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Some Problems Around the Configuration of Eight Circles

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Abstract. We introduce some problems around the configuration of eight circles Theorem

Keywords. Concyclic, circles, chain of circles, six points circle, Tucker circle, Bundle theorem, Miquel theorem, Eight circles theorem

1. Introduction

The problem 3845 in Crux Mathematicorum [1] is one nice configuration of the eight circles. The problem states that let six points A_1, A_2, $\cdots$, A_6 lie on a circle and the six points $B_1, B_2, \cdots, B_6$ lie on an another circle. If the quadruples A_i, A_{i+1}, B_{i+1}, B_i lie on a circles with centers O_i for $i = 1, 2, \cdots, 5$. Then four points A_6, A_1, B_1, B_6 lie on a circle. If O_6 is the center of the new circle, we have three lines O_1O_4, O_2O_5, O_3O_6 are concurrent (Figure 1).

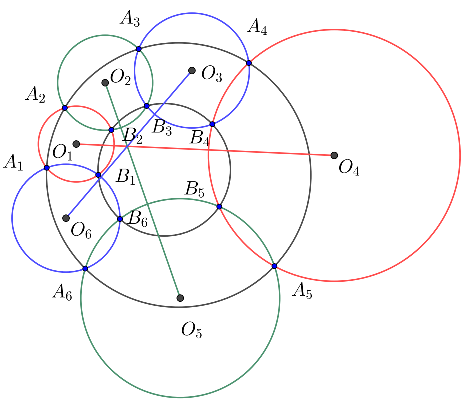

Figure 1.

There are some proof of the problem 3845 in [2] [3], [4], more problem on this configuration by Le Viet An, you can see in [5].

One special case of the problem 3845 is as follows:

Theorem 1.1. *Let six circles* (O_1), (O_2), (O_3), (O_4), (O_5), (O_6) *such that* (O_i) *tangent to* (O_{i+1}) *at* A_i *for* $i = 1, \cdots, 5$. *Here we take the subscripts modulo 6. If six points* A_1, A_2, $\cdots$, A_6 *lie on a circle. Then three lines* O_1O_4, O_2O_5, O_3O_6 *are concurrent (See Figure 2).*

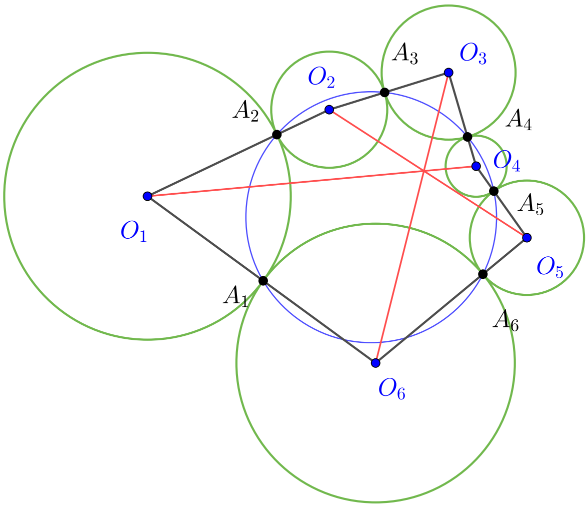

FIGURE 2.

Remark. *The special case of Brianchon theorem* [14] *is also special case of Theorem 1.1.*

On the Problem 3845 configuration, let (O_i) is the circle though A_i, A_{i+1}, B_{i+1}, B_i for $i = 1, 2, ..., 6$ (Figure 1). We have the dual theorem following:

Problem 1.1. *(*[6]*). If* (O_1) *meets* (O_4) *at two points* C_1, C_4*;* (O_2) *meets* (O_5) *at two points* C_2, C_5, (O_3) *meet* (O_6) *at two points* C_3, C_6 *then six points* C_1, C_2, C_3, C_4, C_5, C_6 *lie on a circle with center* O *(The point of concurrence in Problem 3845).*

The problem 3845 and its dual is also a generalization of some famous theorem and Dao-symmedial circle [6], [7]. There are many problems around the configuration of problem 3845 so we should call the Problem 3845 is the Eight circles theorem. In this paper, we introduce some theorems, problems, special cases, generalizations, variants around the configuration of the eight circles theorem.

2. SOME PROBLEMS AROUND THE CONFIGURATION OF EIGHT CIRCLES

Converse of Problem 1.1 is as follows:

Theorem 2.1. *(*[8]*). Let six points* A_1, A_2, A_3, A_4, A_5, A_6 *lying on a circle. Define* $C(A_i, A_j)$ *be a circle through points* A_i, A_j. *Let* $C(A_i, A_{i+1}) \cap C(A_{i+3}, A_{i+4}) = B_i, B_{i+3}$ *we take modulo 6. Let* B_1, B_2, B_3, B_4, B_5 *lie on a circle* (C). *Let* $C(A_i, A_{i+1}) \cap C(A_{i+1}, A_{i+2}) = C_{i+1}$. *Show that* B_6 *also lie on the circle* (C) *and six points* C_1, C_2, C_3, C_4, C_5, C_6 *lie on a circle (Figure 3).*

You can see the proof of Theorem 2.1 by **Futurologist** on *Math Stack Exchange* forum [8].

