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An Extension of the Steiner Line Theorem and Application

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Abstract. We extend the Steiner line with its synthetic proof as well as introduce an application.

Keywords. Steiner line, anti-steiner, proof.

1. Introduction

The Steiner line theorem is a well known old theorem ([1], [2] and [3]). In [5], it is formulated in the following form.

Theorem 1.1. *If P is a point belonging to the circumcircle of triangle ABC, then the images of P through the reflections with axes BC, CA and AB, respectively lie on the same line that passes through the orthocenter of ABC.*

This line is called the **Steiner line** of P with respect to triangle ABC.

And in [4] and [5], we have the following concerned result.

Theorem 1.2. (*N.S. Collings*). *If a line $\mathcal{L}$ passes through the orthocenter of ABC, then the images of $\mathcal{L}$ through the reflections with axes BC, CA and AB are concurrent at one point on the circumcircle of ABC.*

This point is named the **anti-Steiner point** of $\mathcal{L}$ with respect to ABC. Of course, $\mathcal{L}$ is Steiner line of P with respect to ABC if and only if P is the anti-Steiner point of $\mathcal{L}$ with respect to ABC.

Theorem 1 can be extended as follows.

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Theorem 1.3. ([6]). *Given a triangle ABC inscribed in a circle (O) and the orthocenter H. A line ℓ passes through H. Let P be a point lying on the circle (O) and Q be a point lying on ℓ (Q can be a point at infinity). Lines AQ, BQ, CQ meet (O) at A', B', C', respectively. Line PA', PB', PC' meet ℓ at A_P, B_P, C_P, respectively. Let A_0, B_0, C_0 be the symmetric points of A_P under a symmetry about BC, B_P about CA, C_P about AB, respectively. Then four points A_0, B_0, C_0 and H lie on a line.*

Clearly, when Q belongs to PH or Q belongs to (O), we always obtain the theorem 1.

In this article, we present a synthetic proof of Theorem 1. We use (O), (XYZ) to denote the circle with center O, and the circumcircle of triangle XYZ, respectively. As in [7, p.12], the directed angle from the line a to the line b denoted by (a, b). It measures the angle through which a must be rotated in the positive direction in order to become parallel to, or to coincide with, b. Therefore,
(i) $-90^\circ \leq (a, b) \leq 90^\circ$,
(ii) If c is a line then $(a, b) = (a, c) + (c, b)$,
(iii) Two lines a and a' are parallel or coincident if and only if $(a, b) = (a', b)$,
(iv) If a' and b' are the images of a and b respectively under a reflection, then $(a, b) = (b', a')$,
(v) Four non-collinear points A, B, C, D are concyclic if and only if $(AC, AD) = (BC, BD)$.

2. PROOF OF THEOREM 3

We need a following lemma:

Lemma 2.1. *Given a triangle ABC inscribed in a circle (O) and a line ℓ. Let P be a point lying on (O) and Q be a point lying on ℓ (Q can be a point at infinity). Lines PA, PB, PC meet ℓ at A_P, B_P, C_P, respectively. Then the circles $(BCA_P), (CAB_P)$ and (ABC_P) have a common point lying on ℓ.*

Proof. (See figure 1). Let $A_1 := BC_P \cap CB_P$. Applying the converse of the Pascal theorem for six points $\begin{pmatrix} B & P & C \\ C' & A_1 & B' \end{pmatrix}$ with the note that three points $C_P = BA_1 \cap PC'$, $Q = BB' \cap CC'$, $B_P = PB' \cap CA_1$ lie on the same line ℓ and five points B, P, C, C', B' lie on the same circle (O). If follows that A_1 belongs to (O).

