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An Iterative Geometrical Approach for a Problem in the International Mathematical Olympiad 2017

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Abstract. In this paper we attempt to solve an international mathematical olympiad problem based on a geometrical approach. In each iteration we choose an appropriate setting in order to increase the distance between two points. Then we can give a negative answer to the problem at the final iterations.

Keywords. International mathematical olympiad; Non-cooperative game; Iteration.

Mathematics Subject Classification (2010). 51-04, 68T01, 68T99.

1. Introduction

In the international mathematical olympiad - 2017 [1], the third problem in the first day is stated as follows.

"A hunter and an invisible rabbit play a game in the Euclidean plane. The rabbit's starting point, A_0, and the hunter's starting point, B_0, are the same. After $n-1$ rounds of the game, the rabbit is at point A_{n-1} and the hunter is at point B_{n-1}. In the n^{th} round of the game, three things occur in order.
(i) The rabbit moves invisibly to a point A_n such that the distance between A_{n-1} and A_n is exactly 1.
(ii) A tracking device reports a point P_n to the hunter. The only guarantee provided by the tracking device to the hunter is that the distance between P_n and A_n is at most 1.
(iii) The hunter moves visibly to a point B_n such that the distance between B_{n-1} and B_n is exactly 1.
Is it always possible, no matter how the rabbit moves, and no matter what points are reported by the tracking device, for the hunter to choose her moves so that

after 10^9 rounds she can ensure that the distance between her and the rabbit is at most 100?"

The problem belongs to the field of non-cooperative game theory with two players. Interesting readers can refer to the book of Ritzberger [3] for fundamental concepts and technniques. In the given problem the rabbit (player 1) tries to keep as far as possible from the hunter. On the other hand, the hunter (player 2) want to catch the rabbit base on the report of the tracking device. Therefore, in order to solve this problem, we show that there exists a strategy of the rabbit, with respect to a specific report of tracking device, which yields indeed the disadvantage to the hunter.

This problem is quite challeging and needs effort to solve. Recent solution is rather complicated and there is no numerical result to visualize the distance between the hunter and the rabbit. Thus, we discuss in this paper a simple iterative geometrical approach for tackling this problem and the data set with respect to the first 200 iterations.

2. SOLUTION APPROACH

Let us consider a strategy of the rabbit with respect to the report of the tracking device as follows.

Iteration 1: The rabbit moves to position A_1 with $A_0A_1 = 1$. The tracking device reports position $P_1 \equiv A_0$ and we get $B_1 \equiv A_0$. After this step, the distance between the hunter and the rabbit is $A_1B_1 = 1$.

Iteration k: ($k = 2, 3, ..., 10^9$). The rabbit move to the new point A_k on the opposite of the ray $A_{k-1}B_{k-1}$. Then we obviously know that $A_{k-1}A_k = 1$. The tracking device report a position P_k such that $A_kP_k \perp A_{k-1}A_k$ and $A_kP_k = 1$. Consequently, the hunter move to the point B_k such that $B_k \in B_{k-1}P_k$ and $B_{k-1}B_k = 1$.

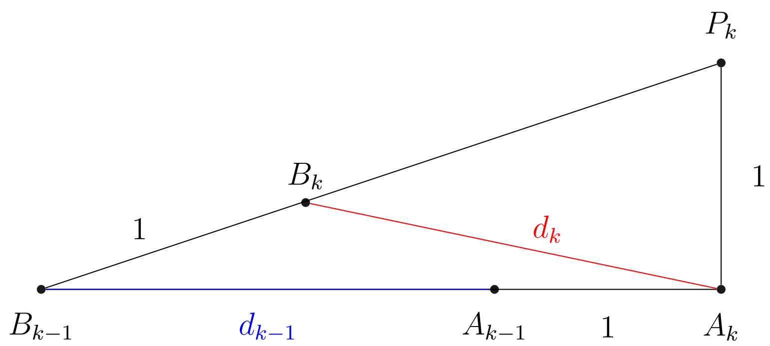

FIGURE 1. The movements of hunter and rabbit in iteration k.

We aim to prove that, the given strategy yields a negative answer to the problem. Let us denote by $d_k := A_kB_k$ the distance between the hunter and the rabbit after the k^{th} iteration. Recall that $d_1 = 1$.

Applying the law of cosin in $\Delta B_{k-1}A_kB_k$, we get

$$A_kB_k^2 = A_kB_{k-1}^2 + B_{k-1}B_k^2 - 2 \cdot A_kB_{k-1} \cdot B_{k-1}B_k \cdot \cos B_{k-1} \quad (1)$$

Replace $\cos B_{k-1} = \frac{A_k B_{k-1}}{B_{k-1} P_k}$ to (1), we obtain

$$d_k^2 = (d_{k-1}+1)^2 + 1 - \frac{2(d_{k-1}+1)^2}{\sqrt{(d_{k-1}+1)^2+1}}. \tag{2}$$

We consider the property of the sequence $\{d_k\}_{k=1,\ldots}$ by the following two phases:

Phase 1: We prove that $d_1 < d_2 < d_3 < \ldots < d_{10^9}$. Indeed, by the elementary inequality

$$\frac{a^2}{\sqrt{a^2+1}} < a, \forall a > 0$$

and replace $a = d_{k-1}+1$ in (2) we obtain $d_k^2 > (d_{k-1}+1)^2 + 1 - 2(d_{k-1}+1) = d_{k-1}^2$ or $d_k > d_{k-1}$ for all $k = 2, 3, \ldots, 10^9$.

