International Journal of Computer Discovered Mathematics (IJCDM)
ISSN 2367-7775 ©IJCDM
Volume 2, 2017, pp.185-200.
Received 10 September 2017. Published on-line 5 October 2017
web: http://www.journal-1.eu/

Computer Discovered Mathematics: Problems for Students about Excentral Triangle

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Abstract. We give new theorems about the Excentral triangle, discovered by the computer program "Discoverer". We present the theorems as problems for researchers and students.

Keywords. Excentral triangle, triangle geometry, Euclidean geometry, computer discovered mathematics, "Discoverer".

Mathematics Subject Classification (2010). 51-04, 68T01, 68T99.

1. Introduction

The excentral triangle of a triangle ABC is the triangle whose vertices are the excenters of ABC. It is the anticevian triangle with respect to the Incenter, and also the antipedal triangle with respect to the Incenter. The circumcircle of the Excentral triangle is the Excentral circle (also called the Bevan circle). See Excentral triangle in [8].

In this note we give new theorems about the Excentral triangle, discovered by the computer program "Discoverer", created by the authors. We present the theorems as problems for researchers and students. We encourage the researchers and students to solve the problems and to publish the solutions.

The problems are suitable for higher students in the area of pedagogy of mathematics - for home works, essays for students, etc.

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2. Notable Points

The "Discoverer" has studied 195 notable points of the Excentral triangle. Of them 72 points are available in Kimberling's ETC [6] and the rest of 123 points are new points (that is, points not available in [6]). See the Supplementary material,

The "Discovered" has discovered new properties of the points of the Excentral triangle, available in [6]. Below we give three new properties of the Euler reflection point of the Excentral triangle, which is the point X(1768) in [6]. We encourage the reader to find new properties of the notable points of the Excentral triangle, which are available in [6].

References for Problem 2.1: Points X(68) ans X(110) in [6], Prasolov Point in [8].

Problem 2.1. *Denote by $JaJbJc$ the Excentral triangle. Denote by X the Euler reflection point of triangle $JaJbJc$. Denote by D the inverse of the Incenter I wrt the circumcircle. Denote by Pa the reflection of point D about line BC, by Pb the reflection of point D about line CA, and by Pc the reflection of point D about line AB. Then point X is the Prasolov point of triangle $PaPbPc$.*

Figure 1 illustrates Problem 2.1.

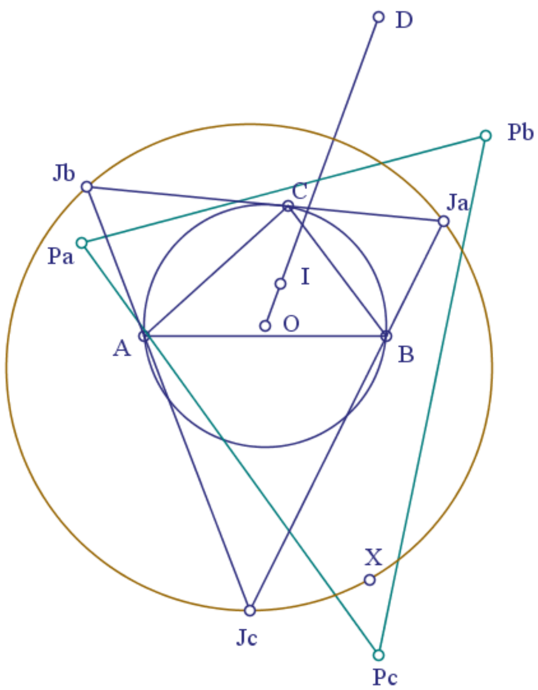

Figure 1.

References for Problem 2.2: For Outer Yff Triangle see Yff Circles in [6].

Problem 2.2. *Denote by $JaJbJc$ the Excentral triangle. Denote by X the Euler reflection point of triangle $JaJbJc$. Denote by c_1 the circumcircle of the Cevian triangle of Nagel point. Denote by $YaYbYc$ the Outer Yff Triange. Denote by c_2 the circumcircle of the Cevian triangle of Nagel point of triangle $YaYbYc$. Then point X is the external center of similitude of circles c_1 and c_2.*

Figure 2 illustrates a part of Problem 2.2.

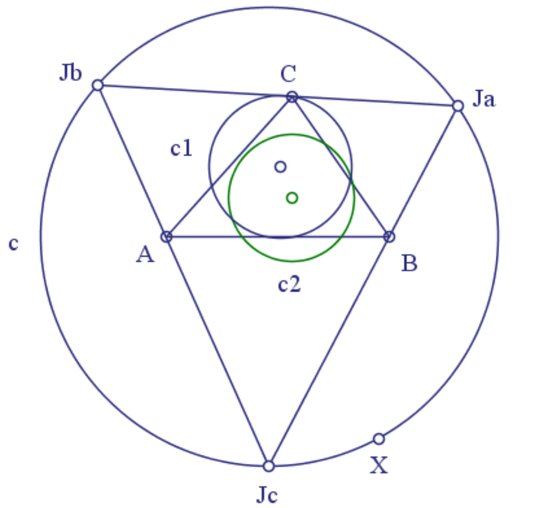

Figure 2.

