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# A Purely Synthetic Proof of Dao's Theorem On A Conic And Its Applications

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**Abstract.** We will give a purely synthetic proof of Dao's theorem on a conic. Then we use the Dao's theorem to prove eight famous theorems in Euclidean geometry and projective geometry.

**Keywords.** Collinear, hexagon, conic section, pole, polar

## 1. Introduction

In 2013-2014, Dao Thanh Oai published without proof the following remarkable theorem:

**Theorem 1.1** (Dao). *Let $ABC$ be a triangle inscribed in a conic $(S)$, $P$ be a point in the plane of $ABC$. Let $AP$, $BP$, $CP$ meet $(S)$ again at $A_1$, $B_1$, $C_1$ respectively. Let $D$ be a point lies on $(S)$ or $D$ lies on the polar line of $P$ respect to $(S)$. Let $A_2 = DA_1 \cap BC$, $B_2 = DB_1 \cap CA$, $C_2 = DC_1 \cap AB$, then $A_2$, $B_2$, $C_2$ are collinear. Further more, $A_2$, $B_2$, $C_2$ and $P$ are collinear if and only if $D$ lies on $(S)$.*

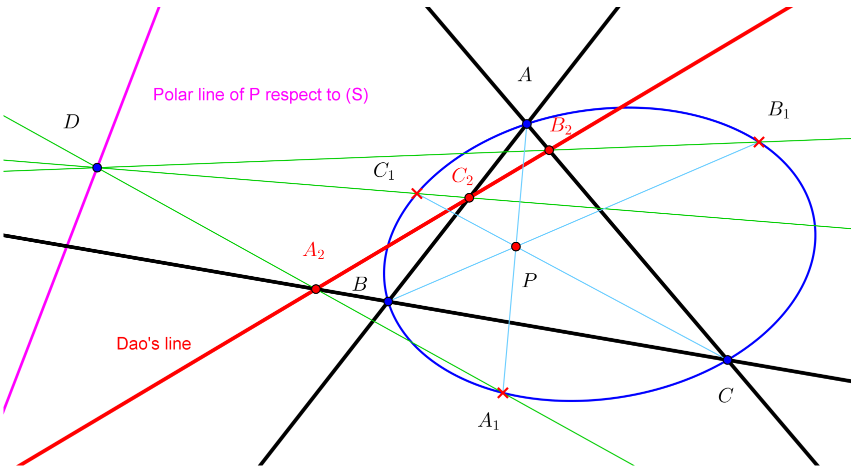


Figure 1.

Theorem 1.1. is a synthesis of some Dao's statements. Firstly, the case $D$ lie on $(S)$ you can see in [1] [2]. With the case $P$ at infinity, polar line of $P$ is line throgh center of the conic $(S)$ and $D$ lie on the polar line of $P$ in [3], Tran Hoang Son given a synthetic proof of this case [4], another proof by Nguyen Minh Ha and Pham Nam Khanh [5]. Finally, the complete statement of theorem 1.1 in [6][7]. Nguyen Ngoc Giang give first proof of the case $D$ lie on the polar line of $P$ by coordinate [8].

Anyway, Theorem 1.1 has really a long history. The case $D$ lies on the conic was also found by Geoff Smith independently. However, he confirmed the similarity to Dao Thanh Oai [9]. When the conic is a circle and $D$ lies on the circle, the converse of theorem were published by Petrisor Neagoe in 2010 [10]. But, in the case conic is a circle and $D$ lies on the circle this is Aubert-Neuberg's theorem [2][11].

So there are many people found some special cases of this theorem independent. But Dao Thanh Oai who found the general case. We can use Dao's statement to prove many theorem, for example Droz-Farny's theorem, Goormaghtigh's theorem, Zaslavsky's theorem, Dao-Tran's theorem, Colling's theorem, Carnot's theorem, the Simson line theorem, Bliss' theorem, and Nixon's theorem. So I call this theorem is the Dao's theorem on a conic.

