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Computer Discovered Mathematics: Triangles homothetic with the Orthic triangle

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Abstract. By using the computer program "Discoverer" we study triangles homothetic with the Orthic triangle.

Keywords. Orthic triangle, homothety, triangle geometry, notable point, computer discovered mathematics, Euclidean geometry, "Discoverer".

Mathematics Subject Classification (2010). 51-04, 68T01, 68T99.

1. Introduction

Gerry Leversha in his book "The Geometry of the Triangle" [10] has studied a set of triangles homothetic with the Orthic triangle (see Table 13.2, [10]): Circum-Orthic triangle, Kosnita triangle and Tangential triangle. In this paper we extend the set by adding two new triangles homothetic with the Orthic triangle: the Extangents triangle and Intangents triangle. In Section 3, we study the centers and the ratios of the homotheties of the extended set.

In Section 4 we give a theorems about triangles homothetic with the Orthic triangle. The theorems are presented here as problems. We encourage the students and researchers to solve the problems and to submit them for publication in our journal.

We use the computer program "Discoverer" created by the authors.

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2. Preliminaries

We use barycentric coordinates. See [1]-[16]. We study the set of six homothetic triangles. Below we remind the barycentric coordinates of these triangles.

The Orthic triangle is the cevian triangle of the Orthocenter, so that the barycentric coordinates of the Orthic triangle $T_1 = TaTbTc$ are as follows:

$$Ta = (0, v, w), \quad Tb = (u, 0, w), \quad Tc = (u, v, 0),$$

where u, v, w are the barycentric coordinates of the Orthocenter.

The Circum-Orthic triangle is the Circumcevian triangle of the Orthocenter, so that the barycentric coordinates of the Circum-Orthic triangle $T_2 = TaTbTc$ are as follows (see [3]):

$$Ta = \left(\frac{-a^2vw}{c^2v + b^2w}, v, w\right), \quad Tb = \left(u, \frac{-b^2wu}{a^2w + c^2u}, w\right), \quad Tc = \left(u, v, \frac{-c^2uv}{b^2u + a^2v}\right),$$

where u, v, w are the barycentric coordinates of the Orthocenter.

The barycentric coordinates of the Kosnita triangle $T_3 = TaTbTc$ are as follows (see [4]):

$$Ta = (a^2(a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2), -b^2(a^4 + b^4 - 2a^2b^2 - a^2c^2 - b^2c^2),$$
$$-c^2(a^4 + c^4 - 2a^2c^2 - a^2b^2 - b^2c^2)).$$
$$Tb = (a^2(a^4 + b^4 - 2a^2b^2 - a^2c^2 - b^2c^2), -b^2(a^4 + b^4 + c^4 - 2a^2b^2 - 2b^2c^2),$$
$$c^2(b^4 + c^4 - 2b^2c^2 - a^2b^2 - a^2c^2)).$$
$$Tc = (a^2(a^4 + c^4 - 2a^2c^2 - a^2b^2 - b^2c^2), b^2(b^4 + c^4 - 2b^2c^2 - a^2b^2 - a^2c^2),$$
$$-c^2(a^4 + b^4 + c^4 - 2a^2c^2 - 2b^2c^2)).$$

The Tangential triangle $T_4 = TaTbTc$ is the Anticevian triangle of the Symmedian point, so that its barycentric coordinates are as follows:

$$Ta = (-a^2, b^2, c^2), \quad Tb = (a^2, -b^2, c^2) \quad Tc = (a^2, b^2, -c^2).$$

The barycentric coordinates of the Extangents triangle $T_5 = TaTbTc$ are as follows (see [14], Extangents triangle):

$$Ta = (-a(1 + cos(A)), b(cos(A) + cos(C)), c(cos(A) + cos(B))),$$
$$Tb = (a(cos(B) + cos(C)), -b(1 + cos(B)), c(cos(B) + cos(A))),$$
$$Tc = (a(cos(C) + cos(B)), b(cos(C) + cos(A)), -c(1 + cos(C))).$$

The barycentric coordinates of the Intangents triangle $T_6 = TaTbTc$ are as follows (see [14], Intangents triangle):

$$Ta = (a(1 + cos(A)), b(cos(A) - cos(C)), c(cos(A) - cos(B))),$$
$$Tb = (a(cos(B) - cos(C)), b(1 + cos(B)), c(cos(B) - cos(A))),$$
$$Tc = (a(cos(C) - cos(B)), b(cos(C) - cos(A)).c(1 + cos(C))).$$

3. Triangles homothetic with the Orthic Triangle

Here we study homotheties of the set of triangles, given in the previous section. This section extends the corresponding results given in [10].