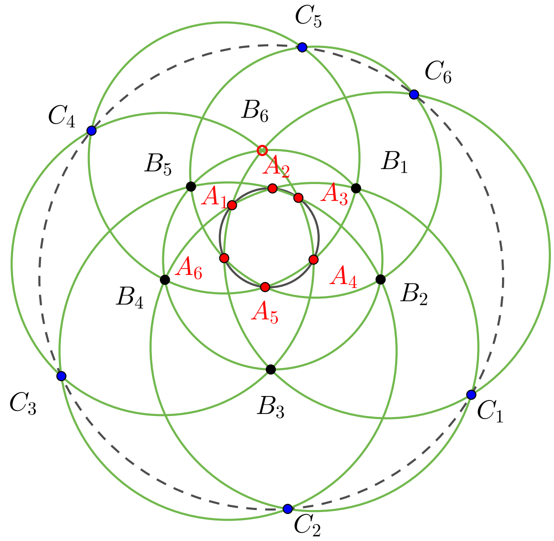

FIGURE 3.

Theorem 2.2. *([10], [12]). Let six points A_1, A_2, A_3, A_4, A_5, A_6 lying on a circle. Define $C(A_1, A_4)$ be a circle through points A_1, A_4, $C(A_2, A_5)$ be a circle through points A_2, A_5, $C(A_3, A_6)$ be a circle through points A_3, A_6. Let B_1 be arbitrary point in $C(A_1, A_4)$, the circle $(B_1A_1A_2)$ meets $C(A_2, A_5)$ again at B_2, the circle $(B_2A_2A_3)$ meets $C(A_3, A_6)$ again at B_3, the circle $(B_3A_3A_4)$ meets $C(A_1, A_4)$ again at B_4, the circle $(B_4A_4A_5)$ meets $C(A_2, A_5)$ again at B_5, the circle $(B_5A_5A_6)$ meets $C(A_3, A_6)$ again at B_6, then B_6, A_6, A_1, B_1 lie on a circle and six points B_1, B_2, B_3, B_4, B_5, B_6 lie on a circle (Figure 4).*

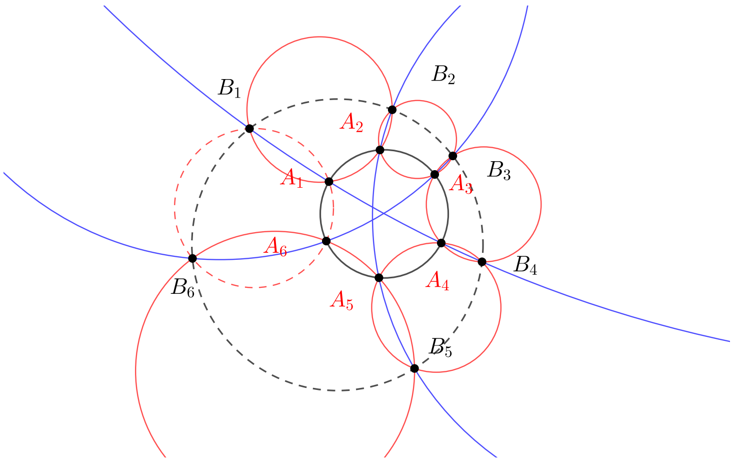

FIGURE 4.

You can see the proof of Theorem 2.2 by **Futurologist** on *Math Stack Exchange* forum [10].

Theorem 2.3. *([10]). Let $ABCDEF$ be a cyclic hexagon. Let A_1 be any point on AD. Let the circle (A_1AB) meets BE again at B_1, the circle (B_1BC) meets CF again at C_1, the circle (C_1CD) meets AD again at D_1, the circle (D_1DE) meets BE again at E_1, the circle (E_1EF) meets CF again at F_1, then six lines A_1B_1, CD, E_1F_1, AB, C_1D_1, EF formed a cyclic hexagon and six points A, A_1, A', F, F_1, F' lie on a circle.*

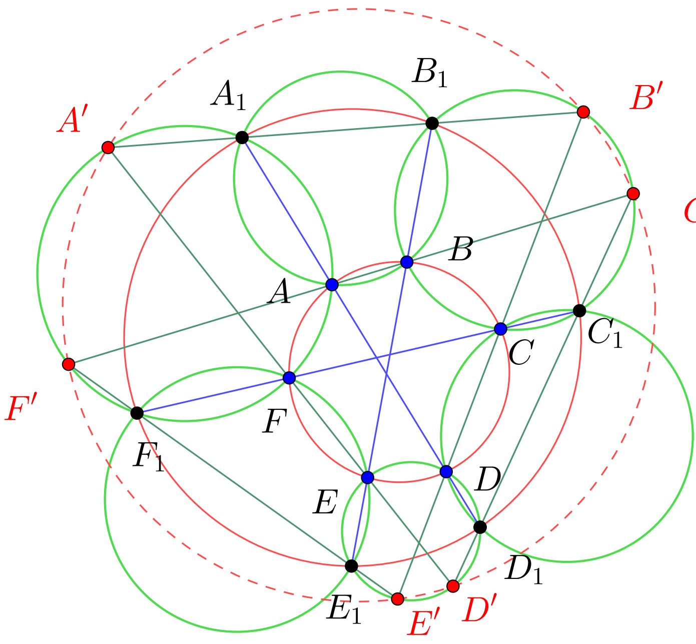

FIGURE 5.

