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Three extentions of Kosnita's theorem

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Abstract. In this paper, we refer to three generalizations of Kosnita's theorem that are one old generalization and the solutions of two generalizations not having any proof.

Keywords. Power of a point, Kosnita's theorem, Newton-Gauss Line, Desargues's theorem, Kiepert's and Jacobi's theorem, Gauss-Bodenmiller theorem.

1. Introduction

In classical geometry, Kosnita's theorem is one of interesting results on the property of certain circles associated with an arbitrary triangle. The statement of Konista's theorem is as follows:

Theorem 1.1. (*Kosnita*). *Let O be the circumcenter of triangle ABC and let O_a, O_b, O_c be the circumcenters of triangles BOC, COA, AOB respectively. Then the lines AO_a, BO_b, CO_c are concurrent at the point K called the Konista's point of triangle ABC.*

In [2], Ion Pătraşcu gives a generalization of Konista's theorem and a proof using Barycentric Coordinates. This theorem is as follows:

Theorem 1.2. ([2]). *Given a point P lying on the triangle plane (ABC) which does not lie on the circumcircle and lines containing sides of this triangle. Let A', B', C' be the orthogonal projections from P onto BC, CA, AB, and points A_1, B_1, C_1 satisfy $\overline{PA_1}.\overline{PA'} = \overline{PB_1}.\overline{PB'} = \overline{PC_1}.\overline{PC'} = k$, $k \in \mathbb{R}^*$. Then the lines AA_1, BB_1, CC_1 are concurrent or the pair of these lines are parallel.*

In this article, we give a new synthetic proof for theorem 1.2 and two another generalizations for theorem 1.1 and their proofs. Two generalizations are as follows.

Theorem 1.3. (*Dao*-[3]). *Given six points A, B, C, D, E and F lying on the same circle (O). Denote by $X = CD \cap EF$, $Y = EF \cap AB$ and $Z = AB \cap CD$. Let O_1, O_2 and O_3 be the circumcenters of triangles OAB, OBC and OEF. Then the lines XO_1, YO_2 and ZO_3 are concurrent and the pair of these lines are parallel.*

Theorem 1.4. ([4]). *Given six points A, B, C, D, E and F lying on the same circle (O). Denote by $U = CE \cap DF$, $V = EA \cap FB$ and $W = AC \cap BD$. Let O_1, O_2, O_3, O_4, O_4 and O_6 be the circumcenters of triangles OAB, OBC, OEF, UDE, VFA and WBC. Then the lines O_1O_4, O_2O_5 and O_3O_6 are concurrent or the pair of these lines are parallel.*

- Clearly, when $B \equiv C$, $D \equiv E$ and $F \equiv A$ then two theorems become to the Theorem 1.1.

2. PRELIMINARIES

2.1. **Proof of the Theorem 1.2.** In order to prove the Theorem 1.2, we give some important theorems that use in this proof.

Theorem 2.1. (*Coxeter,* [5]). (a) *If circles are constructed on two cevians (via different vertices) as diameters, their radical axis passes through the orthocenter of the triangle.*
(b) *The radical center of any three non-coaxal circles having cevians (via different vertices) for diameters coincides with the orthocenter of the triangle.*

Theorem 2.2. (*Gauss-Bodenmiller,* [5]). *The three circles constructed on the diagonals of a complete quadrilateral as diameters are coaxal: their centers are collinear, and, when taken two by two, their radical axes coincide.*

Theorem 2.3. (*Newton-Gauss Line,* [6]). *Midpoints of the diagonals of a complete quadrilateral are collinear on a line.*

Theorem 2.4. (*Desargue,* [7]). *Two triangles are in axial perspective if and only if they are in central perspective.*

The following is the proof of theorem 1.2.

Proof. (see figure 1 and 2). Let H be the orthocenter of triangle ABC.
Applying the Intersecting-chords theorem, from $\overline{PB_1}.\overline{PB'} = \overline{PC_1}.\overline{PC'}$ we get the points B', C', B_1, C_1 being concyclic.

Chasing angle, we have

$$\begin{aligned}(PA, B_1C_1) &= (PA, PB') + (PB', B_1C_1)\\ &= (C'A, C'B) + (B_1B', B_1C_1)\\ &= (C'A, C'B) + (C'B', C'C_1)\\ &= (C'A, C'C_1) = \frac{\pi}{2}.\end{aligned}$$

It follows that PA is perpendicular to B_1C_1.
Similarly, PB is perpendicular to C_1A_1, and PC is perpendicular to A_1B_1.