Problem 2.3. *Denote by $JaJbJc$ the Excentral triangle. Denote by X the Euler reflection point of triangle $JaJbJc$. Denote by I the Incenter. Denote by Ha the Orthocenter of triangle IBC, by Hb the Orthocenter of triangle ICA, and by Hc the Orthocenter of triangle IAB. Denote by $PaPbPc$ the Antimedial triangle of triangle $HaHbHc$. Then point X lies on the circumcircle of triangle $PaPbPc$.*

Figure 3 illustrates a part of Problem 2.3.

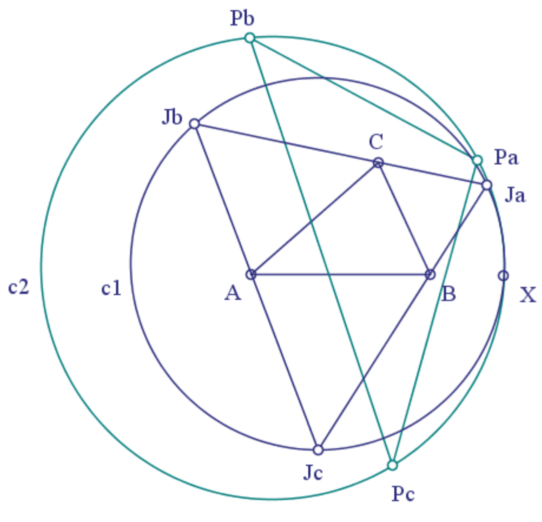

Figure 3.

Below we give properties of new notable points of the Excentral triangle. We encourage the reader ti study the new points of the Excentral triangle available in the Supplementary material.

For Problems 2.4 and 2.5 see the Feuerbach Perspector X(12) in [6].

Problem 2.4. *The Feuerbach Perspector of the Excentral Triangle is the Internal Center of Similitude of the Circumcircle of Triangle ABC and the Incircle of the Excentral Triangle.*

Figure 4 illustrates Problem 2.4. In figure 4:

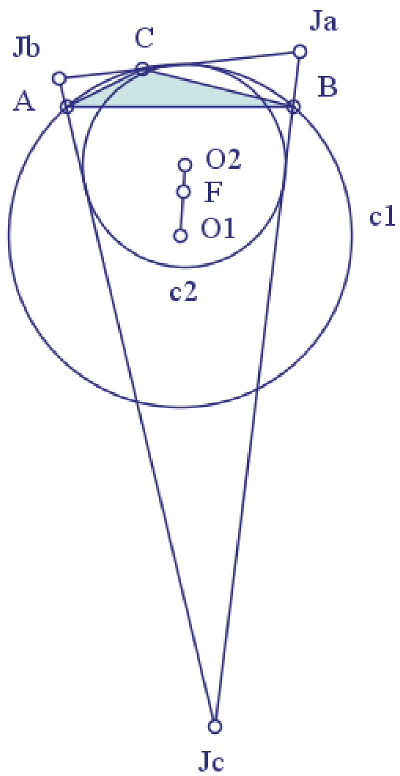

FIGURE 4.

- $JaJbJc$ is the Excentral triangle,
- $c_1 = (O_1)$ is the circumcircle of triangle ABC,
- $c_2 = (O_2)$ is the incircle of triangle $JaJbJc$,
- F is the Feuerbach Perspector of triangle $JaJbJc$. In addition, F is the Internal center of similitude of circles c_1 and c_2.

Problem 2.5. *The Feuerbach Perspector of the Excentral Triangle is the Internal Center of Similitude of the Excentral Circle and the Inner Johnson-Yff Circle of the Excentral Triangle.*

Figure 5 illustrates Problem 2.5. In figure 5:

- $JaJbJc$ is the Excentral triangle,
- O is the center of the Excentral circle,,
- $c = (I)$ is the inner Johnson-Yff circle of Excentral triangle,
- F is the Feuerbach Perspector of the Excentral Triangle. In addition, F is the Internal center of similitude of the Excentral circle and circle c.

Problem 2.6. *Denote by S the Internal Center of Similitude of the Incircle of the Excentral Triangle and Circumcircle of the Excentral Triangle. Denote by $JaJbJc$ the Excentral triangle. Denote by Ga the Gergonne point of triangle $JaBC$, by Gb the Gergonne point of triangle $JbCA$, and by Gc the Gergonne point of triangle $JcAB$. Then*

(1) *Lines $JaGa, JbGb$ and $JcGc$ concur at point S,*
(2) *Point S is the Orthocenter of triangle GaGbGc.*

Figure 6 illustrates Problem 2.6.