A generalization of Theorem 2.3 is as follows:

Theorem 2.4. *([11]). Let a chain of six circles (O_1), (O_2), (O_3), (O_4), (O_5), (O_6), such that two neighbors circles (O_i) meets (O_{i+1}) at two points A_i, B_i for $i = 1, 2, ..., 6$; Such that six points A_1, A_2, A_3, A_4, A_5, A_6 lying on a circle. Let C_1 be a point on the circle (O_6), the circle $(C_1A_1A_2)$ meets the circle (O_2) at C_2, the circle $(C_2b_3b_4)$ meets the circle (O_4) at C_3, the circle $(C_3A_5A_6)$ meets the circle (O_6) at C_4, the circle $(C_4B_1B_2)$ meets the circle (O_2) at C_5, Let the circle $(C_5A_3A_4)$ meets the circle (O_4) at C_6, then*

1. Four points C_6, B_5, B_6, C_1 lie on a circle.

2. Six points C_1, C_2, C_3, C_4, C_5, C_6 lie on a circle.

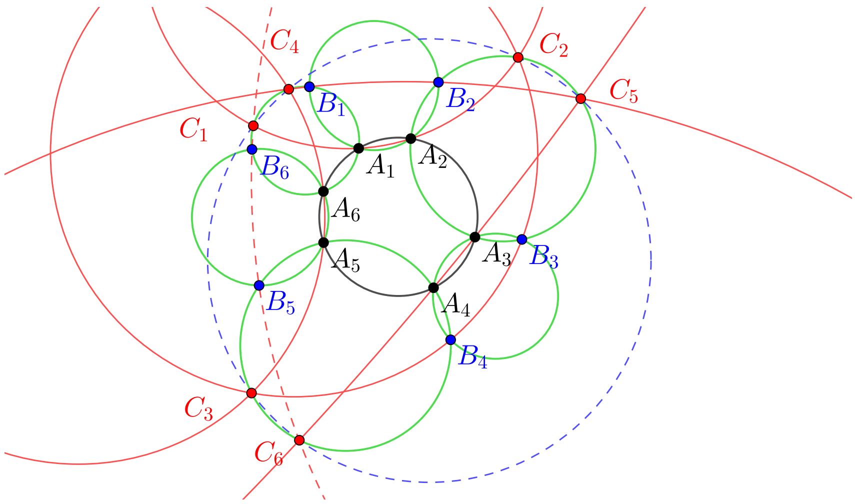

FIGURE 6.

You can see the proof of Theorem 2.4 by **Futurologist** on *Math Stack Exchange* forum [11].

On the Problem 3845 configuration, let (O_i) is the circle though A_i, A_{i+1}, B_{i+1}, B_i for $i = 1, 2, ..., 6$ (Figure 1). By the Miquel's six circles theorem we get $A_i, B_i, B_{i+3}, A_{i+3}$ lie on a circle for $i = 1, 2, 3$. Define the circle through B_1, A_1, A_4, B_4 are (O_7), the circle through B_2, A_2, A_5, B_5 are (O_8), the circle through B_3, A_3, A_6, B_6 are (O_9)

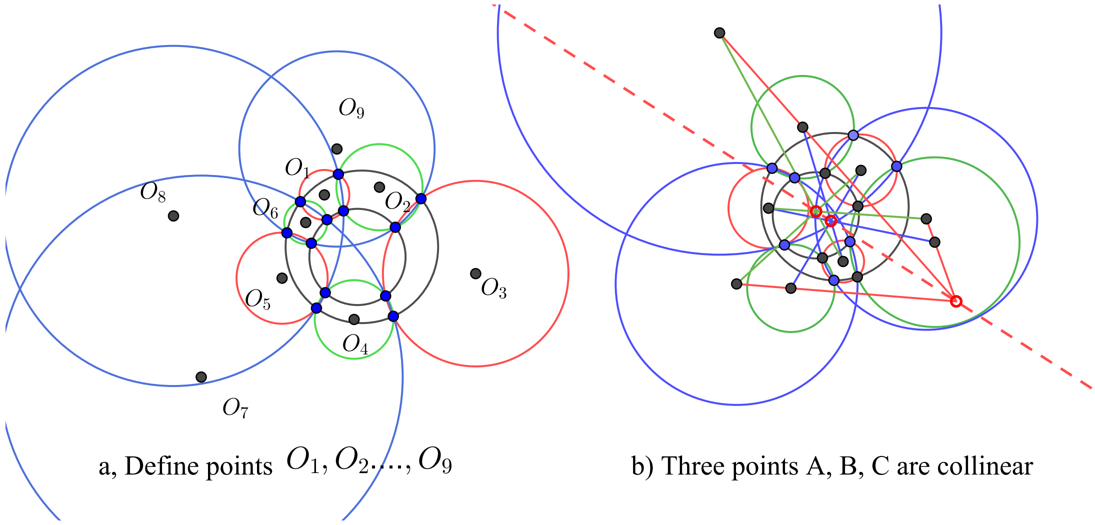

FIGURE 7.

By the eight circles theorem we have:

1. O_1O_4, O_2O_5, O_3O_6 are concurrent, denote the point of concurrence is A.
2. O_1O_7, O_2O_8, O_3O_9 are concurrent, denote the point of concurrence is B.
3. O_4O_7, O_5O_8, O_6O_9 are concurrent, denote the point of concurrence is C.