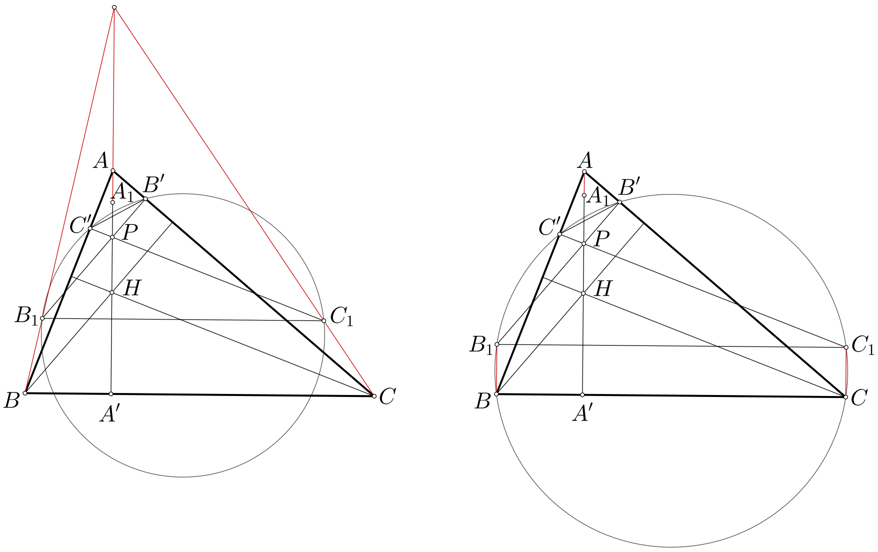

Figure 1A Figure 1B

FIGURE 1.

This thing means that

(1) PA, PB, PC are perpendicular to B_1C_1, C_1A_1, A_1B_1 respectively.

•*Case 1.* If P is coincident with H. Then clearly, AA_1, BB_1 and CC_1 are concurrent at H. Theorem 1.2 is correct in this case.

•*Case 2.* If P is different from H and belongs to one of the altitudes of triangle ABC. Without loss of generality, suppose that P lies on AH, then A' and A_1 belong to the same line AH (see figure 1A and 1B).

From (1) we get two lines B_1C_1 and BC are parallel or coincident. And hence, the pair of lines containing the sides of two triangles PB_1C_1 and HBC are parallel or these lines are concurrent so they are the images by themselves under a homothety or a translation. It follows that PH, B_1B, C_1C are concurrent or the pair of these lines are parallel. Since AA_1 and A_1H are coincident, AA_1, BB_1, CC_1 are concurrent or the pair of these lines are parallel. The theorem 1.2 is also correct in this case.

•*Case 3.* If P does not lie on the lines containing the altitudes of triangle ABC. Then let $A_2 = BC \cap B_1C_1$, $B_2 = CA \cap C_1A_1$, $C_2 = AB \cap A_1B_1$; and let M_a, M_b, M_c be the midpoints of AA_2, BB_2, CC_2 (see figure 2).
Let $K_a = AP \cap B_1C_1$, then AP is perpendicular to B_1C_1 at K_a by (1). It follows that K_a belongs to circles (AB_1) and (AA_2). Applying the Intersecting-chords theorem and the Power-of-a-point theorem, we get

$$\mathcal{P}_{P/(AA_2)} = \overline{PA}.\overline{PK_a} = \overline{PB_1}.\overline{PB'} = k.$$

Similarly, $\mathcal{P}_{P/(BB_2)} = k$, and $\mathcal{P}_{P/(CC_2)} = k$.
This thing means that

(2) $$\mathcal{P}_{P/(AA_2)} = \mathcal{P}_{P/(BB_2)} = \mathcal{P}_{P/(CC_2)}.$$