## 2. A SYNTHETIC PROOF OF DAO'S THEOREM ON A CONIC

Firstly, we give a proof of the lemma as follows:

**Lemma 2.1.** *Let $ABC$ be a triangle inscribed in a conic $(S)$ and $P$ be a point on the plane of $ABC$. Let $AP$, $BP$, $CP$ meet the conic $(S)$ again at $A_1$, $B_1$, $C_1$ respectively. Let $A_2$, $B_2$, $C_2$ be three points on $(S)$ and $A_3 = A_1A_2 \cap BC$, $B_3 = B_1B_2 \cap CA$, $C_3 = C_1C_2 \cap AB$, then $AA_2$, $BB_2$, $CC_2$ are concurrent if and only if $A_3$, $B_3$, $C_3$ are collinear.*

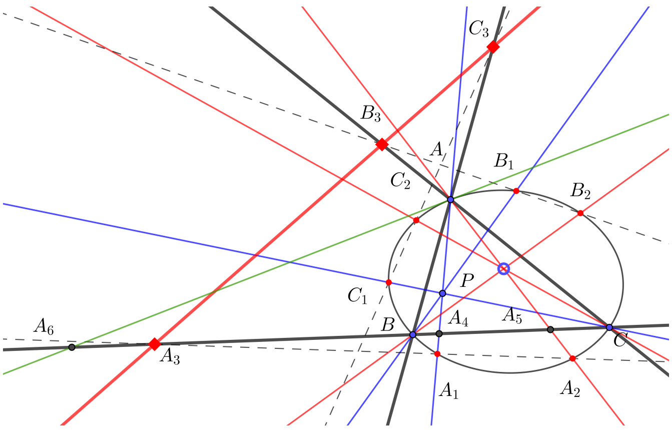


FIGURE 2.

*Proof.* Let be $AA_1$, $AA_2$ and the tangent at $A$ of $(S)$ meets $BC$ at $A_4$, $A_5$, $A_6$ respectively. Define $B_4$, $B_5$, $B_6$, $C_4$, $C_5$, $C_6$ cyclically. By the Pascal theorem for six points $A$, $A$, $B$, $B$, $C$, $C$, we get $A_6$, $B_6$, $C_6$ are collinear.

Now, we have: $(A_3A_4BC) = (A_3A_5BC) . (A_5A_4BC)$

But $(A_3A_5BC) = A_2 (A_3A_5BC) = (A_1ABC) = A (A_1ABC) = (A_4A_6BC)$

Thus, $(A_3A_4BC) = (A_4A_6BC) . (A_5A_4BC) = (A_5A_6BC)$

Similarly, we can prove:

$$\begin{cases} (B_3B_4CA) = (B_5B_6CA) \\ (C_3C_4AB) = (C_5C_6AB) \end{cases}$$

Hence,

$$(A_3A_4BC)\,(B_3B_4CA)\,(C_3C_4AB) = (A_5A_6BC)\,(B_5B_6CA)\,(C_5C_6AB)$$

On the other hand, we have:

$$\left(\frac{\overline{A_3B}}{\overline{A_3C}}\cdot\frac{\overline{B_3C}}{\overline{B_3A}}\cdot\frac{\overline{C_3A}}{\overline{C_3B}}\right)\cdot\left(\frac{\overline{A_4C}}{\overline{A_4B}}\cdot\frac{\overline{B_4A}}{\overline{B_4C}}\cdot\frac{\overline{C_4B}}{\overline{C_4A}}\right) = \left(\frac{\overline{A_5B}}{\overline{A_5C}}\cdot\frac{\overline{B_5C}}{\overline{B_5A}}\cdot\frac{\overline{C_5A}}{\overline{C_5B}}\right)\cdot\left(\frac{\overline{A_6C}}{\overline{A_6B}}\cdot\frac{\overline{B_6A}}{\overline{B_6C}}\cdot\frac{\overline{C_6B}}{\overline{C_6A}}\right)$$

By Ceva's theorem for triangle and three lines are concurrent, we get

$$\frac{\overline{A_4C}}{\overline{A_4B}}\cdot\frac{\overline{B_4A}}{\overline{B_4C}}\cdot\frac{\overline{C_4B}}{\overline{C_4A}} = -1$$