Phase 2: We consider the difference:

$$\begin{aligned} d_k^2 - d_{k-1}^2 &= 2(d_{k-1}+1)\big(1 - \frac{d_{k-1}+1}{\sqrt{(d_{k-1}+1)^2+1}}\big) \\ &= \frac{2(d_{k-1}+1)}{\sqrt{(d_{k-1}+1)^2+1}(\sqrt{(d_{k-1}+1)^2+1} + d_{k-1}+1)}. \end{aligned}$$

Let us make an assumption that $d_k \le 100$ for all $k = 1, \ldots, 10^9$. Denote by $t = d_{k-1} + 1 \le 101$, we get

$$\begin{aligned} d_k^2 - d_{k-1}^2 &= \frac{2t}{\sqrt{t^2+1}(\sqrt{t^2+1}+t)} \\ &= \frac{2}{\sqrt{t^2+1}(\sqrt{1+\frac{1}{t^2}}+1)} \\ &> \frac{2}{303}. \end{aligned}$$

By the analysis in the two phases, $d_k^2 > d_{k-1}^2 + \frac{2}{303}$ for all $k = 2, \ldots, 10^9$. By recursive computation, we obtain $d_{10^9}^2 > d_1^2 + \frac{2}{303}(10^9 - 1) > 100^2$. This contradicts to the assumption.

In summary, after 10^9 iterations, hunter can not be sure to get close to the rabbit around 100 after 10^9 iteration.

3. Appendix

This section contributes to the computation of our approach. We use Python version 3.6.2 [2] to set up the calculation of d_k in (2). The result for the first 200 iterations can be observed in Table 3.

TABLE 1. Distance between the hunter and the rabbit in first 200 iterations.

Iter. 1-50	***Iter. 51-100***	***Iter. 101-150***	***Iter. 151-200***
1	3.750385114	4.822567684	5.582336067
1.192598523	3.777452856	4.839958287	5.595696337
1.348206712	3.804185898	4.857237327	5.60899873
1.480930199	3.830594405	4.874406575	5.622243868
1.597791684	3.856688055	4.891467758	5.635432363
1.702873477	3.882476072	4.908422561	5.648564815
1.798789222	3.907967255	4.925272626	5.661641817
1.887327447	3.933170005	4.942019555	5.67466395
1.969773378	3.958092349	4.958664912	5.687631786
2.047085872	3.98274196	4.975210222	5.700545886
2.120001894	4.007126182	4.991656975	5.713406805
2.189101733	4.031252047	5.008006626	5.726215085
2.254851569	4.055126294	5.024260594	5.738971261
2.317632301	4.078755383	5.040420267	5.751675861
2.377759689	4.102145516	5.056487	5.764329403
2.435498783	4.125302643	5.072462118	5.776932395
2.491074521	4.148232485	5.088346915	5.789485339
2.544679644	4.170940537	5.104142657	5.80198873
2.596480725	4.193432087	5.119850581	5.814443053
2.646622831	4.215712224	5.135471897	5.826848787
2.695233172	4.237785848	5.151007789	5.839206402
2.742423993	4.259657678	5.166459415	5.851516362
2.788294904	4.281332265	5.181827908	5.863779123
2.832934751	4.302813997	5.197114377	5.875995136
2.876423163	4.324107107	5.212319907	5.888164844
2.91883182	4.345215683	5.22744556	5.900288681
2.960225499	4.366143671	5.242492378	5.912367078
3.000662968	4.386894885	5.257461378	5.924400458
3.04019772	4.407473011	5.27235356	5.936389239
3.078878606	4.427881613	5.2871699	5.94833383
3.116750374	4.448124142	5.301911357	5.960234636
3.153854128	4.468203934	5.316578869	5.972092057
3.190227724	4.488124221	5.331173356	5.983906485
3.225906113	4.507888132	5.345695721	5.995678309
3.26092164	4.527498702	5.360146847	6.007407908
3.295304304	4.54695887	5.374527601	6.019095661
3.329081986	4.566271487	5.388838834	6.030741939
3.362280651	4.585439317	5.403081379	6.042347106
3.394924521	4.604465045	5.417256054	6.053911524
3.427036236	4.623351274	5.431363662	6.065435548
3.45863699	4.642100533	5.44540499	6.076919529
3.489746659	4.660715279	5.459380811	6.088363813
3.520383905	4.679197899	5.473291883	6.099768742
3.550566283	4.697550711	5.48713895	6.111134651
3.580310323	4.715775971	5.500922745	6.122461872
3.609631614	4.733875872	5.514643983	6.133750733
3.638544878	4.751852547	5.528303371	6.145001557
3.667064029	4.769708072	5.541901599	6.156214662
3.695202239	4.787444467	5.555439348	6.167390364
3.722971988	4.805063699	5.568917285	6.178528971

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