Problem 2.1. *([13]). Three points A, B, C are collinear (See Figure 7).*

Problem 2.2. *Let six points $A_1, A_2, ...A_6$ lie on a circle, and the six points $B_1, B_2, ..., B_6$ lie on another circle. If the quadruples $A_i, A_{i+1}B_{i+1}B_i$ lie on a circles with centers O_i for $i = 1, 2, ..., 5$ then A_6, A_1, B_1, B_6 lie on a circle. Six lines A_1B_1, A_2B_2, A_3B_3, A_4B_4, A_5B_5, A_6B_6 formed a Brianchon hexagon by Brianchon's theorem on conics (Figure 8).*

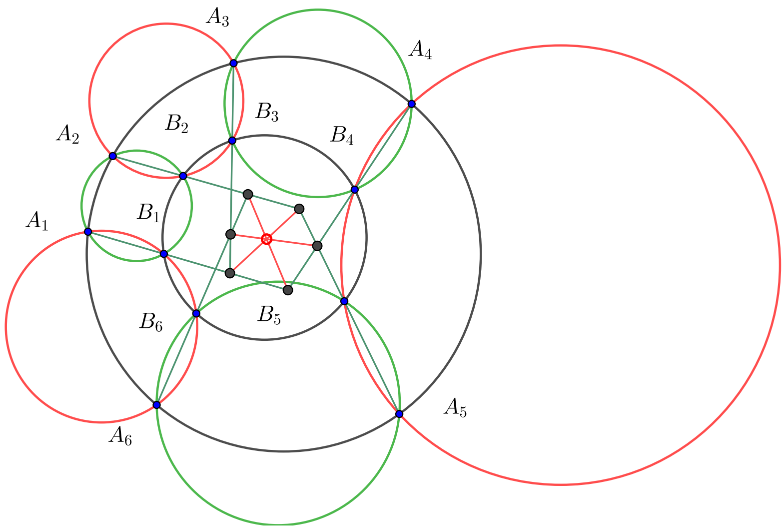

FIGURE 8.

Problem 2.3. *([15]). Let (C_1), (C_2), (C_3) be three circles in a plane. Let A_{i1}, A_{i4} be points lie on circle (C_1); A_{i2}, A_{i5} be points lie on circle (C_2); A_{i3}, A_{i6} be lie on circle (C_3) for $i = 1, 2, \cdots n$ (Figure 9). So that six points A_{11}, A_{12} , A_{13}, A_{14}, A_{15}, A_{16}, lie on a*

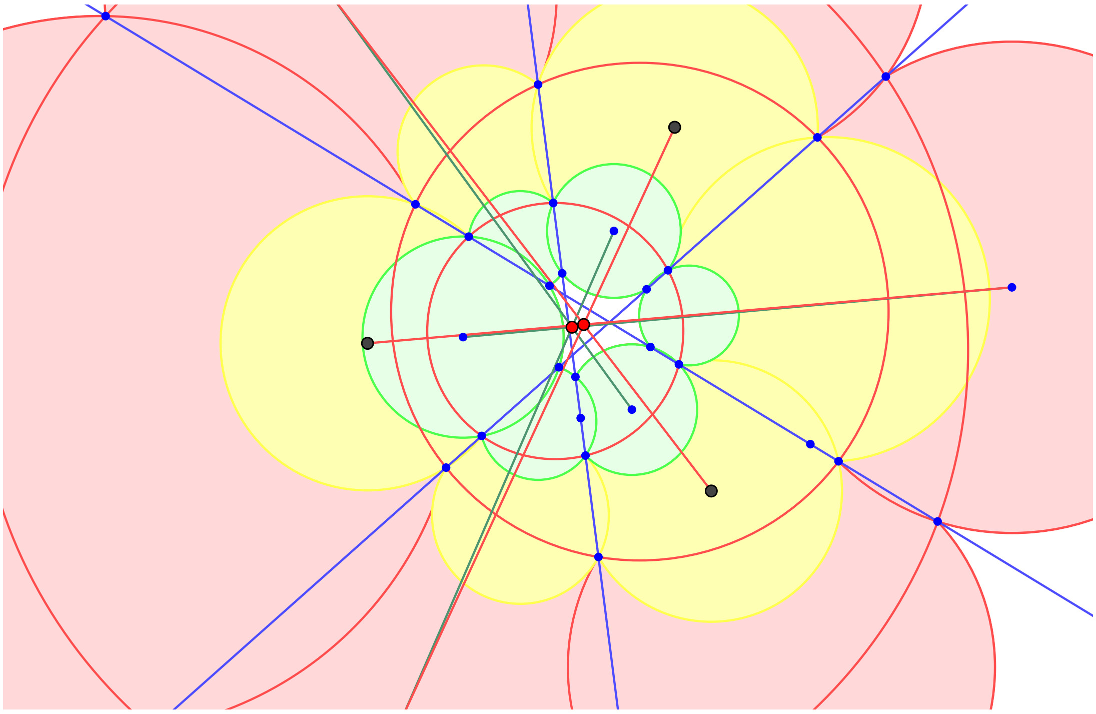

FIGURE 9.

circle and four poins A_{ij}, A_{ij+1}, A_{i+1j+1}, A_{i+1j} lie on a circle for $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, 6$

1. Then A_{i1}, A_{i2}, A_{i3}, A_{i4}, A_{i5}, A_{i6} lie on a circle for $i = 1, 2, \cdots, n$.

Denote (O_{ij}) is the circle through A_{ij}, A_{ij+1}, A_{i+1j+1}, A_{i+1j} for $i = 1, 2, \cdots, n$ and $j = 1$, 2, 3, 4, 5, 6