Using the directed angle between two lines, we have

$$\begin{aligned}
(RA, RB_P) &= (RA, RC_P) && \text{(by } R, B_P, C_P \text{ are collinear)} \\
&= (BA, BC_P) && \text{(by } B \in (ARC_P)) \\
&= (BA, BA_1) && \text{(by } B, C_P, A_1 \text{ are collinear)} \\
&= (CA, CA_1) && \text{(by } C \in (ABA_1)) \\
&= (CA, CB_P) && \text{(by } C, B_P, A_1 \text{ are collinear).}
\end{aligned}$$

This thing proves that R belongs to the circle (CAB_P).
Similarly, we also have R belonging to the circle (BCA_P).
Lemma 4 is proved.

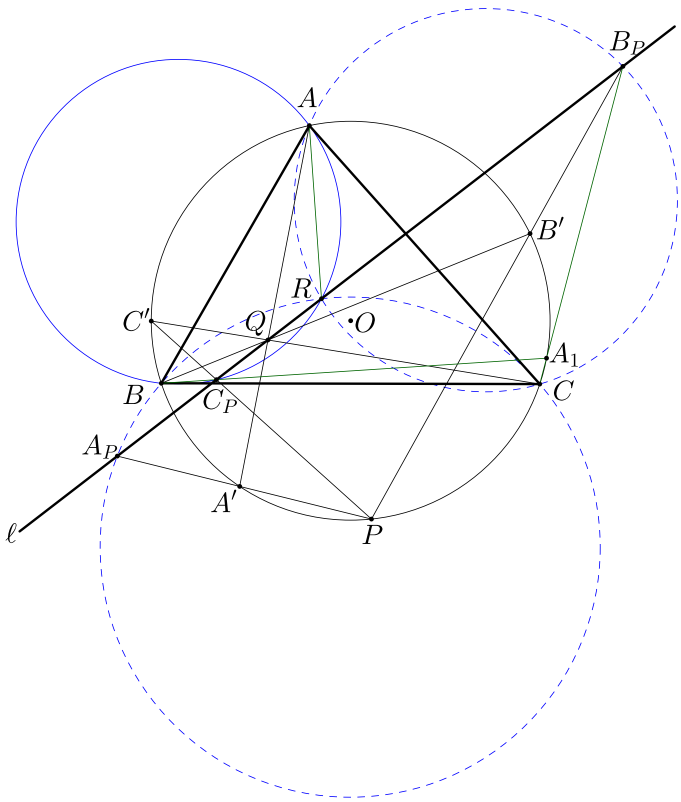

Figure 1.

The proof of theorem 3. (see figure 2). According to the lemma 4, we have $R := \ell \cap (BCA_P) \cap (CAB_P) \cap (ABC_P)$.
Since H belongs to ℓ, by the theorem 2, ℓ has the anti-Steiner point S with respect to triangle ABC.
Line AH meets (O) at A and A_2. We easily see that A_2 is the symmetric point of H under a symmetry about line BC. Hence,

(1) SA_2 is the symmetric line of ℓ under a symmetry about line BC.

It follows that three lines BC, ℓ and SA_2 are either concurrent or pairs of them are parallel each other.
• If BC, ℓ and SA_2 are parallel then we note that each set of four points (B, C, R, A_P) and (B, C, S, A_2) also belongs to a circle so R, S are the symmetric points of A_P, A_2 under a symmetry about the perpendicular bisector of segment BC. It follows that R, S, A_P, A_2 lie on the same circle. Conversely, if $I := BC \cap \ell \cap SA_2$ then by the intersecting chords theorem, we have $\overline{IR}.\overline{IA_P} = \overline{IB}.\overline{IC} = \overline{IS}.\overline{IA_2}$. It follows four points R, S, A_P, A_2 belonging to a circle. Thus, in any case, we always have

(2) R, S, A_P, A_2 belonging to a circle.

On the other hand, we easily see that

(3) A_2A_P is the symmetric line of HA_0 under a symmetry about the line BC.