Theorem 3.1. *The Center of the homothety of the Orthic triangle and Circum-Orthic triangle is the X(4) Orthocenter. The ratio of the homothety k_{12} is*

$$k_{12} = 2.$$

Theorem 3.2. *The Center of the homothety of the Orthic triangle and Kosnita triangle is the point X(24). The ratio of the homothety k_{13} is*

$$k_{13} = \frac{2a^2b^2c^2}{(b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2)}.$$

We see that if the triangle ABC is acute, then $k_{13} > 0$ and if the triangle ABC is obtuse, then $k_{13} < 0$.

Theorem 3.3. *The Center of the homothety of the Orthic triangle and Tangential triangle is the point X(25). The ratio of the homothety k_{14} is*

$$k_{14} = \frac{4a^2b^2c^2}{(b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2)}.$$

We see that if the triangle ABC is acute, then $k_{14} > 0$ and if the triangle ABC is obtuse, then $k_{14} < 0$.

Theorem 3.4. *The Center of the homothety of the Orthic triangle and Extangents triangle is the X(19) Clawson Point. The ratio of the homothety k_{15} is*

$$k_{15} = \frac{2abc(2abc+a^2b+ab^2+ca^2+b^2c+c^2a+bc^2-a^3-b^3-c^3)}{(b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2)}.$$

We see that if the triangle ABC is acute, then $k_{15} > 0$ and if the triangle ABC is obtuse, then $k_{15} < 0$.

Theorem 3.5. *The Center of the homothety of the Orthic triangle and Intangents triangle is the point X(33). The ratio of the homothety k_{16} is*

$$k_{16} = \frac{-2abc(b+c-a)(c+a-b)(a+b-c)}{(b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2)}.$$

We see that if the triangle ABC is acute, then $k_{16} < 0$ and if the triangle ABC is obtuse, then $k_{16} > 0$.

Theorem 3.6. *The Center of the homothety of the Circum-Orthic triangle and Kosnita triangle is the point X(186). The ratio of the homothety k_{23} is*

$$k_{23} = \frac{a^2b^2c^2}{(b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2)}.$$

We see that if the triangle ABC is acute, then $k_{23} > 0$ and if the triangle ABC is obtuse, then $k_{23} < 0$.

Theorem 3.7. *The Center of the homothety of the Circum-Orthic triangle and Tangential triangle is the point X(24). The ratio of the homothety k_{24} is*

$$k_{24} = \frac{2a^2b^2c^2}{(b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2)}.$$

We see that if the triangle ABC is acute, then $k_{24} > 0$ and if the triangle ABC is obtuse, then $k_{24} < 0$.

Theorem 3.8. *The Center of the homothety of the Circum-Orthic triangle and Extangents triangle is the point X(6197). Denote by* k_{25} *the ratio of the homothety. If the triangle* ABC *is acute, then* $k_{25} > 0$ *and if the triangle* ABC *is obtuse, then* $k_{25} < 0$.

Theorem 3.9. *The Center of the homothety of the Circum-Orthic triangle and Intangents triangle is the point X(6198). Denote by* k_{26} *the ratio of the homothety. If the triangle* ABC *is acute, then* $k_{26} < 0$ *and if the triangle* ABC *is obtuse, then* $k_{26} > 0$.

Theorem 3.10. *The Center of the homothety of the Kosnita triangle and Tangential triangle is is the X(3) Circumcenter. The ratio of the homothety* k_{34} *is*

$$k_{34} = 2.$$

Theorem 3.11. *The Center of the homothety of the Kosnita triangle and Extangents triangle is the point X(10902). The ratio of the homothety* k_{35} *is*

$$k_{35} = \frac{2abc + a^2b + ab^2 + ca^2 + b^2c + c^2a + bc^2 - a^3 - b^3 - c^3)}{abc}.$$

It is easy to see that $k_{35} > 0$.

Theorem 3.12. *The Center of the homothety of the Kosnita triangle and Intangents triangle is the point X(35).*

Theorem 3.13. *The Center of the homothety of the Tangential triangle and Extangents triangle is the point X(55) Internal Similitude Center of the Circumcircle and Incircle. The ratio of the homothety* k_{45} *is*

$$k_{45} = \frac{2abc + a^2b + ab^2 + ca^2 + b^2c + c^2a + bc^2 - a^3 - b^3 - c^3)}{2abc}.$$

We see that $k_{45} > 0$.