2. Three lines $O_{i1}O_{k4}$, $O_{i2}O_{k5}$, $O_{i3}O_{k6}$ are concurrent for all $i, k = 1, 2, \cdots$

3. Three lines $O_{i1}O_{k1}$, $O_{i3}O_{k3}$, $O_{i5}O_{k5}$ are concurrent, for all $i, k = 1, 2, \cdots$

4. Three lines $O_{i2}O_{k2}$, $O_{i4}O_{k4}$, $O_{i6}O_{k6}$ are concurrent for all $i, k = 1, 2, \cdots$

5. If (O_{i1}) meets (O_{k1}) at wo points P_1, P'_1; (O_{i2}) meets (O_{k2}) at wo points P_2, P'_2, ..., similarly (O_{i6}) meets (O_{k6}) at wo points P_6, P'_6 then 12 points $P_1, \cdots, P_6, P'_1, \cdots, P'_6$ lie on a circle.

6. If (O_{i1}) meets (O_{k4}) at wo points P_1, P_4, (O_{i2}) meets (O_{k5}) at wo points P_2, P_5, (O_{i3}) meets (O_{k6}) at wo points P_3, P_6. Then six the six points lie on a circle $P_1, \cdots, P_6$ lie on a circle. $i = 1, 2, \cdots$

Remark. *Problem 2.3 is a generalization of the configuration of Eight circles*

Problem 2.4. *Let (O_a) meets (O'_a) at A_1, A_2; (O_b) meets (O'_b) at B_1, B_2; (O_c) meets (O'_c) at C_1, C_2 such that A_1, A_2, B_1, B_2, C_1, C_2 lie on a circle. Let (O_a) meets (O_b) at two points A_b, B_a; (O'_a) meets (O'_b) at two points A'_b, B'_a then by Bundle theorem we have quadruple A_b, B_a, A'_b, B'_a are concyclic; Define similarly we have two quadruples $\{B_c, C_b, B'_c, C'_b\}$ and $\{C_a, A_c, C'_a, C'_b\}$ are concyclic. Show that three circles $(A_bB_aA'_bB'_a)$, $(B_cC_bB'_cC'_b)$, $(C_aA_cC'_aC'_b)$ are coaxial (Figure 10).*

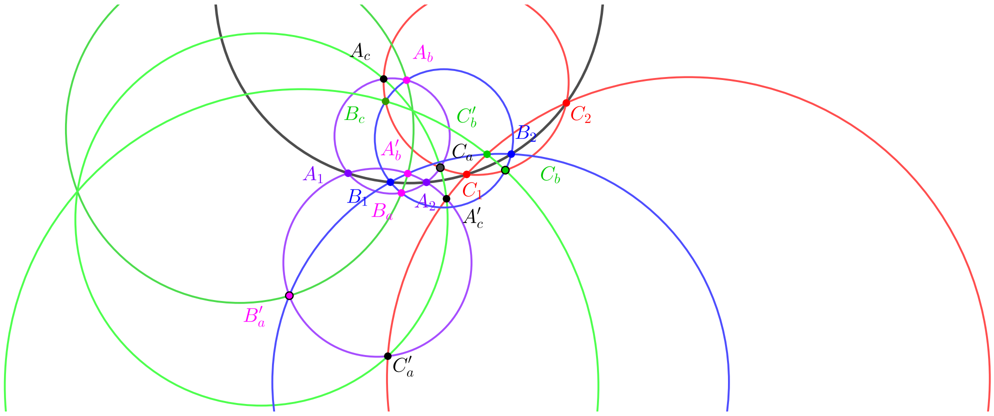

Figure 10.

Theorem 2.5. *([16]). Let $A_1, ..., A_6$ and $B_1, ..., B_6$ be 12 points lying on a conic α, and suppose that for $i = 1, ..., 5$ through $A_i, A_{i+1}, B_{i+1}, B_i$ passes a circle (C_i). Then through A_6, B_6, A_1, B_1 as well passes a circle (C_6) . Let P_1, P_4 be intersection points of (C_1) and (C_4); the same for P_2, P_5 and P_3, P_6, and O_i is the center of circle (C_i) (Figure 11), then*