We have

$$\begin{aligned}(RS, \ell) &= (RS, RA_P) && \text{(by } R, A_P \in \ell)\\ &= (A_2S, A_2A_P) && \text{(by (2))}\\ &= (HA_0, \ell) && \text{(by (1) and (3)).}\end{aligned}$$

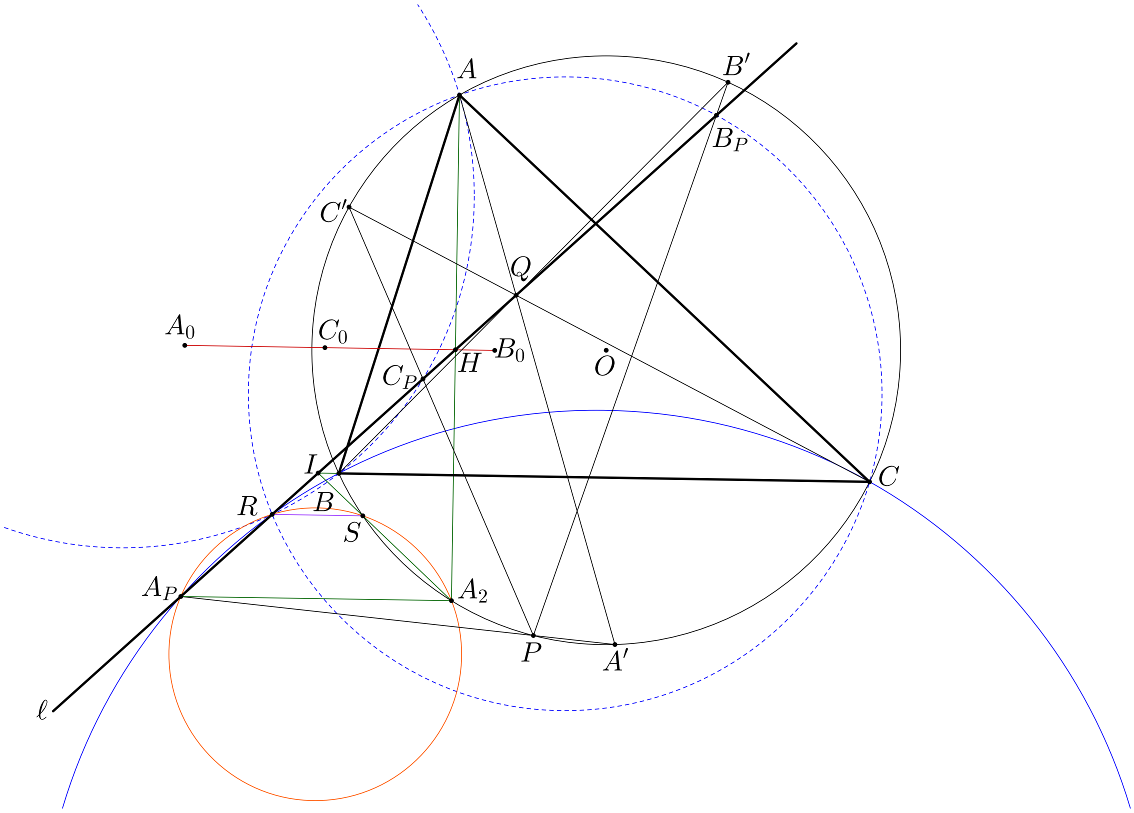

Figure 2.

It follow RS being parallel to or coincident with HA_0.
Similarly, we also have RS being parallel to or coincident with HB_0 and HC_0.
Hence, four points A_0, B_0, C_0 and H lie on the same line being parallel to or coincident with RS. This means that theorem 3 is proved.