Theorem 3.14. *The Center of the homothety of the Tangential triangle and Intangents triangle is point X(55) Internal Similitude Center of the Circumcircle and Incircle. The ratio of the homothety* k_{46} *is*

$$k_{46} = \frac{-(b + c - a)(c + a - b)(a + b - c)}{2abc}.$$

We see that $k_{46} < 0$.

Theorem 3.15. *The Center of the homothety of the Extangents triangle and Intangents triangle is point X(55) Internal Similitude Center of the Circumcircle and Incircle. The ratio of the homothety* k_{56} *is*

$$k_{56} = \frac{-(b + c - a)(c + a - b)(a + b - c)}{2abc + a^2b + ab^2 + ca^2 + b^2c + c^2a + bc^2 - a^3 - b^3 - c^3}.$$

We see that $k_{56} < 0$.

4. Problems about Triangles homothetic with the Orthic triangle

The problems below are discovered by the computer program "Discoverer". We encourage the students and researchers to solve the problems and to submit the solutions to our journal.

In the problems below ABC as a triangle with side lengths $BC = a, CA = b$ and $AB = c$, $HaHbHc$ is the Orthic triangle and triangle $XaXbXc$ is given it the statement of the problem. Prove that the triangles $HaHbHc$ and $XaXbXc$ are homothetic. Find the center and the ratio of the homothety as functions of a, b and c. Identify the center of homothety as Kimberling center.

Problem 1. *Let $PaPbPc$ be the Cevian triangle of the X(69) Retrocenter. Denote by $XaXbXc$ the triangle whose vertices are as follows: Xa is the X(20) de Longchamps Point of triangle $APbPc$, Xb is the de Longchamps Point of triangle $BPcPa$ and Xc is the de Longchamps Point of triangle $CPaPb$.*

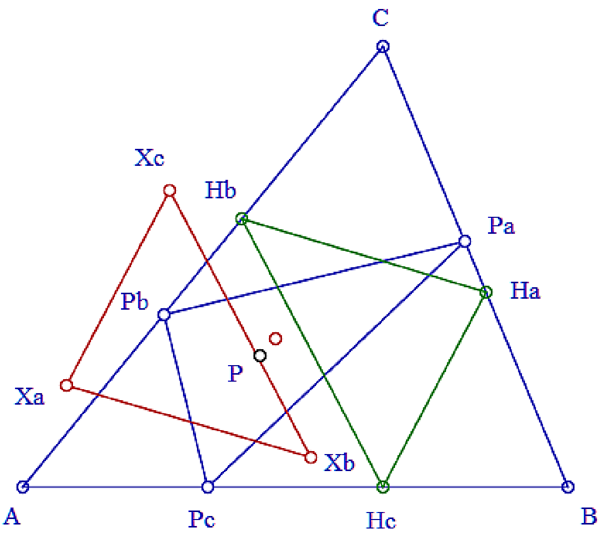

FIGURE 1.

Figure 1 illustrates Problem 1. In figure 1, $HaHbHc$ is the Orthic triangle, P is the Retrocenter, $PaPbPc$ is the Cevian triangle of P, Xa is the de Longchamps Point of triangle $APbPc$, Xb is the de Longchamps Point of triangle $BPcPa$, and Xc is the de Longchamps Point of triangle $CPaPb$. Then triangle $HaHbHc$ and $XaXbXc$ are homothetic. The red point is the center of the homothety.

Problem 2. *Let P be the reflection of the Circumcenter in the Orthocenter and let $PaPbPc$ be the Pedal triangle of P. Denote by $XaXbXc$ the triangle whose vertices are as follows: Xa is the Nine-Point Center of triangle $APbPc$, Xb is the Nine-Point Center of triangle $BPcPa$, and Xc is the Nine-Point Center of triangle $CPaPb$.*

Problem 3. *Let $MaMbMc$ be the Medial triangle. Denote by $XaXbXc$ the triangle whose vertices are as follows: Xa is the reflection of the Circumcenter in the line $MbMc$, Xb is the reflection of the Circumcenter in the line $McMa$, and Xc is the reflection of the Circumcenter in the line $MaMb$.*

Figure 2 illustrates Problem 3. In figure 2, $HaHbHc$ is the Orthic triangle, $MaMbMc$ is the Medial triangle, O is the Circumcenter, Xa is the reflection of the Circumcenter in the line $MbMc$, Xb is the reflection of the Circumcenter in the line $McMa$, and Xc is the reflection of the Circumcenter in the line $MaMb$. Triangles $HaHbHc$ and $XaXbXc$ are homothetic and the red point is the center of the homothety.