1. Three lines O_1O_4, O_2O_5, and O_3O_6 have a common point O.

2. Six points $P_1, ..., P_6$ lie on a circle with center O.

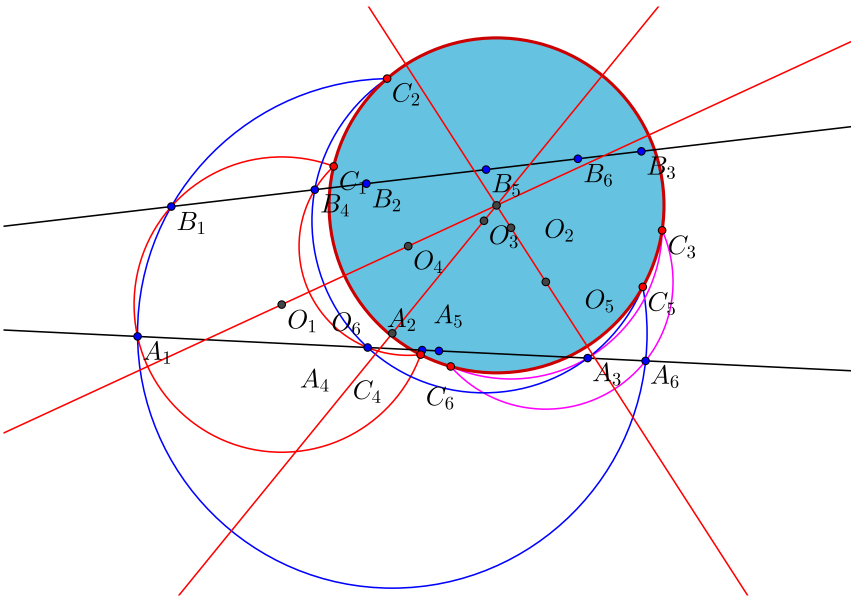

Figure 11.

Now, we reformulate the Theorem 2.5 to the Theorem 2.6 is as follows. Easily we can see that Theorem 2.6 is a generalization of Pascal theorem and Pappus theorem.

Theorem 2.6. *([17]). Let A_1, A_2, A_3, A_4, A_5, A_6 be six arbitrary points in a hyperbola. Let B_1 be arbitrary point in the hyperbola. The circle $(A_iA_{i+1}B_i)$ meets the hyperbola at point B_{i+1} for $i = 1, 2, \cdots, 6$. Let circle $(A_1A_2B_1)$ meets the circle $(A_4A_5B_4)$ at A, B, circle $(A_2A_3B_2)$ meets the circle $(A_5A_6B_5)$ at C, D, circle $(A_3A_4B_3)$ meets the circle $(A_6A_1B_1)$ at E, F, then six points A, B, C, D, E, F lie on a circle. (Figure 12)*

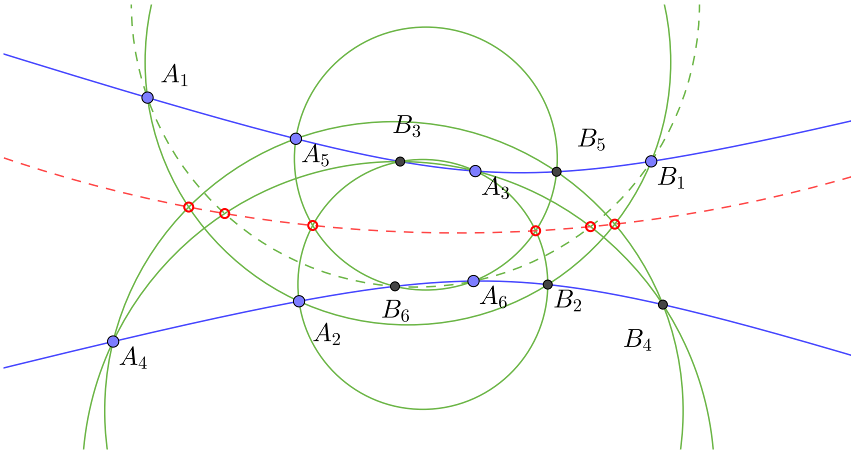

FIGURE 12.

1. If B_1 at ∞, six circles are six lines, so the theorem is Pascal theorem.

2. If the hyperbola is two lines, and B_1 at ∞ then six circles are six lines, the theorem is Pappus theorem.

You can see the proof of Theorem 2.5 by **Arseniy Akopyan** on *Mathoverflow* forum [16].

Theorem 2.7. *([18]). Let L_1, L_2 be two parallel lines, let A, B, C, D be four points in a plane. Let E be a point lie on the line L_1, F be the point lie on line L_2 such that $EF \parallel AB$. Let circle (E, EC) meets the circle (F, FD) at two points H, G. Then locus of H, G is a conic section when E be moved on line L_1 (Figure 13).*

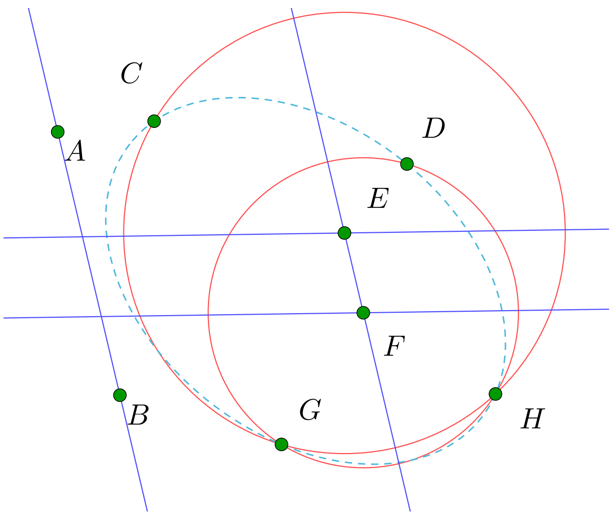

FIGURE 13.

