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Triangles Homothetic with the Extouch Triangle

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Abstract. By using the computer program "Discoverer" we study the Grebe triangles.

Keywords. triangle geometry, remarkable point, computer discovered mathematics, Euclidean geometry, "Discoverer".

Mathematics Subject Classification (2010). 51-04, 68T01, 68T99.

1. Introduction

The Extouch triangle is the triangle formed by the points of tangency of a triangle ABC with its excircles. See Extouch triangle in [7].

The Extouch triangle is the Cevian triangle of the Nagel point (See the Nagel point in [7], and the pedal triangle of the Bevan point (See the Bevan point in [7].

The computer program "Discoverer" created by the authors [4], [5], has discovered many theorems about triangles homothetic with the Extouch triangle. Here we present a few of these theorems as problems for students.

The reader may find problems about triangles similar (but not homothetic) with the Extouch triangle in [6].

Given triangle ABC, we denote the side lengths as follows: $a = BC, b = CA$ and $c = AB$.

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2. PROBLEMS ABOUT HOMOTHETIC TRIANGLES

In this paper we denote by $PaPbPc$ the Extouch triangle and by $QaQbQc$ a triangle homothetic with the Extouch triangle.

References for Problem 1:

- Anticevian triangle in [7].
- Mittenpunkt in [7].

Problem 1. *The Extouch triangle is homothetic with the Anticevian Triangle of the Mittenpunkt. Denote by*

- L_1 *the Line through the Incenter and Centroid,*
- L_2 *the Line through the Mittenpunkt and the Internal center of Similitude of the Incenter and the Circumcenter,*
- X *the intersection point of lines* L_1 *and* L_2*,*

Point X *is the center of the homothety. The ratio of the homothety is*

$$k = \frac{4abc}{(b+c-a)(c+a-b)(a+b-c)} > 0.$$

Figure 1 illustrates Problem 1.

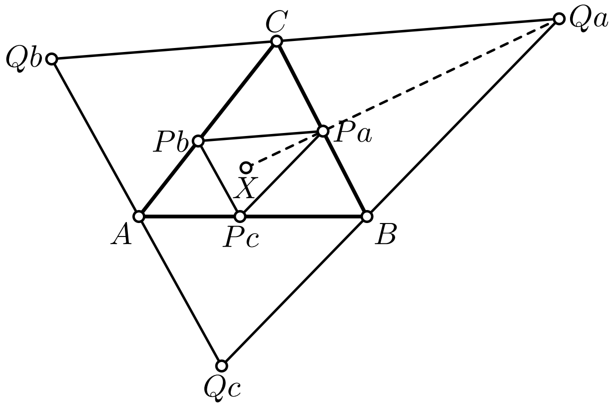

FIGURE 1.

Proof. We will use barycentric coordinates. See e.g. [3].

The Nagel point $Na = (u, v, w)$ has barycentric coordinates $u = b + c - a, v = c + a - b, w = a + b - c$, and the Cevain traingle of the Nagel point, that is, the Extouch triangle, $PaPbPc$, has barycentric coordinates

$$Pa = (0, v, w), Pb = (u, 0, w), Pc = (u, v, 0).$$

The Mittenpunkt has barycentric coordinates

$$u_1 = a(b + c - a), \quad v_1 = b(c + a - b), \quad w_1 = c(a + b - c),$$

and the Anticevian triangle of the Mittenpunkt $QaQbQc$ has barycentric coordinates

$$Qa = (-u_1, v_1, w_1), \quad Qb = (u_1, -v_1, w_1), \quad Qc = (u_1, v_1, -w_1).$$

First, we will prove that triangles $PaPbPc$ and $QaQbQc$ are homothetic. If points Pa, Pb and Pc are in normalized barycentric coordinates, then the infinity points of lines $PbPc, PcPa$ and $PaPb$ are as follows:

$$\begin{aligned} I_{PbPc} &= Pb - Pc = ((b-c)(b+c-a), b(a+c-b), -c(a+b-c)), \\ I_{PcPa} &= Pc - Pa = (a(b+c-a), (a-c)(a+c-b), -c(a+b-c), \\ I_{PaPb} &= Pa - Pb = (a(b+c-a), -b(a+c-b), (a-b)(a+b-c). \end{aligned}$$

In a similar way, we find the infinity points of the side lines of triangle $QaQbQc$. We see that the infinity points of the corresponding side lines of these triangles are the same. Hence, the triangles are homothetic.

