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A New Proof of the Feuerbach Theorem

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Abstract. We present a proof of the famous Feuerbach theorem based on the Paskalev-Tchobanov distance formula.

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The famous Feuerbach theorem states (See [4]):

Theorem 1. *The nine-point circle of any triangle is tangent internally to the incircle and tangent externally to the three excircles.*

Proof. Given triangle ABC with side-lengths $a = BC, b = CA, c = AB$. Denote

- $s = \dfrac{a+b+c}{2}$ the semiperimeter of triangle ABC.
- Δ the area of triangle ABC.
- r the radius of the Incircle.
- R the radius of the Circumcircle.
- r_N the radius of the Nine-Point Circle.
- r_a the radius of the A-Excircle.
- r_b the radius of the B-Excircle.
- r_c the radius of the C-Excircle.

We use the following well-known relations:

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)},$$

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$$r = \frac{\Delta}{s},\ R = \frac{abc}{4\Delta},\ r_N = \frac{R}{2},\ r_a = \frac{\Delta}{s-a},\ r_b = \frac{\Delta}{s-b},\ r_c = \frac{\Delta}{s-c}.$$

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Denote

$$\begin{array}{lcl} E & = & a^3+b^3+c^3+3bca-ba^2-b^2a-ca^2-c^2a-c^2b-b^2c, \\ E_1 & = & a^3-b^3-c^3+3abc+ba^2-b^2a+ca^2-c^2a+c^2b+b^2c, \\ E_2 & = & -a^3+b^3-c^3+3abc-ba^2+b^2a+ca^2+c^2a+b^2c-c^2b, \\ E_3 & = & -a^3-b^3+c^3+3abc+ba^2+b^2a-ca^2+c^2a-b^2c+c^2b. \end{array}$$

Note that the above forms $E_1 - E_3$ are known [2].

It is easy to see that

$$r_N - r = \frac{E}{8\Delta},\ r_N + r_a = \frac{E_1}{8\Delta},\ r_N + r_b = \frac{E_2}{8\Delta},\ r_N + r_c = \frac{E_3}{8\Delta}.$$

The Nine-Point Circle is tangent internally to the Incircle if and only if $NI = r_N - r$ where NI is the segment with endpoints the Nine-Point Center N and the Incenter I.

The Nine-Point Circle is tangent externally to the A-Excircle if and only if $NJ_a = r_N + r_a$ where J_a is the center of the A-Excircle. Similar results hold for B-Excircle with center J_b and C-Excircle with center J_c.

The barycentric coordinates of the centers of the above circles are as follows:

$$N = (a^2(b^2+c^2)-(b^2-c^2)^2, b^2(c^2+a^2)-(c^2-a^2)^2, c^2(a^2+b^2)-(a^2-b^2)^2).$$

$$I = (a,b,c),\ J_a = (-a,b,c),\ J_b = (a,-b,c),\ J_c = (a,b,-c).$$

To find the length of segment NI we use barycentric coordinates of points N and I and the Paskalev-Tchobanov distance formula [3]. See also formula (9) in [1]. We obtain $NI = \dfrac{E}{8\Delta}$. Hence

$$r_N - r = NI.$$

By using the same approach and the Paskalev-Tchobanov distance formula we see that

$$r_N + r_a = NJ_a,\ r_N + r_b = NJ_b,\ r_N + r_c = NJ_c.$$

This completes the proof. □

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