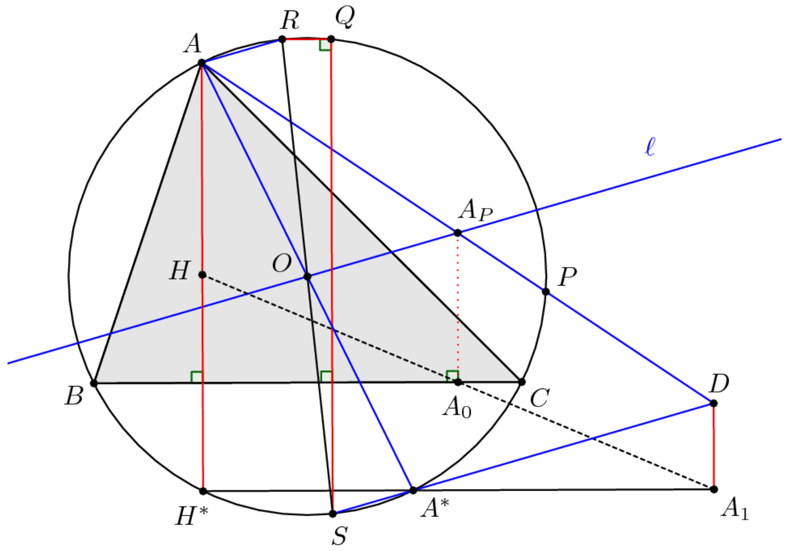

FIGURE 5.

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