By Menelaus theorem to triangle $ABC$ and three points $A_6$, $B_6$, $C_6$ are collinear, we get

$$\frac{\overline{A_6C}}{\overline{A_6B}}\cdot\frac{\overline{B_6A}}{\overline{B_6C}}\cdot\frac{\overline{C_6B}}{\overline{C_6A}} = 1$$

Consequently,

$$-\frac{\overline{A_3B}}{\overline{A_3C}}\cdot\frac{\overline{B_3C}}{\overline{B_3A}}\cdot\frac{\overline{C_3A}}{\overline{C_3B}} = \frac{\overline{A_5B}}{\overline{A_5C}}\cdot\frac{\overline{B_5C}}{\overline{B_5A}}\cdot\frac{\overline{C_5A}}{\overline{C_5B}}$$

Again, by Menelaus theorem and Céva theorem, we get: $AA_2$, $BB_2$, $CC_2$ are concurrent if and only if $A_3$, $B_3$, $C_3$ are collinear.

Now, we back to proof of Theorem 1.

**Case 1:** $D$ lies on $(S)$.

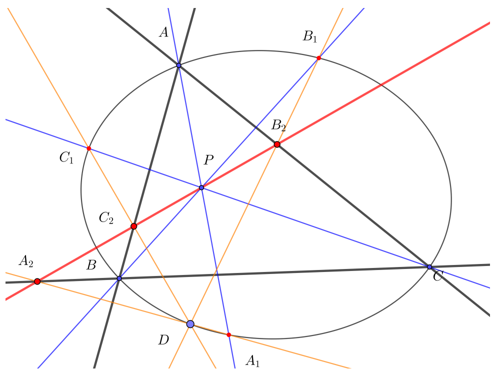


Figure 3.

We apply the lemma to the triangle $ABC$ with $AA_1$, $BB_1$, $CC_1$ are concurrent and $AD$, $BD$, $CD$ are concurrent. Then $DA_1 \cap BC$, $DB_1 \cap CA$, $DC_1 \cap AB$ are collinear.

**Case 2:** $D$ lies on on the polar line of $P$ in $(S)$.

Let $A_2$, $A_3$ be the second intersections of $DA_1, DA$ with $(S)$ respectively, and $E$ be the intersection of $AA_2$ with $A_1A_3$. Since $D$ lies on the polar line of $P$ in $(S)$, $D$ is conjugate to $P$ in $(S)$. If we denote by $P$ the intersections of $AA_1$, $A_2A_3$, $D$ will be conjugate to $P'$ in $(S)$. Thus, $P \equiv P'$. In other words, $A_3, P, A_1$ are collinear. Hence, $E$ is the pole of $DP$

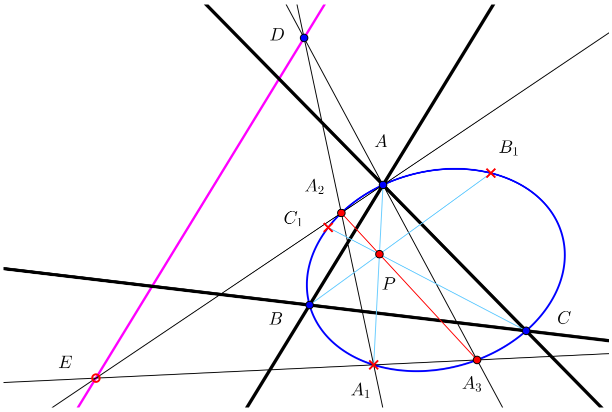


FIGURE 4.

in $(S)$. It means that $AA_2$ passes through the pole of $DP$ in $(S)$. Similarly, we define $B_2$, $C_2$ and prove that $BB_2$, $CC_2$ pass through the pole of $DB$ in $(S)$. Consequently, $AA_2$, $BB_2$, $CC_2$ are concurrent.