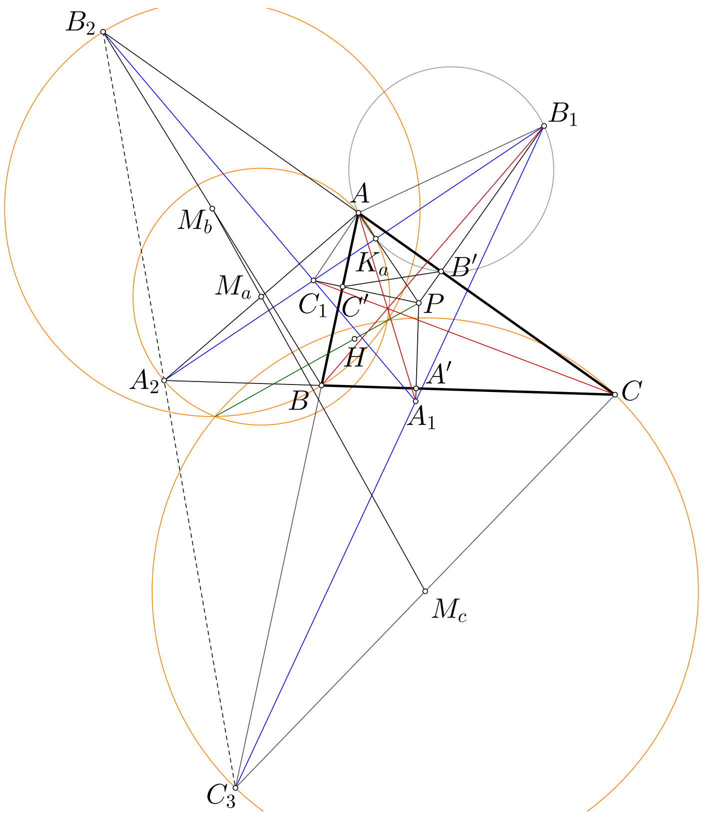

FIGURE 2.

Applying theorem 2.1 (a), from (2) we get HP being the common radical axis of each pair of two circles from three circles (AA_2), (BB_2), (CC_2). Hence, their centers belong to the line perpendicular to PH, i.e, points M_a, M_b, M_c are collinear. Applying theorems 2.2 and 2.3, we easily have that three points A_2, B_2, C_2 are collinear. Hence, by theorem 2.4 then AA_1, BB_1 and CC_1 are concurrent or the pair of these lines are parallel. Hence, theorem 1.2 is also correct in this case.

Theorem 1.2 is proved. □

2.2. **The proof of theorem 1.3.**

Proof. Suppose that circles (OCD) and (OEF) meet at P and K; similarly to points L and M as figure 3.

We have that CD, EF and OK are concurrent at the radical center of three circles $(OCD), (OEF)$ and (O). It follows that K belongs to OX.
Similarly, L belongs to OY and M belongs to OZ.

Note that $OE = OF$ so $(EO, EF) = (FE, FO)$. It follows

$$(KE, KX) = (KE, KO) = (FE, FO) = (EO, EF) = (EO, EX).$$

This thing means that OE is tangent to (KEX). Hence, applying the Power-of-a-point theorem, we have