You can see the proof of Theorem 2.7 by Cherng-tiao Perng in *Mathoverflow* forum [18]:

Problem 2.5. *Let four circles (O_1), (O_2), (O_3), (O_4). Let (O_1) meets (O_2) at points P_1, P_2 ; (O_2) meets (O_3) at points P_3, P_4; (O_3) meets (O_4) at points P_5, P_6; (O_4) meets (O_1) at points P_7, P_8. Then $O_1O_2O_3O_4$ is a parallelogram if only if P_1, P_2, P_3, P_4, $\cdots$,P_8 lie on a conic (Figure 14).*

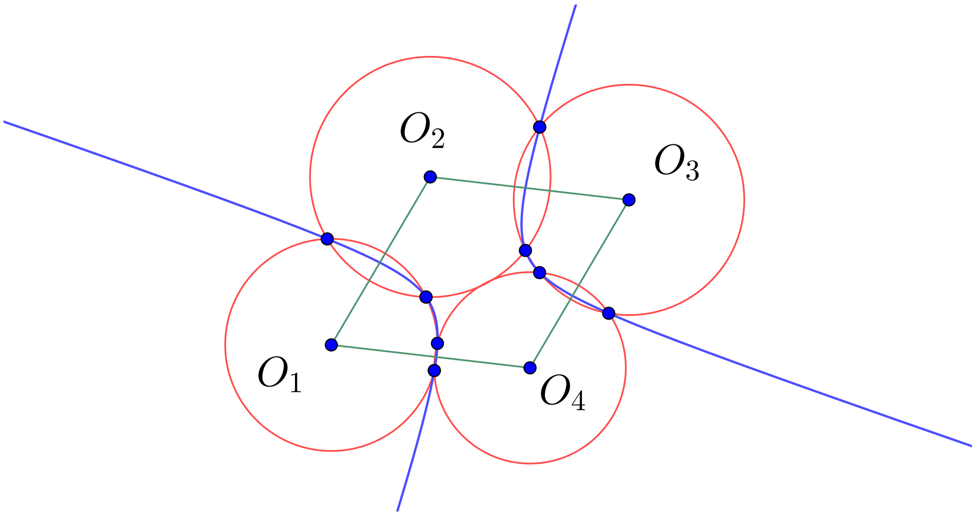

Figure 14.

Problem 2.6. *Let A_1, A_2, A_3, A_4, A_5, A_6 be a cyclic hexagon. The circle with diameter A_iA_{i+1} meets the circle with diameter $A_{i+1}A_{i+2}$ at B_i for $i = 1, 2, .., 6$. Then six points B_1, B_2, B_3, B_4, B_5, B_6 lie on a conic (Figure 15).*

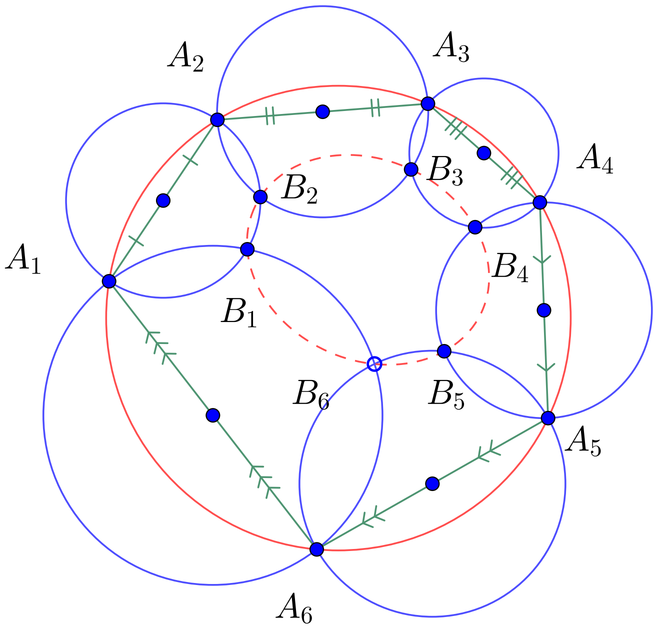

Figure 15.

Theorem 2.8. *Let ABC be a triangle P be a point in the plane, and O be the circumcenter. O_a, O_b, O_c be centers of three circles (BCP), (CAP), (ABP). O_bO_c meets OA at A'. Define B', C' cyclically. Then $A'O_a, B'O_b, C'O_c$ are concurrent (Figure 16).*

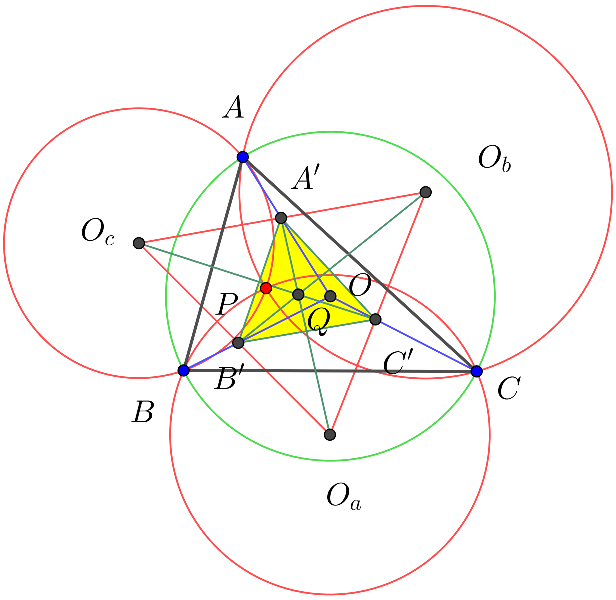

FIGURE 16.