In order to find the center of the homothety, we find the intersection point X_1 of the lines $PaQa$ and $PbQb$. By using formula (3) in [3], we find the barycentric equations of lines $PaQa$ and $PbQb$, and then by using formula (5) in [3], we obtain

$$X_1 = (a(b+c-a)^2, b(c+a-b)^2, c(a+b-c)^2).$$

Next we find the ratio of the homothety. Point Qa lies on the ray with initial point X_1 and through point Pa, so that the ratio is positive. By using the distance formula (9) in [**?**], we find the lengths of segments X_1Pa and X_1Qa, and then we find the ratio:

$$k = \frac{X_1Qa}{X_1Pa} = \frac{4abc}{(b+c-a)(c+a-b)(a+b-c)}.$$

To complete the solution, we have to prove that the intersection X of the lines L_1 and L_2 coincides with the center of the homothety X_1. By using formula (3) in [Grozdev & Dekov, 2016-A] we find the barycentric equations of lines L_1 and L_2, and then, by using formula (5) in [3], we find the intersection X of these lines. We see that the points X and X_1 coincide. This completes the proof. □

References for Problem 2:

- Euler Anticevian triangle in [2],
- Mittenpunkt in [7],
- Schiffler Point in [7].

Problem 2. *The Extouch triangle is homothetic with the Euler Anticevian Triangle of the Mittenpunkt. Denote by*

- L_1 *the line through the Incenter and the Centroid,*
- L_2 *the line through the Mittenpunkt and Schiffler point,*
- P *the intersection point of lines* L_1 *and* L_2*,*
- L_3 *the line through the Mittenpunkt and the Internal Center of Similitude of the Incircle and Circumcircle,*
- L_4 *the line through point* P *and the External Center of Similitude of the Incircle and Circumcircle.*
- X *the intersection point of lines* L_3 *and* L_4*,*

Then point X *is the center of the homothety. The ratio of the homothety is*

$$k = \frac{2abc}{(b+c-a)(c+a-b)(a+b-c)} > 0.$$

Figure 2 illustrates Problem 2. In figure 2 $MaMbMc$ is the Anticevian triangle of the Mittenpunkt.

References for Problem 3: Bevan point in [7].

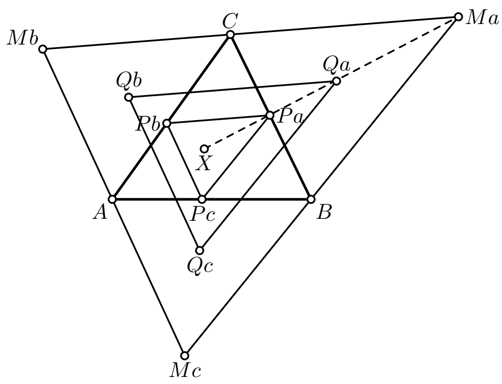

FIGURE 2.

Problem 3. *The Extouch triangle is homothetic with the Triangle of Reflections of the Bevan Point in the Sidelines of Triangle ABC. Then the Bevan point is the center of the homothety. The ratio of the homothety is* $k = 2$.

Figure 3 illustrates Problem 3.

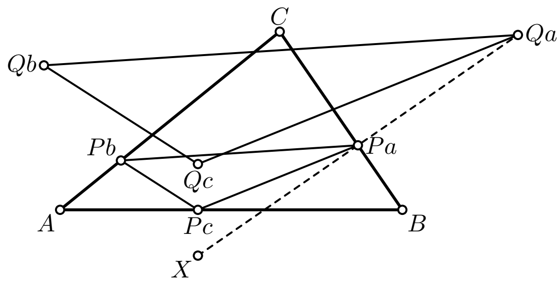

FIGURE 3.

References for Problem 4:

- Gergonne point in [7],
- Half-Cevian Triangle in [1].

Problem 4. *The Extouch triangle is homothetic with the Half-Cevian Triangle of the Gergonne point. The center of the homothety is the Centroid. The ratio of the homothety is* $k = -\dfrac{1}{2}$.

Figure 4 illustrates Problem 4.

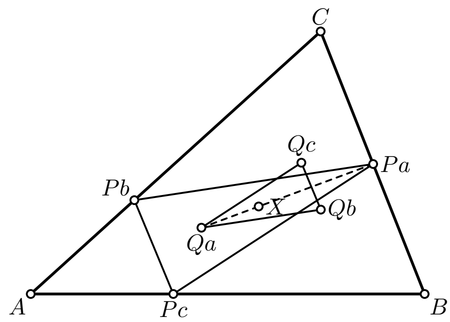

FIGURE 4.