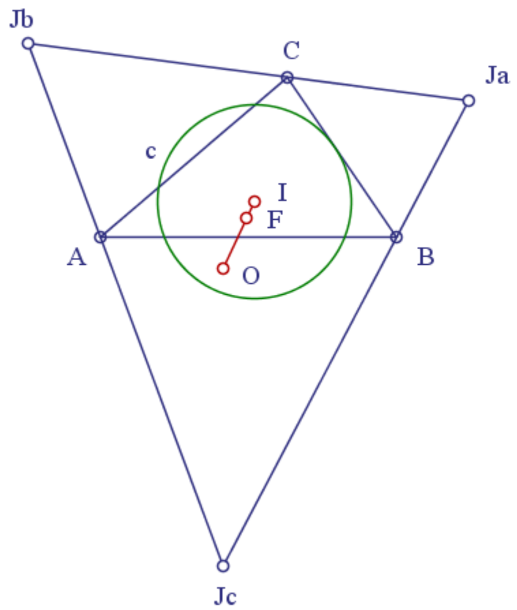

Figure 5.

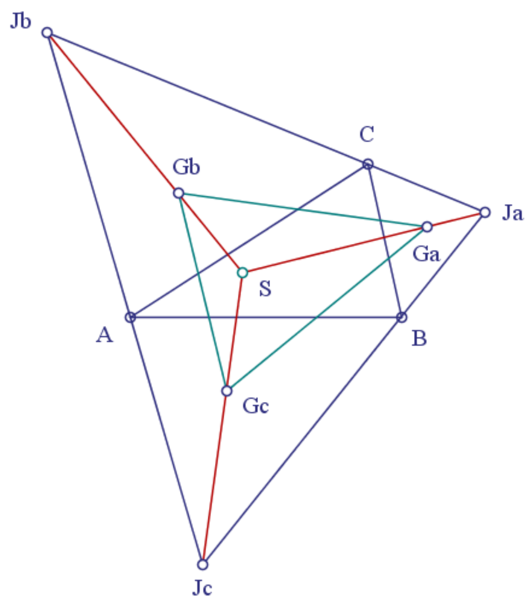

Figure 6.

Problem 2.7. *Denote by Se the External Center of Similitude of the Incircle of the Excentral Triangle and Circumcircle of the Excentral Triangle. Denote by $JaJbJc$ the Excentral triangle. Denote by Na the Nagel point of triangle $JaBC$, by Nb the Nagel point of triangle $JbCA$, and by Nc the Nagel point of triangle $JcAB$. Then*

(1) *Lines $JaGa, JbGb$ and $JcGc$ concur at point Se,*
(2) *Point Se is the Orthocenter of triangle $NaNbNc$.*

Figure 7 illustrates Problem 2.7.

References for Problem 2.8: Point X(72) in [6] could be defined as the intersection of line through the Incenter and Symmedian point, and the line through the

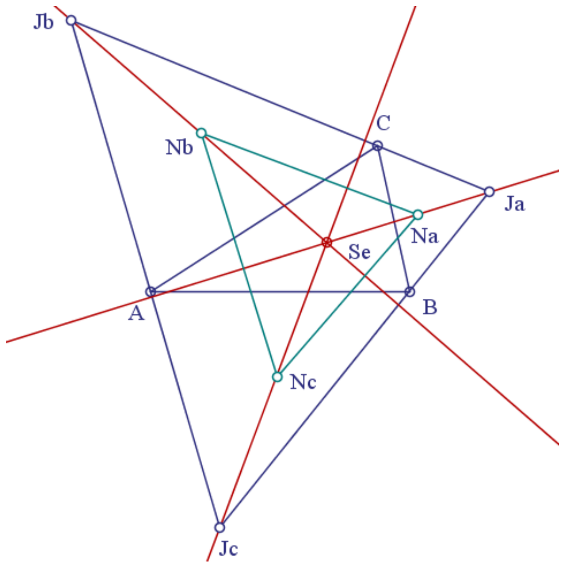

FIGURE 7.

Orthocenter and Nagel point. Point X(177) could be defined as the Incenter of the Intouch triangle.

Problem 2.8. *Denote by $T = JaJbJc$ the Excentral triangle. Denote by X the X(72) point of the Excentral triangle. Denote by $PaPbPc$ the Antipedal triangle of the Incenter of Intouch triangle. Denote by Ha the Orthocenter of triangle $PaBC$, by Hb the Orthocenter of triangle $PbCA$, and by Hc the Orthocenter of triangle $PcAB$. Then*

(1) *Lines $JaHa, JbHb$ and $JcHc$ concur at point H,*
(2) *Point X is the Orthocenter of triangles $HaHbHc$,*
(3) *Triangles ABC and $HaHbHc$ are homothetic.*

Figure 8 illustrates Problem 2.8.

For Problem 2.9 see Fuhrmann Triangle, Fuhrmann Circle, Fuhrmann Center in [8], X(177) = 1st MID-ARC POINT in [6].

Problem 2.9. *Denote by $JaJbJc$ the Excentral triangle. Denote by Pa the Incenter of triangle $JaBC$, by Pb the Incenter of triangle $JbCA$, and by Pc the Incenter of triangle $JcAB$. Then The First Mid-Arc Point F of the Excentral Triangle is the Fuhrmann point of triangle $PaPbPc$.*

Figure 9 illustrates Problem 2.9.