$$OE^2 = \overline{OK}.\overline{OX}.$$

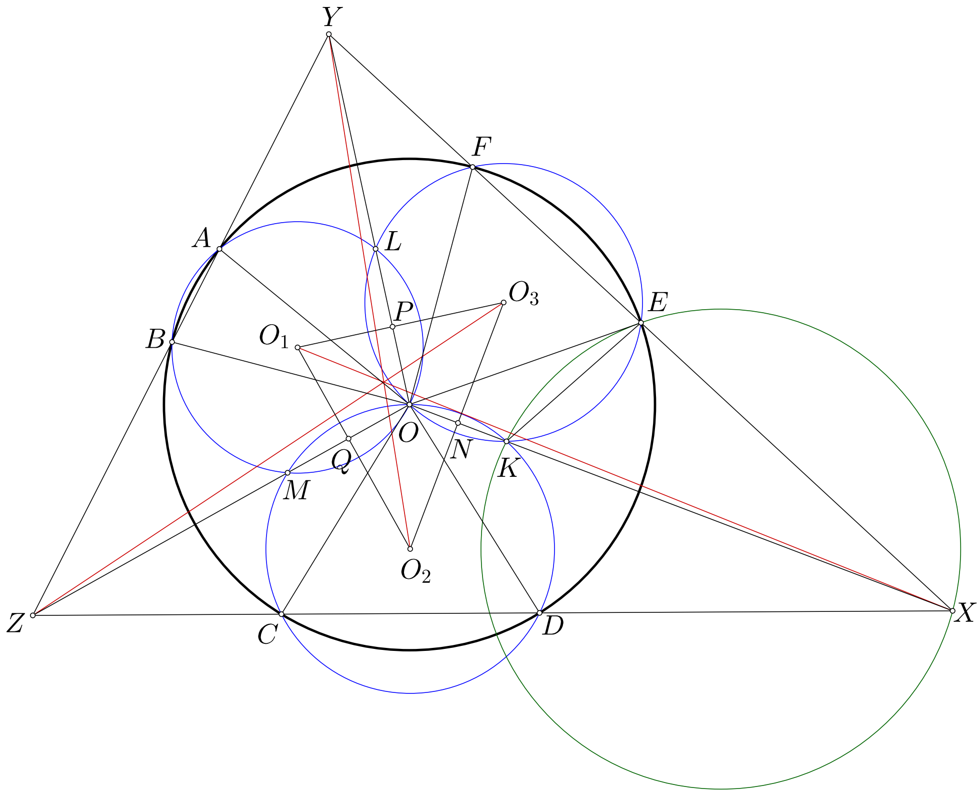

FIGURE 3.

Similarly, we also have $OA^2 = \overline{OL}.\overline{OY}$ and $OC^2 = \overline{OM}.\overline{OZ}$.
Note that $OE = OA = OC$. Hence

(3) $$\overline{OK}.\overline{OX} = \overline{OL}.\overline{OY} = \overline{OM}.\overline{OZ}.$$

Let us denote by $N = O_2O_3 \cap OX, P = O_3O_1 \cap OY$ and $Q = O_1O_2 \cap OZ$. We easily have that

(4) M, N, P are the midpoints of OK, OL, OM, respectively,

and

(5) OX, OY, OZ are perpendicular to O_2O_3, O_3O_1, O_1O_2, respectively.

Since (3) and (4) it follows

(6) $$\overline{ON}.\overline{OX} = \overline{OP}.\overline{OY} = \overline{OQ}.\overline{OZ}.$$

Applying theorem 1.2, since (5) and (6) it follows that O_1X, O_2Y and O_3Z are concurrent or the pair of theses lines are parallel.
Theorem 1.3 is proved. □

2.3. **The proof of theorem 1.4.** In order to prove theorem 1.4, we need the result in [1] as follows.

Theorem 2.5. (*Kiepert and Jacobi,* [8]). *Given three different points A, B, and C. Points A' not lying on BC, B' not lying on CA, and C' not lying on AB satisfy the condition $(AB, AC') = (AB', AC)$, $(BC, BA') = (BC', BA)$ and $(CA, CB') = (CA', CB)$. Then AA', BB' and CC' are concurrent or the pair of these lines are parallel.*

We restate a nice proof of Kiepert's and Jacobi's theorem that is in [9].

Proof. Construct the circles (O_a) passing through B and C, (O_b) passing through C and A, (O_c) passing through A and B such that

$$(O_a) = \left\{X|\frac{1}{2}(\overrightarrow{O_aB}, \overrightarrow{O_aC}) = (AB, AC')\right\},$$

$$(O_b) = \left\{X|\frac{1}{2}(\overrightarrow{O_bC}, \overrightarrow{O_bA}) = (BC, BA')\right\},$$

$$(O_c) = \left\{X|\frac{1}{2}(\overrightarrow{O_cA}, \overrightarrow{O_cB}) = (CA, CB')\right\}.$$

Let the points of intersection of the circle $(A'BC)$ and the line AA' be A' and A''; similarly to points B', B'' and C', C'' (see figure 4).