Remark. *We can use direct the Eight circle theorem to solve Theorem 2.8*

Theorem 2.9. *Let ABC be a triangle (O) be any circle on the plane, P_1 be arbitrary point in (O). (BCP_1) meets (O) again at P_2, (CAP_2) meets (O) again at P_3, (ABP_3) meets (O) again at P_4, (BCP_4) meets (O) again at P_5, (CAP_4) meets (O) again at P_6, (ABP_6) meets (O) again at P_7 P_7 then $P_7 \equiv P_1$ (See Figure 17)*

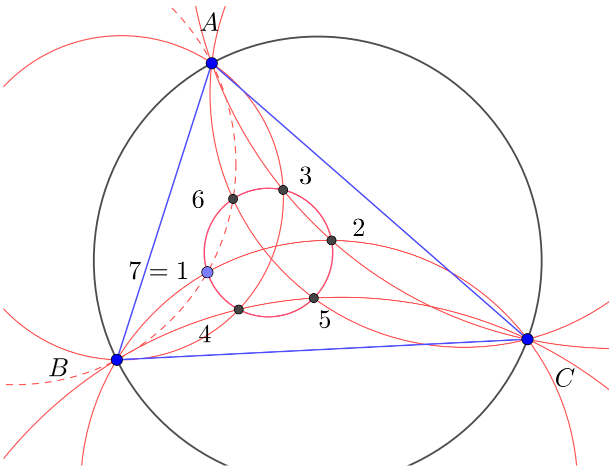

FIGURE 17.

Remark. *We can use direct the Eight circle theorem to solve Theorem 2.9*

Theorem 2.10. *Define is as Theorem 2.9, then circles (BP_1P_4), (CP_2P_5), (AP_3P_6) are tangent to (ABC) (See Figure 18)*

Theorem 2.11. *Define as theorem 18 and Theorem 2.9 (See figure 19).*

1. Let (O_a), (O_b), (O_c) are the center of circles (BP_1P_2C), (CP_5P_3A), (AP_6P_4B) respectively. Then AO_a, BO_b, CO_c are concurrent. Let the point of concurrence is P.

2. Let (O'_a), (O'_b), (O'_c) are the center of circles (BP_4P_5C), (CP_2P_6A), (AP_3P_1B) respectively. Then AO'_a, BO'_b, CO'_c are concurrent. Let the point of concurrence is Q.

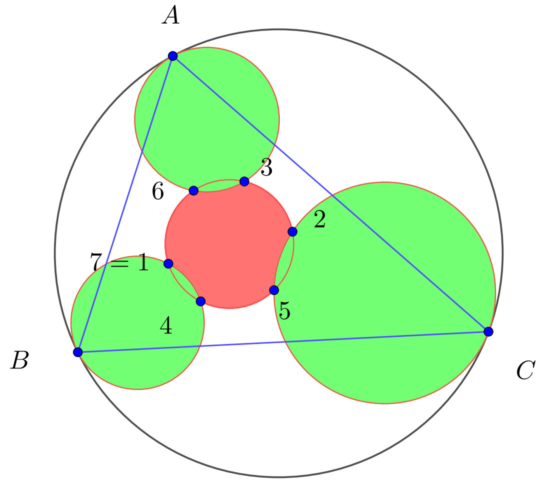

FIGURE 18.

3. Three points P, Q and circumcenter of ABC are collinear.

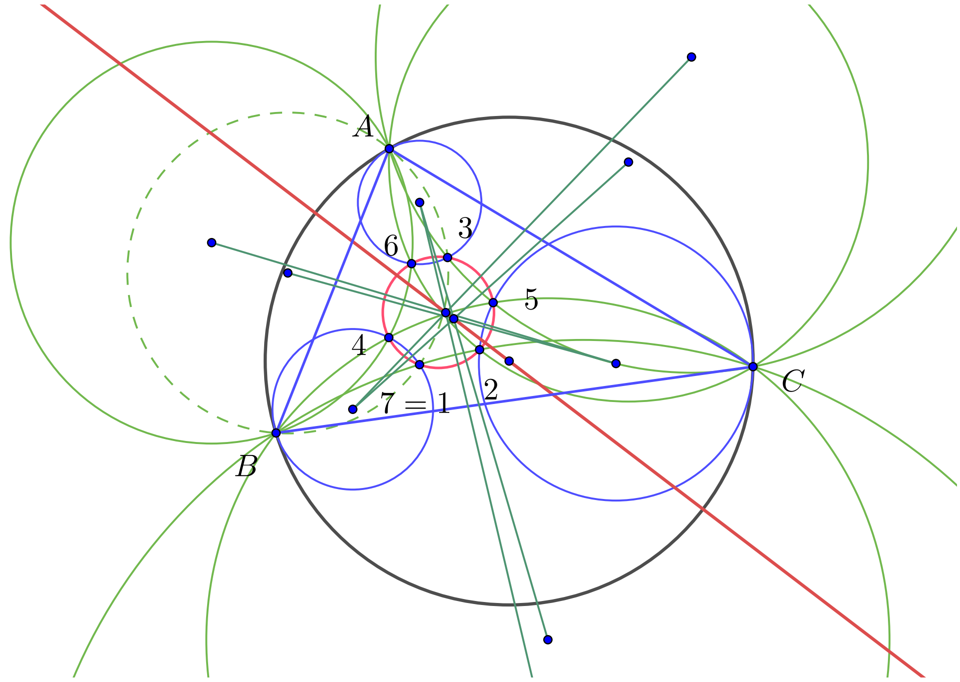

FIGURE 19.

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