For Problems 2.10, 2.11 and 2.12, see X(182) Center of the Brocard Circle.

Problem 2.10. *Denote by $JaJbJc$ the Excentral triangle. Denote by X the Center of the Brocard Circle of triangle $JaJbJc$. Denote by $NaNbNc$ the Cevian triangle of the Nagel point. Denote by Oa the circumcenter of triangle $ANbNc$, by Ob the circumcenter of triangle $BNcNa$, and by Oc the circumcenter of triangle $CNaNb$. Then point X is the Mittenpunkt of triangle $OaObOc$.*

Figure 10 illustrates Problem 2.10.

Problem 2.11. *Denote by $JaJbJc$ the Excentral triangle. Denote by X the Center of the Brocard Circle of triangle $JaJbJc$. Denote by $MaMbMc$ the Pedal*

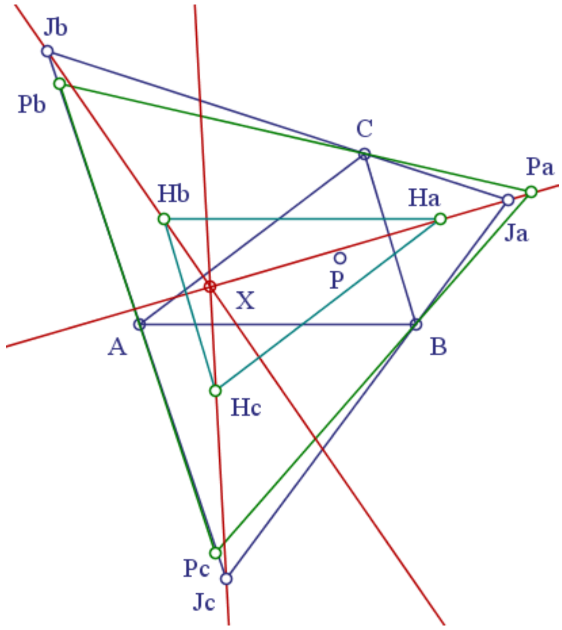

Figure 8.

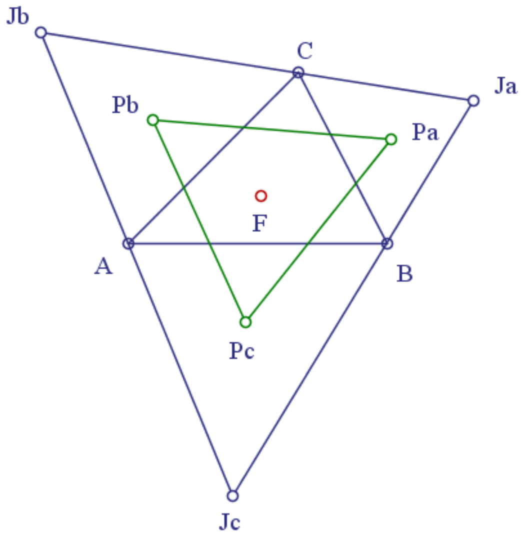

Figure 9.

triangle of the Mittenpunkt. Denote by Oa the circumcenter of triangle $AMbMc$, by Ob the circumcenter of triangle $BMcMa$, and by Oc the circumcenter of triangle $CMaMb$. Then point X is the Bevan point of triangle OaObOc.

Problem 2.12. *Denote by JaJbJc the Excentral triangle. Denote by X the Center of the Brocard Circle of triangle $JaJbJc$. Then point X is the midpoint of the Bevan point and Mittenpunkt.*

For Problem 2.13 see X(354) = WEILL POINT in [6], and Weill Point in [8].

Problem 2.13. *Denote by JaJbJc the Excentral triangle. Denote by X the Weill point of triangle $JaJbJc$. Denote by Pa the Incenter of triangle $JaBC$, by Pb the Incenter of triangle $JbCA$, and by Pc the Incenter of triangle $JcAB$. Then point X is the Center of the Orthocentroidal circle of triangle $PaPbPc$.*

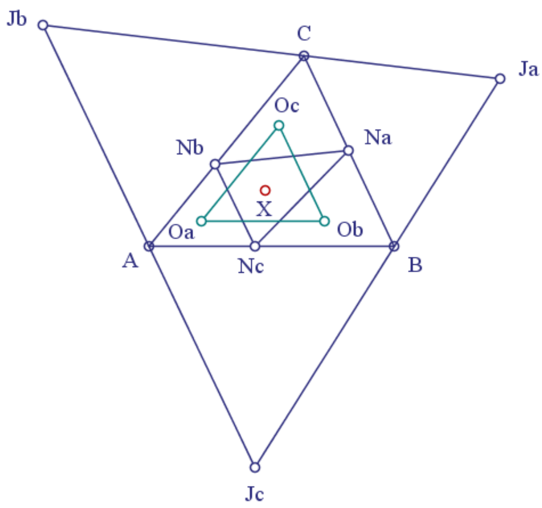

FIGURE 10.

For Problem 2.14 see X(485) = VECTEN POINT in [6]. and Weill Point in [8].