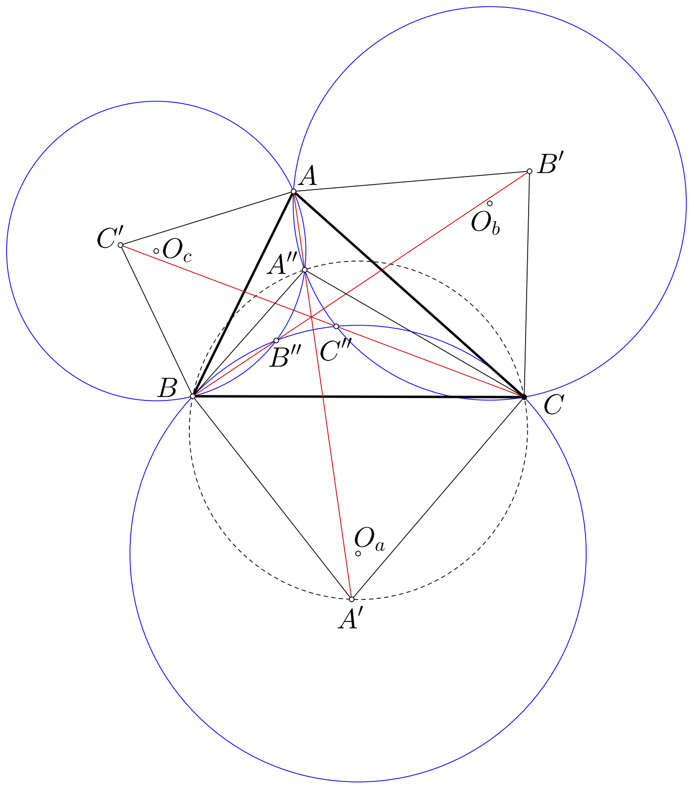

Figure 4.

We have

$$\begin{aligned}(A''B, A''A) &= (A''B, A''A') \\ &= (CB, CA') \\ &= (CA, CB') \\ &= \frac{1}{2}(\overrightarrow{O_cA}, \overrightarrow{O_cB}).\end{aligned}$$

It follows that A'' belongs to (O_c). Similarly, A'' also belongs to (O_b).
Thus, $AA' \equiv A'A''$ is the radical axis of two circles (O_b) and (O_c).
Similarly, BB' is the radical axis of (O_c) and (O_a); and CC' is the radical axis of

(O_a) and (O_b).
It follows that three lines AA', BB' and CC' are concurrent of the pair of these lines are parallel.

Theorem 2.5 is proved. □

The following is the proof of theorem 1.4.

Proof. The circles (OCD) and (OEF) meet at P and K; similarly to points L and M as figure 5.

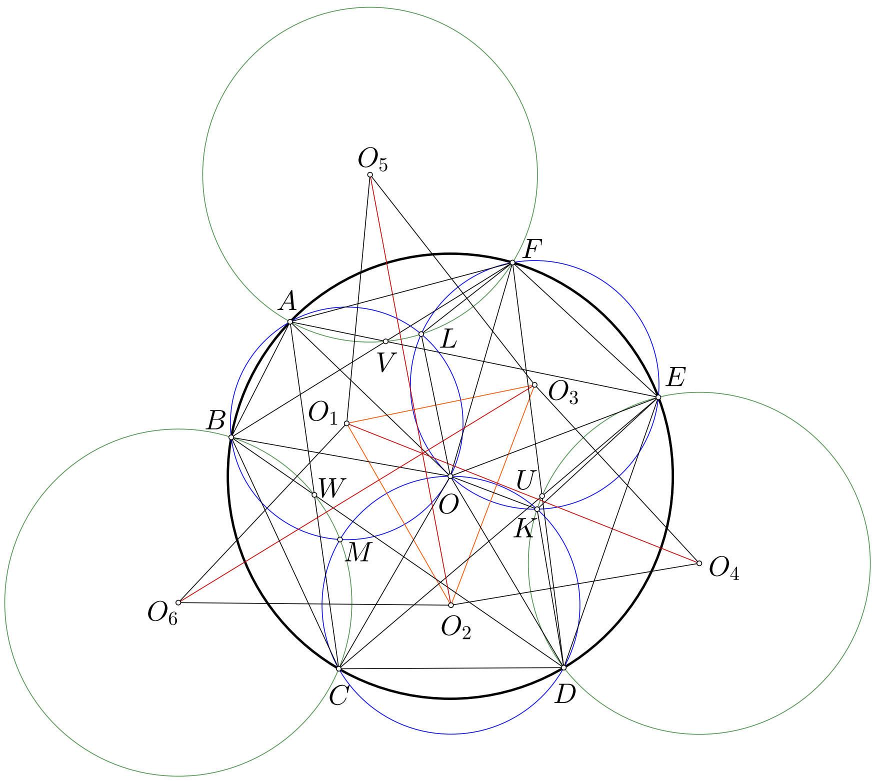

FIGURE 5.