References for Problem 5:

- Bevan point in [7],
- Orthic triangle in [7],
- Mittenpunkt in [7],
- Nagel point in [7].

Problem 5. *Denote by*

- $HaHbHc$ *the Orthic triangle,*
- Qa *the Bevan point of triangle* $AHbHc$*,*
- Qb *the Bevan point of triangle* $HaBHc$*,*
- Qc *the Bevan point of triangle* $HaHbC$*,*
- L_1 *the Line through the Circumcenter and Mittenpunkt,*
- L_2 *the Line through the Orthocenter and Nagel point,*
- X *the intersection point of lines* L_1 *and* L_2*.*

The Extouch triangle is homothetic with triangle $QaQbQc$*. The center of the homothety is point* X*. The ratio of the homothety is* $k = -1$*.*

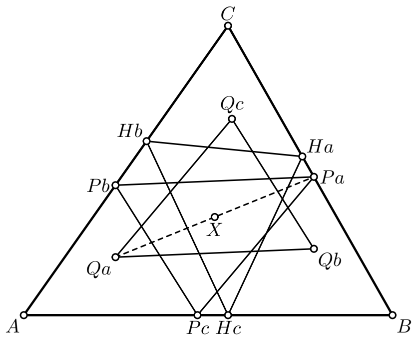

Figure 5.

Figure 5 illustrates Problem 5. In 5,

- $PaPbPc$ is the Extouch triangle,
- $HaHbHc$ is the Orthic triangle,
- $QaQbQc$ is the triangle whose vertices are the Bevan points,
- X is the center of the homothety,

References for Problem 6:

- Intouch triangle in [7],
- de Longchamps point in [7],
- Symmedian point in [7],
- Nine-Point center in [7],
- Spieker center in [7].

Problem 6. *Denote by*

- $MaMbMc$ *the Intouch triangle,*
- Qa *the de Longchamps point of triangle* $AMbMc$*,*
- Qb *the de Longchamps point of triangle* $MaBMc$*,*
- Qc *the de Longchamps point of triangle* $MaMbC$*,*
- L_1 *the Line through the Incenter and Symmedian point,*
- L_2 *the Line through the Nine-Point center and Spieker center,*
- X *the intersection point of lines* L_1 *and* L_2*.*

The Extouch triangle is homothetic with triangle QaQbQc. The center of the homothety is point X. The ratio of the homothety is $k = -1$.

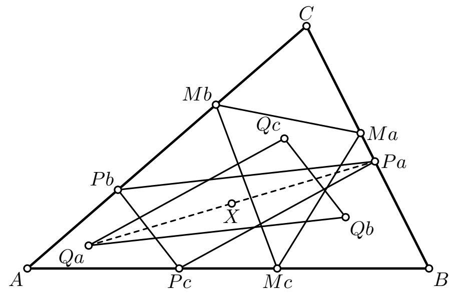

FIGURE 6.

Figure 6 illustrates Problem 6. In 6,

- $PaPbPc$ is the Extouch triangle,
- $MaMbMc$ is the Intouch triangle,
- $QaQbQc$ is the triangle whose vertices are the de Longchamps points,
- X is the center of the homothety,

References for Problem 7:

- Hexyl triangle in [7],
- Perspector in [7],
- External Similitude Center in [7].

Problem 7. *Denote by*

- *point Q the Perspector of triangls ABC and Hexyl triangle,*
- *Qa the de Circumcenter of triangle PQC,*
- *Qb the de Circumcenter of triangle AQC,*
- *Qc the de Circumcenter of triangle ABQ,*
- *point X the reflection of the External center of similitude of Incircle and Circumcircle in Circumcenter.*

The Extouch triangle is homothetic with triangle QaQbQc. The center of the homothety is point X. The ratio of the homothety is

$$k = \frac{2abc}{(b+c-a)(c+a-b)(a+b-c)} > 0.$$

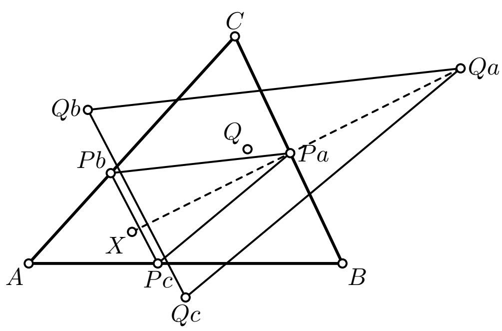

FIGURE 7.

Figure 7 illustrates Problem 7.

Acknowledgements

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