Problem 2.14. *Denote by $JaJbJc$ the Excentral triangle. Denote by X the Vecten point of triangle $JaJbJc$. Denote by O the Circumcenter. Benote by Ia the Incenter of triangle OBC, by Ib the Incenter of triangle OCA, and by Ic the Incenter of triangle OAB. Then point X concur in the lines $JaIa$, $JbIb$ and $JcIc$.*

3. HOMOTHETIC TRIANGLES

The "Discoverer" has discovered a number of triangles, homothetic with the Excentral triangle. Below we give a few problems about homotheties from the Excentral triangle.

Problem 3.1. *The Excentral triangle is homothetic with the Intouch triangle. The center of homothety is the X(57) Isogonal Conjugate of the Mittenpunkt. The ratio of homothety is*

$$k = \frac{(a+b-c)(b+c-a)(c+a-b)}{4abc} > 0.$$

Figure 11 illustrates Problem 3.1. In figure 11,

- $JaJbJc$ is the Excentral triangle,
- $PaPbPc$ is the Intouch triangle, and
- X is the center of homothety.

References for Problem 3.2: Yff Central Triangle in [8]. X(173) = CONGRUENT ISOSCELIZERS POINT in Kimberling [6], Congruent isoscelizers point in Wikipedia [9], A Simple Construction of the Congruent Isoscelizers Point in Danneels [2].

Problem 3.2. *The Excentral triangle is homothetic with the Yff Central triangle. The center of homothety is the X(173) Congruent Isoscelizers Point. The ratio of homothety is*

$$k = \frac{\Delta^2}{2abc(aR_bR_c + bR_cR_a + cR_aR_b + 2a + 2b + 2c)} > 0.$$

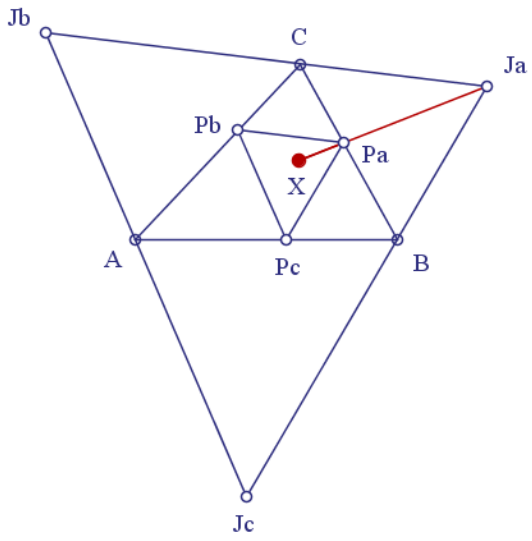

Figure 11.

where

$$R_a = \frac{\sqrt{(a+b+c)(-a+b+c)}}{\sqrt{bc}},$$
$$R_b = \frac{\sqrt{(a+b+c)(a-b+c)}}{\sqrt{ca}},$$
$$R_c = \frac{\sqrt{(a+b+c)(a+b-c)}}{\sqrt{ab}}.$$

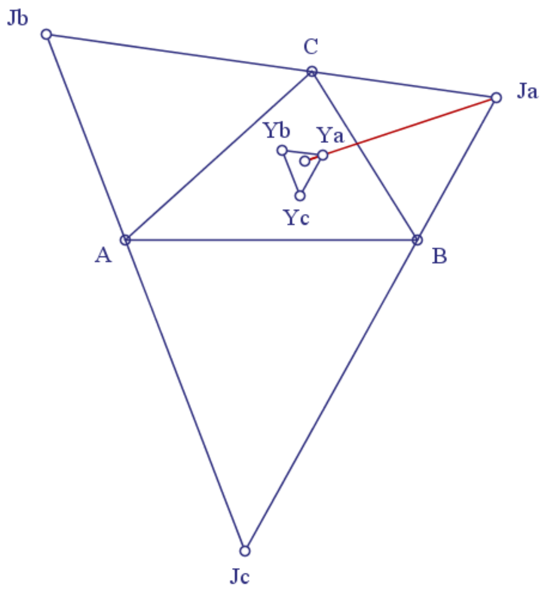

Figure 12.

Figure 12 illustrates Problem 3.2. In figure 12,

- $JaJbJc$ is the Excentral triangle,
- $YaYbYc$ is the Yff Central triangle, and

- the point inside triangle $YaYbYc$ is the center of homothety.

References for Problem 3.3: X(165) Centroid of Excentral triangle in [6], Circum-Anticevian Triangle of a Point in [3].