Note that $OC = OD$ and $OE = OF$ so

$$(CD, CO) = \frac{1}{2}\left(\overrightarrow{OD}, \overrightarrow{OC}\right) + \frac{\pi}{2} \text{ and } (OF, FE) = \frac{1}{2}\left(\overrightarrow{OF}, \overrightarrow{OE}\right) + \frac{\pi}{2}.$$

Hence

$$\begin{aligned}(KD, KE) &= (KD, KO) + (KO, KE)\\ &= (CD, CO) + (FO, FE)\\ &= \frac{1}{2}\left(\overrightarrow{OD}, \overrightarrow{OC}\right) + \frac{1}{2}\left(\overrightarrow{OF}, \overrightarrow{OE}\right)\\ &= (ED, EC) + (DF, DE)\\ &= (DF, EC)\\ &= (UD, UE).\end{aligned}$$

It follows K lies on (UDE).
Similarly, L lies on (VFA), and M lies on (WBC).

From that, we easily have that O_3O_4 is perpendicular to KE, O_3O_2 is perpendicular to OK, O_3O_5 is perpendicular to LF, and O_3O_1 is perpendicular to OL. Chasing angles, we have

$$\begin{aligned}(O_3O_2, O_3O_4) &= (KO, KE)\\ &= (FO, FE)\\ &= (EF, EO)\\ &= (LF, LO)\\ &= (O_3O_5, O_3O_1).\end{aligned}$$

Similarly, $(O_1O_3, O_1O_5) = (O_1O_6, O_1O_2)$, and $(O_2O_1, O_2O_6) = (O_2O_4, O_2O_3)$.
Hence, by theorem 2.5, lines O_1O_4, O_2O_5 and O_3O_6 are concurrent of the pair of these lines are parallel.
Theorem 1.3 is proved. □

References

[1] Michael de Villiers, A Dual to Kosnita's theorem, Mathematics and Informatics Quarterly, 6(3), 169-171, Sept 1996.
http://mzone.mweb.co.za/residents/profmd/kosnita.pdf.

[2] Ion Pătraşcu (2010), A generalization of Kosnita's theorem (in Romanian), available at http://recreatiimatematice.ro/arhiva/processed/22010/14_22010_RM22010.pdf.

[3] T.O. Dao, Advanced Plane Geometry, message 4092, November 26, 2017, available at https://groups.yahoo.com/neo/groups/AdvancedPlaneGeometry/conversations/messages/4209.

[4] V.A. Le, Advanced Plane Geometry, message 4146, October 16, 2017, available at https://groups.yahoo.com/neo/groups/AdvancedPlaneGeometry/conversations/messages/4146.

[5] A. Bogomolny, Circles On Cevians, Interactive Mathematics Miscellany and Puzzles, available at
https://www.cut-the-knot.org/Curriculum/Geometry/CircleOnCevian.shtml.

[6] A. Bogomolny, Theorem of Complete Quadrilateral, Interactive Mathematics Miscellany and Puzzles, available at
https://www.cut-the-knot.org/Curriculum/Geometry/Quadri.shtml.

[7] A. Bogomolny, Desargues' Theorem, Interactive Mathematics Miscellany and Puzzles, available at
https://www.cut-the-knot.org/Curriculum/Geometry/Desargues.shtml.

[8] A. Bogomolny, Kiepert's And Jacobi's Theorems, Interactive Mathematics Miscellany and Puzzles, available at
https://www.cut-the-knot.org/Curriculum/Geometry/Kiepert.shtml.

[9] https://artofproblemsolving.com/community/c6h154396.

[10] Marian Cucoaneş, A new proof to the theorem of Coşniţă, INTERNATIONAL JOURNAL OF GEOMETRY Vol. 6 (2017), No. 1, 24 - 26.

[11] Ion Pătraşcu and Cătălin Barbu, Two new proofs of Goormaghtigh's theorem, INTERNATIONAL JOURNAL OF GEOMETRY Vol. 1 (2012), No. 1, 10 - 19.

[12] H. S. M. Coxeter, S. L. Greitzer, Geometry Revisited, MAA, 1967.