3. An Application of Theorem 3

Theorem 3.1. *Given a triangle ABC inscribed in a circle (O) and the orthocenter H. Line ℓ passes through H. Let P be a point lying on (O) and two points Q and D lie on ℓ. Lines AQ, BQ, CQ meet (O) at A', B', C', respectively. Circles $(A'DP), (B'DP), (C'DP)$ meet ℓ at A_P, B_P, C_P, respectively. Let A_0, B_0, C_0 be the symmetric points of A_P under a symmetry about BC, B_P under a symmetry about CA, C_P under a symmetry about AB. Then four points A_0, B_0, C_0 and H lie on the same line.*

-When P lies on ℓ then line $\overline{A_0B_0C_0H}$ is the Steiner line of P with respect to ABC.
Proof. (see figure 3). Line DP meets (O) at P and E; construct the chord EF of (O) such that it is parallel to or coincident with ℓ.

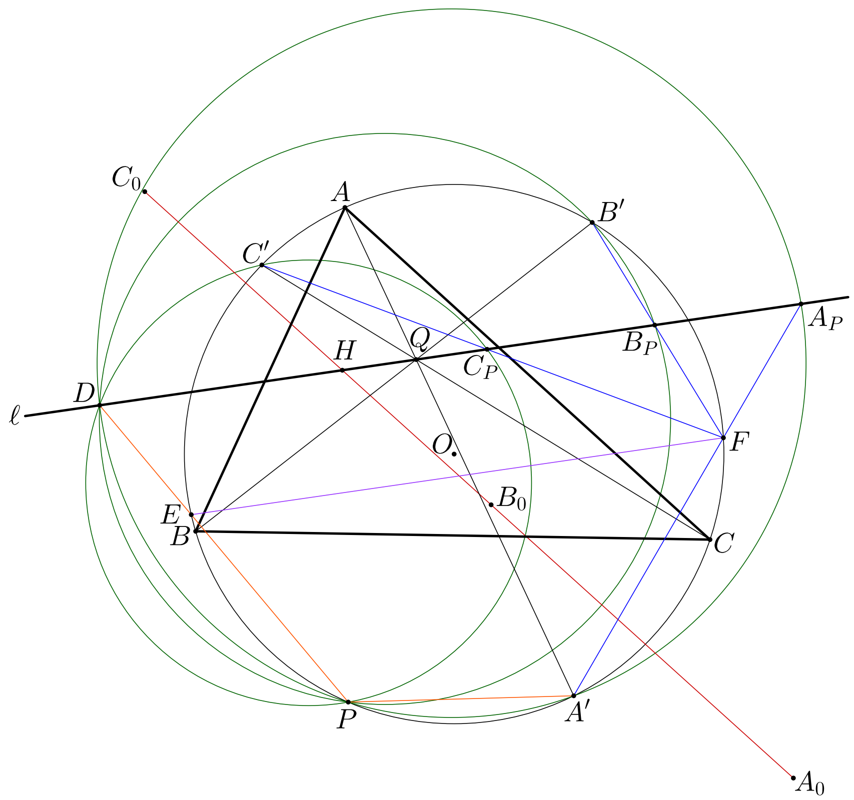

Figure 3.

Using the directed angle, we have

$$\begin{aligned}
(A'A_P, EF) &= (A'A_P, \ell) && \text{(by } EF \text{ is parallel to or coincident with } \ell\text{)}\\
&= (A_PA', A_PD) && \text{(by } A_P, D \in \ell\text{)}\\
&= (PA', PD) && \text{(by } P \in (A'DA_P)\text{)}\\
&= (PA', PE) && \text{(by } E \in DP\text{)}\\
&= (FA', FE) && \text{(by } F \in (A'EP)\text{)}.
\end{aligned}$$

It follows that $A'A_P$ and $A'F$ are coincident. Hence F belongs to $A'A_P$.
Similarly, we also have F belonging to $B'B_P$ and $C'C_P$.
Applying theorem 3 with the note that $A_P = FA' \cap \ell$, $B_P = FB' \cap \ell$ and $C_P = FC' \cap \ell$ then we have that four points A_0, B_0, C_0 and H lie on the same line.
Theorem 5 is proved.

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