Problem 3.3. *The Excentral triangle is homothetic with the Circum-Anticevian Triangle of the Incenter. The center of homothety is the Centroid of Excentral triangle. The ratio of homothety is* $-\frac{1}{2}$

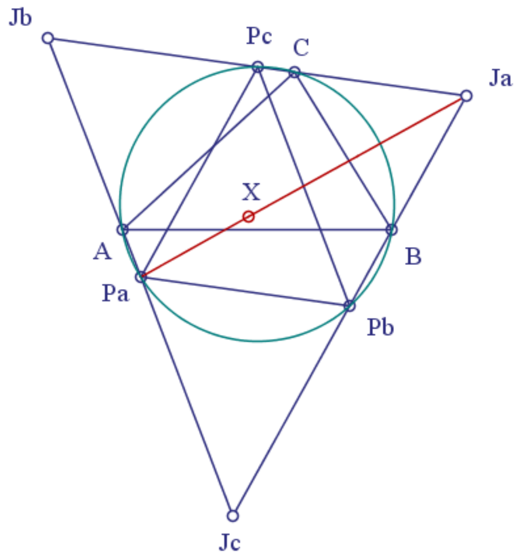

FIGURE 13.

Figure 13 illustrates Problem 3.3. In figure 13,

- $JaJbJc$ is the Excentral triangle,
- $PaPbPc$ is the Circum-Anticevian triangle of Incenter, and
- X is the center of homothety.

References for Problem 3.4: X(24) = PERSPECTOR OF ABC AND ORTHIC-OF-ORTHIC TRIANGLE in [6], X(46) = X(24)-of-excentral-triangle in [6].

Problem 3.4. *The Excentral triangle is homothetic with the Triangle of Reflections of the Incenter in the Sidelines of Triangle ABC. The Center of homothety is point X(46). The ratio of homothety is*

$$k = \frac{(a+b-c)(b+c-a)(c+a-b)}{2abc} > 0.$$

Figure 14 illustrates Problem 3.4. In figure 14,

- $JaJbJc$ is the Excentral triangle,
- I is the Incenter,
- Pa is the reflection of point I about line BC,
- Pb is the reflection of point I about line CA, and
- Pc is the reflection of point I about line AB,
- $PaPbPc$ is the Triangle of Reflections of the Incenter in the Sidelines of Triangle ABC,
- X is the center of homothety.

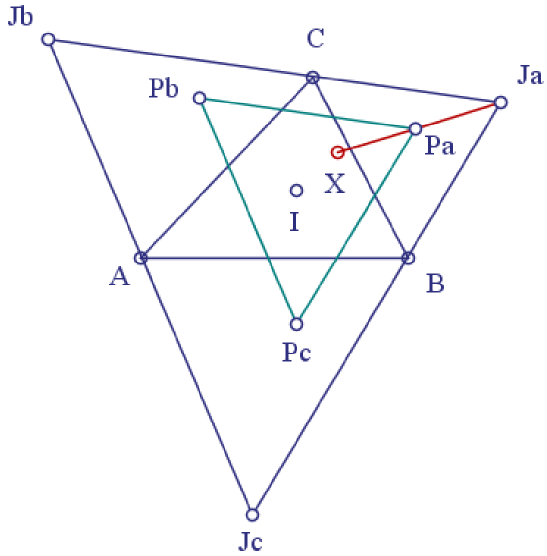

FIGURE 14.

References for Problem 3.5: Point X(12514) in [6].

Problem 3.5. *The Excentral triangle is homothetic with the Triangle of Reflections of the Spieker Center in the Sidelines of Medial Triangle. The Center of homothety is point X(12514). The ratio of homothety is*

$$k = -\frac{(a+b-c)(b+c-a)(c+a-b)}{4abc} < 0.$$

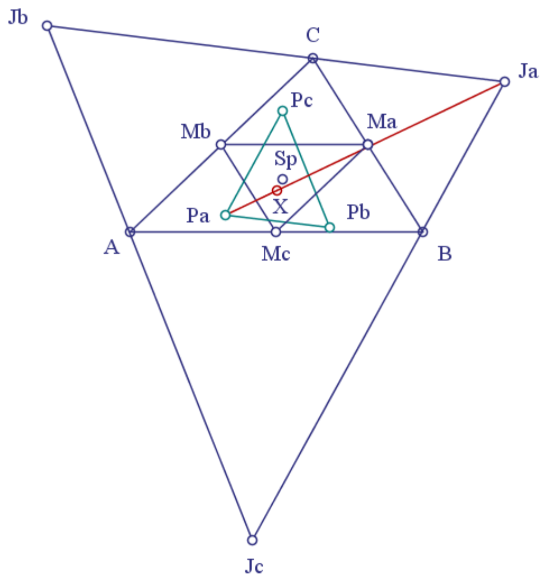

FIGURE 15.

Figure 15 illustrates Problem 3.5. In figure 15,

- $JaJbJc$ is the Excentral triangle,
- $MaMbMc$ is the Medial triangle,
- Sp is the Spieker center,

- Pa is the reflection of point Sp about line $MbMc$,
- Pb is the reflection of point Sp about line $McMa$, and
- Pc is the reflection of point Sp about line $MaMb$,
- $PaPbPc$ is the Triangle of Reflections of point Sp in sidelines of triangle $MaMbMc$,
- X is the center of homothety.

Reference for Problem 3.6: Contact triangle in Weisstein [8].