By the lemma to the triangle $ABC$ inscribed in $(S)$ with $AA_1$, $BB_1$, $CC_1$ are concurrent and $AA_2$, $BB_2$, $CC_2$ are concurrent, we get: $A_1A_2 \cap BC$, $B_1B_2 \cap CA$, $C_1C_2 \cap AB$ are collinear. It follows that: $DA_1 \cap BC$, $DB_1 \cap CA$, $DC_1 \cap AB$ are collinear. □

## 3. SOME SPECIAL CASE OF DAO'S THEOREM ON A CONIC

**Theorem 3.1** (Droz-Farny)**.** *Let $ABC$ be a triangle and $H$ be the its orthocenter. Let $l_1$, $l_2$ be two perpendicular lines through $H$. Let $l_1$ meets $BC$, $CA$, $AB$ at $A_1$, $B_1$, $C_1$ respectively and $l_2$ meets $BC$, $CA$, $AB$ at $A_2$, $B_2$, $C_2$. Then, the midpoints of $A_1A_2$, $B_1B_2$, $C_1C_2$ are collinear.*

The Droz-Farny's theorem is a special case of the Goormaghtigh theorem as follows [12][5].

**Theorem 3.2** (Goormaghtigh-[12])**.** *Let $ABC$ be a triangle and $P$ be a arbitrary point on the plane of $ABC$. Let $l$ be arbitrary line through $P$. Then reflection of lines $AP$, $BP$, $CP$ in $l$ meet $BC$, $CA$, $AB$ at three collinear points. respectively*

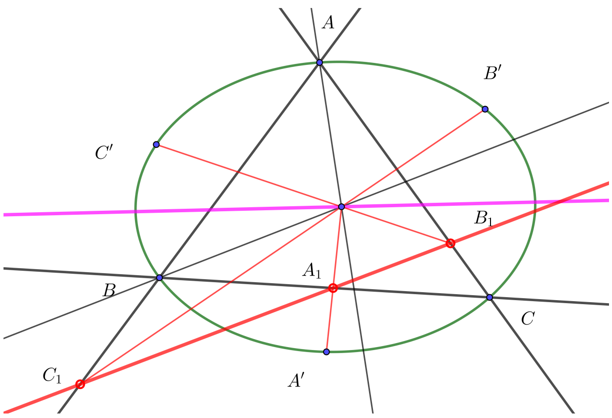


FIGURE 5. Goormaghtigh theorem

*Proof.* Let $A'$, $B'$, $C'$ be the reflections of $A$, $B$, $C$ in $l$ respectively. Then $PA'$, $PB'$, $PC'$ be the reflections of $AP$, $BP$, $CP$ in $l$ respectively and $A$, $B$, $C$, $A'$, $B'$, $C'$ lie on a conic $(S)$ which its major axis is $l$.

By the Dao's theorem on a conic to two triangles $ABC$, $A'B'C'$ inscribed in with $AA'$, $BB'$, $CC'$ concur at an infinite point $\infty$ and $P$ lies on $l$ which is the pole of $\infty$ in $(S)$, we get: $PA' \cap BC$, $PB' \cap CA$, $PC' \cap AB$ at three collinear points. □

**Theorem 3.3** (Zaslavsky - [13]). *Let $ABC$ be a triangle and $P$ be arbitrary point in the plane of $ABC$. Let $A'$, $B'$, $C'$ be the reflections of $A$, $B$, $C$ in $P$ respectively. Three parallel line through $A'$, $B'$, $C'$ meet $BC$, $CA$, $AB$ at $A_0$, $B_0$, $C_0$ respectively. Then $A_0$, $B_0$, $C_0$ are collinear.*

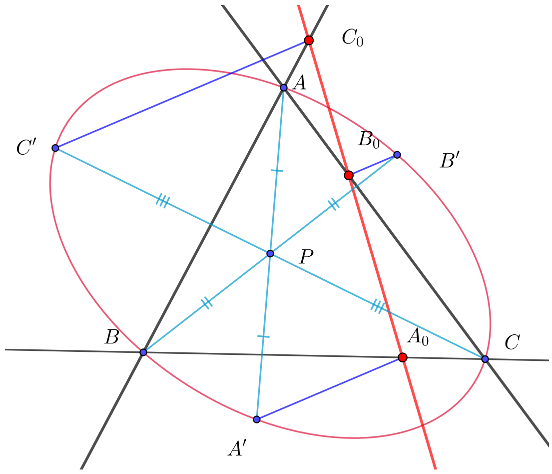


FIGURE 6. Zaslavsky theorem

*Proof.* We have $A$, $B$, $C$, $A'$, $B'$, $C'$ lie on a conic $(S)$ which its major axis is $l$.