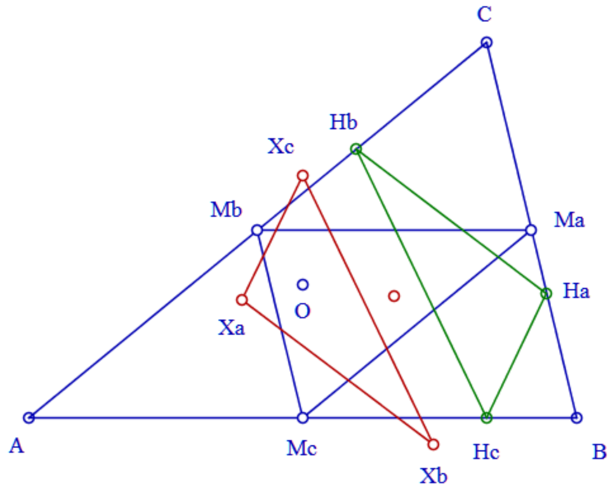

FIGURE 2.

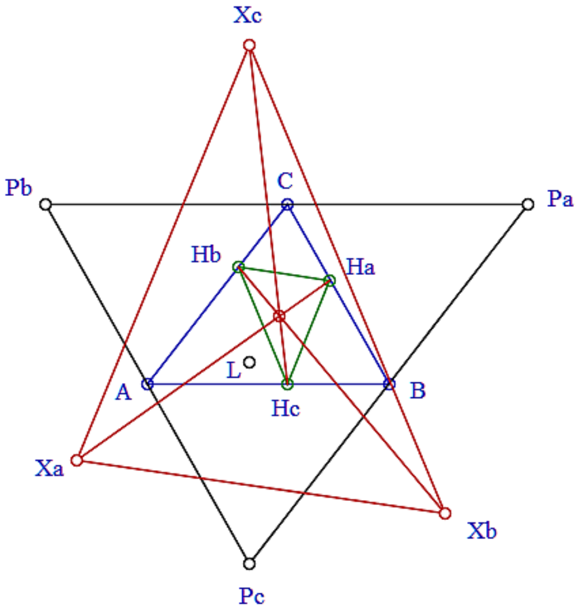

FIGURE 3.

Problem 4. *Denote by $XaXbXc$ the triangle whose vertices are as follows: Xa is the reflection of the Nine-Point Center in the line $HbHc$, Xb is the reflection of the Nine-Point Center in the line $HcHa$, and Xc is the reflection of the Nine-Point Center in the line $HaHb$.*

Problem 5. *Let $PaPbPc$ be the Antimedial triangle. Denote by $XaXbXc$ the triangle whose vertices are as follows: Xa is the reflection of the de Longchamps Point in the line $PbPc$, Xb is the reflection of the de Longchamps Point in the line $PcPa$, and Xc is the reflection of the de Longchamps Point in the line $PaPb$.*

Figure 3 illustrates Problem 5. In figure 3, $HaHbHc$ is the Orthic triangle, L is de Longchamps point, $PaPbPc$ is the Antimedial triangle, Xa is the reflection of the de Longchamps Point in the line $PbPc$, Xb is the reflection of the de Longchamps Point in the line $PcPa$, and Xc is the reflection of the de Longchamps Point in the line $PaPb$. Triangles $HaHbHc$ and $XaXbXc$ are homothetic. The point of intersection of the lines $HaXa, HbXb$ and $HcXc$ is the center of the homothety.

Problem 6. *Denote by O the circumcenter and by $XaXbXc$ the triangle whose vertices are as follows: Xa is the Parry Reflection Point of triangle OBC, Xb is the Parry Reflection Point of triangle OCA, and Xc is the Parry Reflection Point of triangle OAB.*

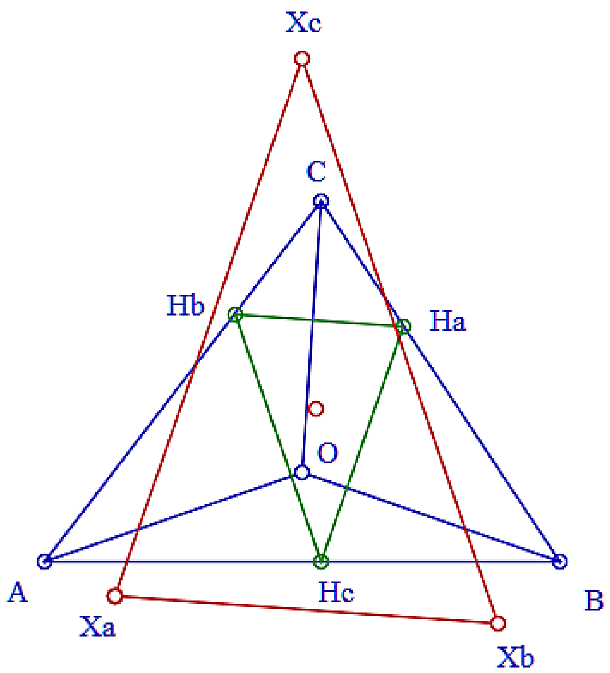

FIGURE 4.