Problem 3.6. *The Excentral triangle is homothetic with the Medial triangle of Contact triangle. The Center of homothety is the Incenter. The ratio of homothety is*

$$k = -\frac{(a+b-c)(b+c-a)(c+a-b)}{8abc} < 0.$$

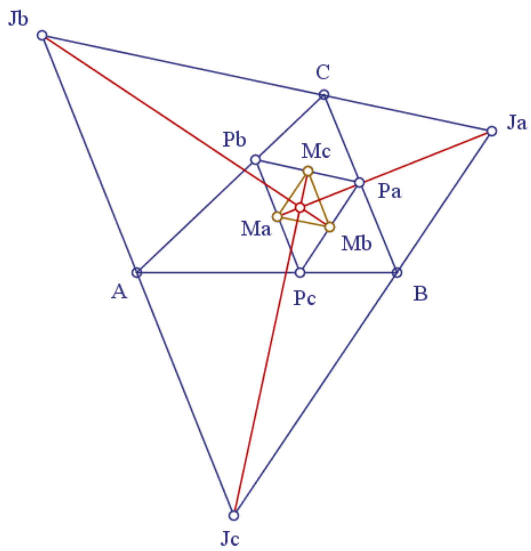

FIGURE 16.

Figure 16 illustrates Problem 3.6. In figure 16,

- $JaJbJc$ is the Excentral triangle,
- $PaPbPc$ is the Contact triangle,
- $MaMbMc$ is the Medial triangle of triangle $PaPbPc$,
- the problem inside triangle $MaMbMc$ is the center of homothety.

References for Problem 3.7: de Longchamps point in [8], X(20) DE LONGCHAMPS POINT and point X(12565) in Kimberling [6].

Problem 3.7. *The Excentral triangle is homothetic with the Triangle of the de Longchamps Points of the Cevian Corner Triangles of the Nagel Point. The center of homothety is the point X(12565). Find the ratio of homothety.*

Figure 17 illustrates Problem 3.7. In figure 17,

- $JaJbJc$ is the Excentral triangle,
- $PaPbPc$ is the Cevian triangle of Nagel point,
- Qa is the de Longchamps point of triangle $APbPc$,
- Qb is the de Longchamps point of triangle $BPcPa$,

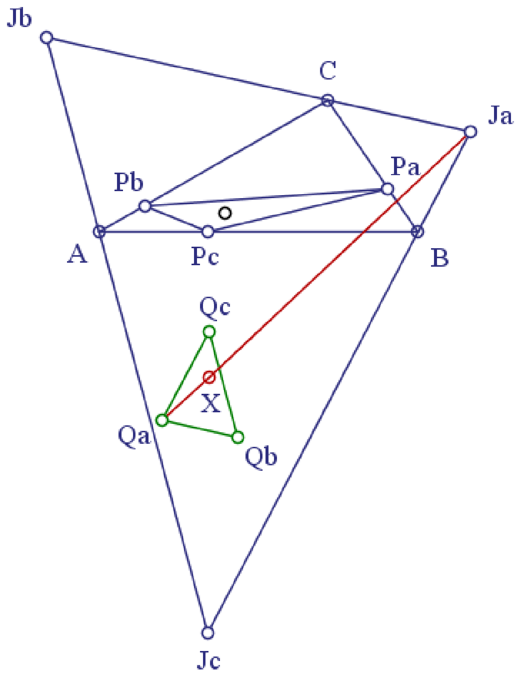

FIGURE 17.

- Qc is the de Longchamps point of triangle $CPaPb$,
- $QaQbQc$ is the Triangle of the de Longchamps Points of the Cevian Corner Triangles of the Nagel Point,
- X is the center of homothety.

Reference for Problem 3.8: Half-Cevian Triangle of the Nagel Point in [5].

Problem 3.8. *The Excentral triangle is homothetic with the Half-Cevian Triangle of the Nagel Point. The center of homothety is the Mittenpunkt. Find the ratio of homothety.*

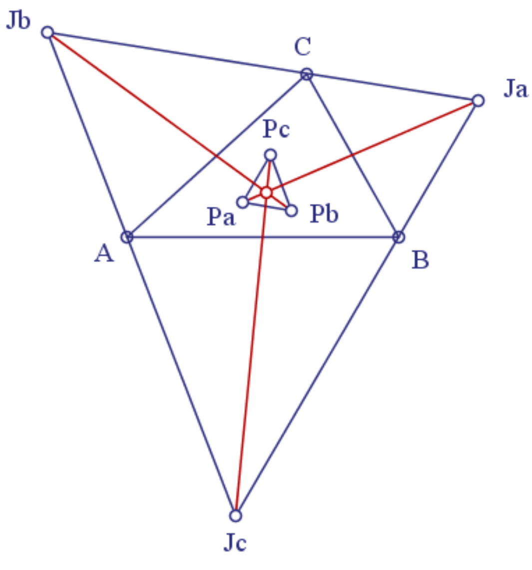

FIGURE 18.

Figure 18 illustrates Problem 3.8. In figure 18,

- $JaJbJc$ is the Excentral triangle,
- $PaPbPc$ is the Half-Cevian triangle of Nagel point,
- The point inside triangle $PaPbPc$ is the center of homothety.