By the Dao's theorem on a conic to two triangles $ABC$, $A'B'C'$ inscribed in with $AA'$, $BB'$, $CC'$ concur at an infinite point $\infty$ and $P$ lies on $l$ which is the pole of $\infty$ in $(S)$, we get: $PA' \cap BC$, $PB' \cap CA$, $PC' \cap AB$ at three collinear points. □

Moreover, another proof of Zaslavsky theorem, which belongs to Danrij Grinberg, can be found in [14].

**Theorem 3.4** (Dao-Tran [4][5]). *Suppose the midpoints of the parallel segments $AA'$, $BB'$, $CC'$ lie on a line $l$. Let $D$ be a point on $l$. Then $DA'$, $DB'$, $DC'$ intersect $BC$, $CA$, $AB$ at three collinear points respectively*

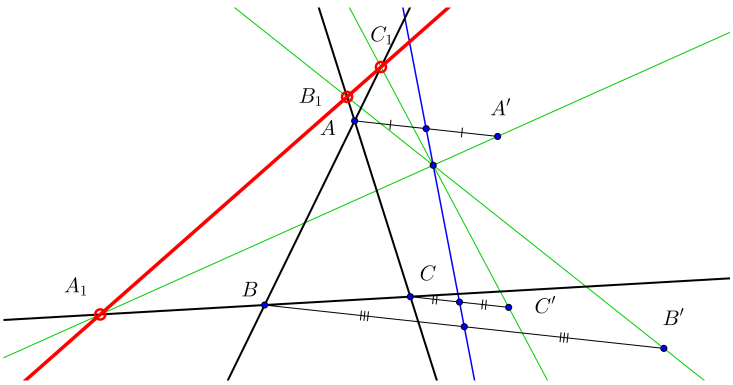


FIGURE 7. Dao-Tran

*Proof.* We have $A$, $B$, $C$, $A'$, $B'$, $C'$ lie on a conic $(S)$ which axis is $l$. By the Dao's theorem on a conic for two triangles $ABC$, $A'B'C'$ inscribed in $(S)$ with $AA'$, $BB'$, $CC'$ concur at an infinite point $\infty$ and $D$ lies on $l$ which is the pole of $\infty$ in $(S)$, we get: $DA'$, $DB'$, $DC'$ meet $BC$, $CA$, $AB$ at three collinear points respectively. □

You can see some other proofs of Theorem 5 in [4][5].

**Theorem 3.5** (Colling-[15])**.** *Let $ABC$ be a triangle and $l$ be a line on the plane of triangle $ABC$. Let $l_A$, $l_B$, $l_C$ be the reflections of $l$ in $BC$, $CA$, $AB$ respectively. Then $l_A$, $l_B$, $l_C$ are concurrent if and only if $l$ through the orthocenter of the triangle $ABC$.*

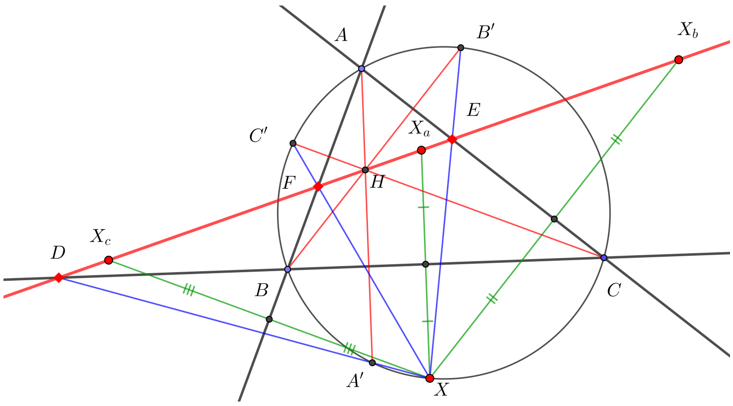


FIGURE 8. Colling theorem

*Proof.* (⇒) Suppose that $l_A$, $l_B$, $l_C$ concur at $X$.