Figure 4 illustrates Problem 6. In figure 4, $HaHbHc$ is the Orthic triangle, O is the Circumcenter, Xa is the Parry Reflection Point of triangle OBC, Xb is the Parry Reflection Point of triangle OCA, and Xc is the Parry Reflection Point of triangle OAB. Triangles $HaHbHc$ and $XaXbXc$ are homothetic. The center of the homothety is the red point.

Problem 7. *Let K be the Symmedian point and let $TaTbTc$ be the Tangential triangle. Denote by $XaXbXc$ the triangle whose vertices are as follows: Xa is the midpoint of points Ta and K, Xb is the midpoint of points Tb and K, and Xc is the midpoint of points Tc and K.*

Figure 5 illustrates Problem 7. In figure 5, $HaHbHc$ is the Orthic triangle, K is the Symmedian point,$TaTbTc$ is the Tangential triangle, Xa is the midpoint of points Ta and K, Xb is the midpoint of points Tb and K, and Xc is the midpoint of points Tc and K. Triangles $HaHbHc$ and $XaXbXc$ are homothetic. The center of the homothety is the red point.

Problem 8. *Let P be the X(69) Retrocenter and let $PaPbPc$ be the Cevian triangle of P. Denote by $XaXbXc$ the triangle whose vertices are as follows: Xa is the midpoint of points A and Pa, Xb is the midpoint of points B and Pb, and Xc is the midpoint of points C and Pc.*

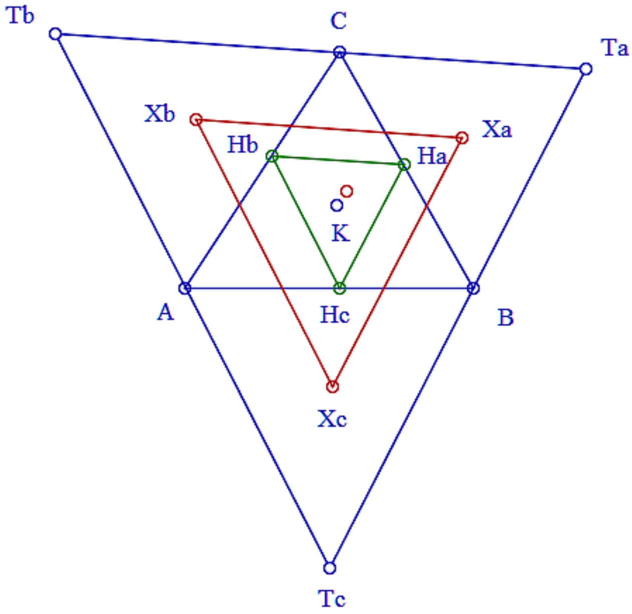

Figure 5.

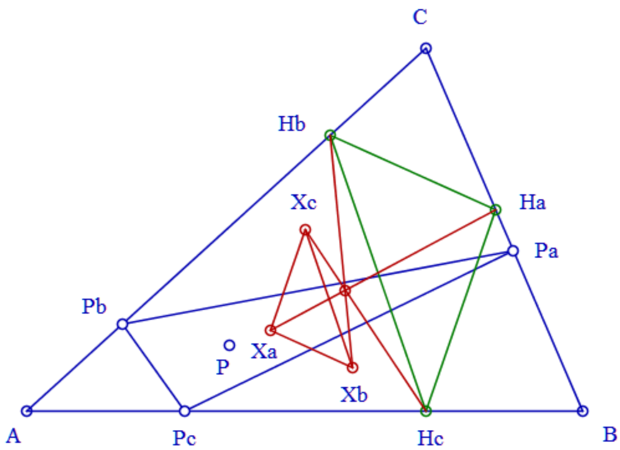

Figure 6.

Figure 6 illustrates Problem 8. In figure 6, $HaHbHc$ is the Orthic triangle, P is the Retrocenter,$PaPbPc$ is the Cevian triangle of P, Xa is the midpoint of points A and Pa, Xb is the midpoint of points B and Pb, and Xc is the midpoint of points C and Pc. Triangles $HaHbHc$ and $XaXbXc$ are homothetic and the center of the homothety is the red point.

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