4. SIMILAR TRIANGLES

The "Discoverer" has discovered a number of triangles, similar with the Excentral triangle. Below we give two problems about similarities from the Excentral triangle.

Problem 4.1. *The Excentral triangle is similar with the Pedal Triangle of the Inverse of the Incenter in the Circumcircle. The Ratio of Similarity is*

$$k = \frac{4\sqrt{abc|-ab^2+a^3-a^2c-ac^2+b^3+3bac-bc^2-cb^2+c^3-ba^2|}}{(b+c-a)(c+a-b)(a+b-c)}$$

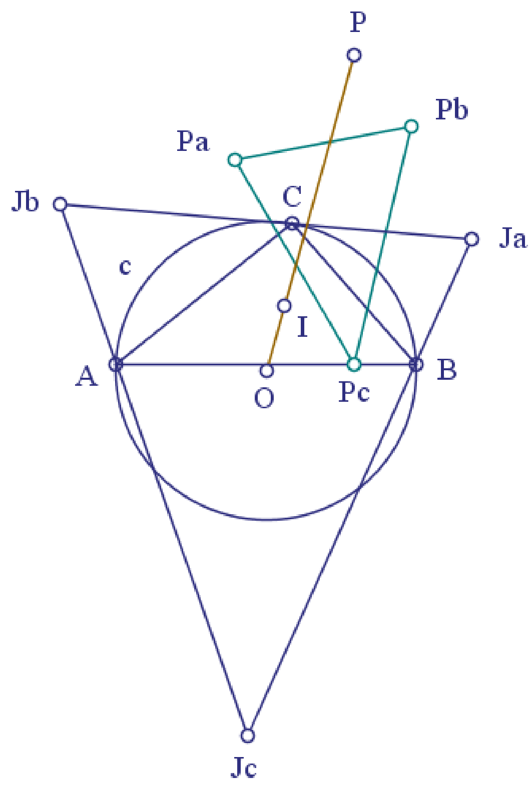

FIGURE 19.

Figure 19 illustrates Problem 4.1. In figure 19,

- $JaJbJc$ is the Excentral triangle,
- c is the circumcircle and O is its center,
- I is the Incenter,
- P is the Inverse of the Incenter in the Circumcircle.
- $PaPbPc$ is the Pedal triangle of point P.

Then triangles $JaJbJc$ and $PaPbPc$ are similar.

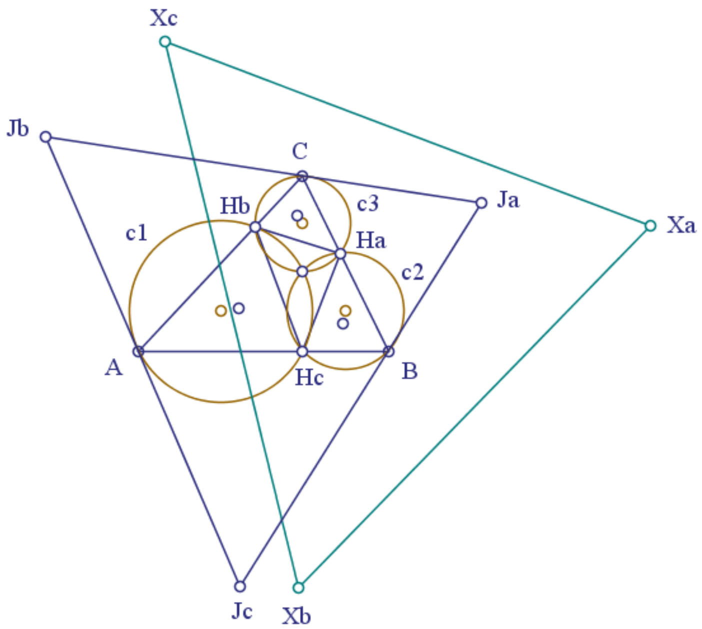

Figure 20.

Problem 4.2. *Denote by $JaJbJc$ the Excentral triangle. Denote by $HaHbHc$ the Orthic triangle. Denote by $c1$ the Circumcircle of triangle $AHbHc$. Denote by $c2$ the Circumcircle of triangle $BHcHa$. Denote by $c3$ the Circumcircle of triangle $CHaHb$. Denote by Xa the inversion point of Incenter of triangle $AHbHc$ wrt circle $c1$. Denote by Xb the inversion point of Incenter of triangle $BHcHa$ wrt circle $c2$. Denote by Xc the inversion point of Incenter of triangle $CHaHb$ wrt circle $c3$. Then triangles $JaJbJc$ and $XaXbXc$ are similar. Find the ratio of similarity.*

Figure 20 illustrates Problem 4.2.

Supplementary Material

The enclosed supplementary material contains theorems related to the topic.

Acknowledgement

The authors are grateful to Professor René Grothmann for his wonderful computer program *C.a.R.* http://car.rene-grothmann.de/doc_en/index.html. See also http://www.journal-1.eu/2016-1/Grothmann-CaR-pp.45-61.pdf.

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