Let $D$, $E$, $F$ be the intersection of $l$ with $BC$, $CA$, $AB$, then $D$, $E$, $F$ lie on $l_A$, $l_B$, $l_C$ respectively. So we have:

$\angle BXC = \angle BXF + \angle FXE + \angle EXC = \angle DBF - 90° + 2(90° - \angle FAE) + 90° - \angle ECD = 90° - \angle ABC + 2(90° - \angle FAE) + 90° - \angle ACB = 180° - \angle BAC$

Therefore, $X$, $A$, $B$, $C$ lie on $(O)$. Let $X_A$, $X_B$, $X_C$ be points reflection of $X$ in $BC, CA, AB$ respectively. Then $X_A$, $X_B$, $X_C$ lie on $l$. According to Steiner theorem, $l$ passes through the orthocenter of the triangle $ABC$

Suppose that $l$ passes through the orthocenter $H$ of the triangle $ABC$. Let $A'$, $B'$, $C'$ be points reflection of $H$ in $BC$, $CA$, $AB$ respectively. Then $A'$, $B'$, $C'$ lie on $l_A$, $l_B$, $l_C$ respectively and on $(O)$.

By the Dao's theorem on a conic for two triangles $ABC$, $A'B'C'$ inscribed in $(O)$ with $AA'$, $BB'$, $CC'$ concur at $H$ and $D$, $E$, $F$ lie on $BC$, $CA$, $AB$ respectively such that $D$, $E$, $F$, $H$ are collinear, we get: $DA' \equiv l_A$, $EB' \equiv l_B$, $FC' \equiv l_C$ concur at a point. □

**Theorem 3.6** (Carnot-[16])**.** *Let $ABC$ be a triangle inscribed in a circle $(O)$ and $M$ be a point lies on $(O)$. Let $D$, $E$, $F$ be points on $BC$, $CA$, $AB$ such that $(MD, BC) \equiv (ME, CA) \equiv (MF, AB) \pmod{\pi}$, Then $D$, $E$, $F$ are collinear.*

*Proof.* Let $A'$, $B'$, $C'$ be the second intersections of $MD$, $ME$, $MF$ with $(O)$. Since $(DM, DC) = (MD, BC) = (ME, CA) = (EM, EC) \pmod{\pi}$, $MDEC$ is an inscribed quadrilateral . So we have: $(CA, CB) = (CE, CD) = (ME, MD) = (MB', MA') \pmod{\pi}$.

Therefore $AB = A'B'$. It means that $AA' \parallel BB'$. Similarly, we get $AA' \parallel BB' \parallel CC'$. By the Dao's theorem on a conic to two triangles $ABC$, $A'B'C'$ inscribed in $(O)$ with $AA'$, $BB'$, $CC'$ concur at a infinite point and $M$ lies on $(O)$ , we get: $D$, $E$, $F$ are collinear □

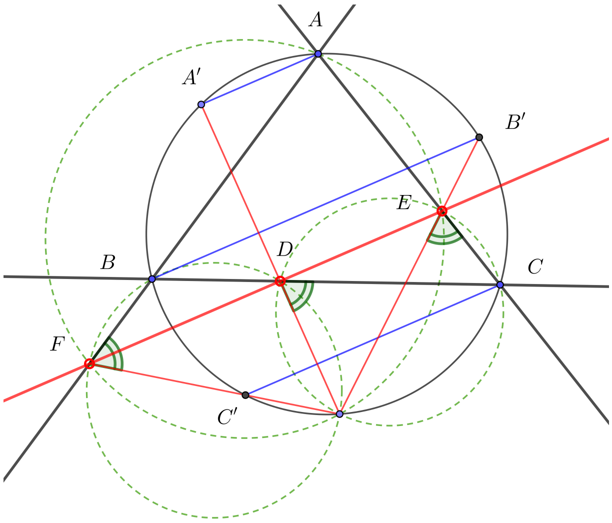


Figure 9.

Noted that the Carnot theorem is a generalization of the Simson line theorem [16].

**Theorem 3.7** (Bliss-[17])**.** *Let $ABC$ be a triangle and $D$, $E$, $F$ are midpoints of $BC$, $CA$, $AB$ respectively. Let $l_A$, $l_B$, $l_C$ be parallel lines through $D$, $E$, $F$ respectively. Then reflections of $BC$, $CA$, $AB$ in $l_A$, $l_B$, $l_C$ respectively are concurrent and the point of concurrence lies on the Nine points circle.*

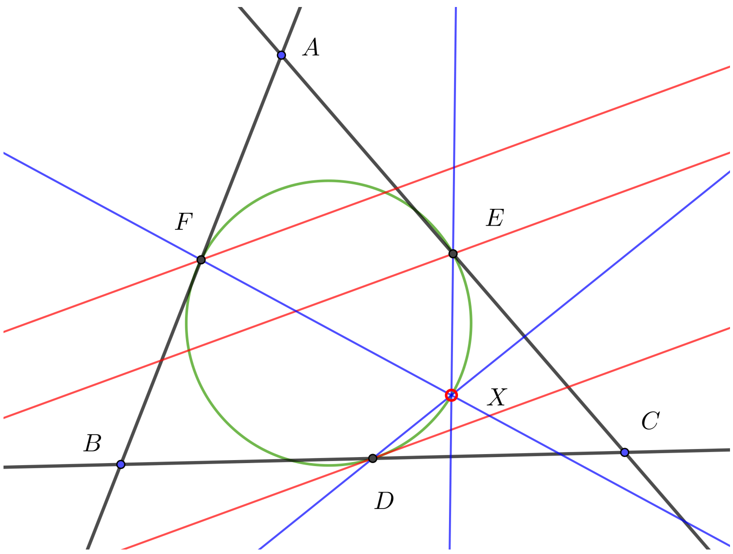


Figure 10.

*Proof.* Let $A'$, $B'$, $C'$ be the second intersections of $l_A$, $l_B$, $l_C$ with the Euler circle of the triangle $ABC$. By the Dao's theorem on a conic to two triangles $DEF$, $A'B'C'$ inscribed in the Euler circle of the triangle $ABC$ with $DA'$, $EB'$, $FC'$ concur at a infinite point, we get three lines, which are reflection of $BC$, $CA$, $AB$ in $l_A$, $l_B$, $l_C$ respectively, concur at a point lie on the Euler circle of the triangle $ABC$. Since $\angle AFC' = \angle A'B'E = \angle DEB' = \angle XFC'$, $FX$ is reflection of $AB$ in $FC' \equiv l_C$. Similarly: $EX$, $DX$ is reflection of $CA$, $BC$ in $l_B$, $l_C$ respectively. We are done.

We omit the proof of the following: $\square$

**Theorem 3.8** (Nixon-[18]). *A circle is tangent internally (or externally) to the circumcircle of a triangle and to two sides of the triangle, the line joining its points of contact with the sides through the incenter (or excenter) of the triangle.*

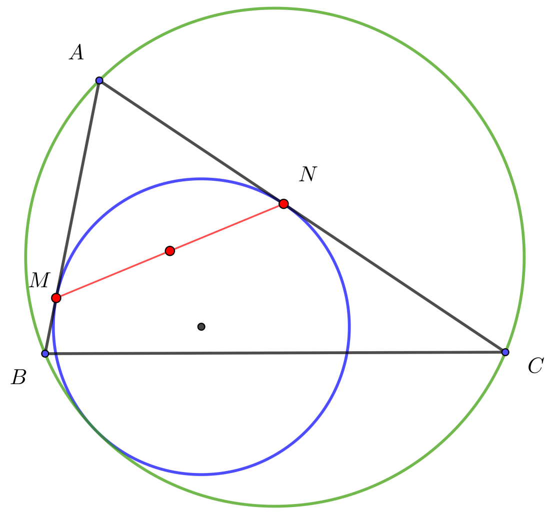


FIGURE 11.

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