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Computer Discovered Mathematics: Incentral Triangle

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Abstract. By using the computer program "Discoverer" we study the Incentral triangle.

Keywords. triangle geometry, remarkable point, computer discovered mathematics, Euclidean geometry, "Discoverer".

Mathematics Subject Classification (2010). 51-04, 68T01, 68T99.

1. Introduction

The Incentral triangle is one of the significant triangles in triangle geometry. In this paper we give results about the Incentral triangle. discovered by the computer program "Discoverer", created by the authors.

The Incentral triangle is the triangle whose vertices are determined by the intersections of the reference triangle's angle bisectors with the respective opposite sides. Also, it is the Cevian triangle of triangle ABC with respect to its Incenter. See e.g. [12], Incentral Triangle.

We use barycentric coordinates. See [1]- [14].

Sometimes the barycentric coordinates of a point of the form $(f(a,b,c), f(b,c,a), f(c,a,b))$ are shortened to $[f(a,b,c)]$.

The area of the reference triangle ABC is denoted by Δ.

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Recall that the Grinberg Point is the point X(37),[7], and the Moses Point is the point X(75),[7].

2. Barycentric coordinates, Area, Side Lengths

The barycentric coordinates of the Incenter I are $I = (a, b, c)$. Since the Incentral triangles is the Cevian triangle of the Incenter, the barycentric coordinates of the Incentral triangle $PaPbPc$ are as follows:

$$Pa = (0, b, c), Pb = (a, 0, c), Pc = (a, b, 0).$$

The normalized barycentric coordinates of the vertices of the Incentral triangle are as follows:

$$Pa = \left(0, \frac{b}{b+c}, \frac{c}{b+c}\right), \quad Pb = \left(\frac{a}{a+c}, 0, \frac{c}{a+c}\right), \quad Pc = \left(\frac{a}{a+b}, \frac{b}{a+b}, 0\right),$$

so that, by using formula (2), [4] we find the area of the Incentral triangle

$$area(PaPbPc) = \begin{vmatrix} 0 & \frac{b}{b+c} & \frac{c}{b+c} \\ \frac{a}{a+c} & 0 & \frac{c}{a+c} \\ \frac{a}{a+b} & \frac{b}{a+b} & 0 \end{vmatrix} \Delta = \frac{2abc\Delta}{(b+c)(c+a)(a+b)}.$$

where Δ is the area of the reference triangle ABC.

The side lengths of the Incentral triangle are as follows (see [12],Incentral Triangle):

$$(1) \qquad \begin{aligned} a' &= \frac{abc\sqrt{3 + 2(-\cos A + \cos B + \cos C)}}{(a+b)(a+c)}, \\ b' &= \frac{abc\sqrt{3 + 2(\cos A - \cos B + \cos C)}}{(b+c)(b+a)}, \\ c' &= \frac{abc\sqrt{3 + 2(\cos A + \cos B - \cos C)}}{(c+a)(c+b)}, \end{aligned}$$

3. Cevians

By using the distance formula (9), [4], we find the lengths of the cevians of the Incentral triangle $PaPbPc$ as follows:

$$|APa| = \frac{\sqrt{bc(a+b+c)(b+c-a)}}{b+c}$$

$$|BPb| = \frac{\sqrt{ca(a+b+c)(c+a-b)}}{c+a}$$

$$|CPc| = \frac{\sqrt{ab(a+b+c)(a+b-c)}}{a+b}$$

4. Homothetic Triangles

Theorem 4.1. *The Incentral Triangle and the Anticevian Triangle of the Grinberg Point are homothetic. The Center of homothety is the Kimberling point X(42) and ratio of the homothety is* $k = \frac{(a+b)(b+c)(c+a)}{2abc}$.

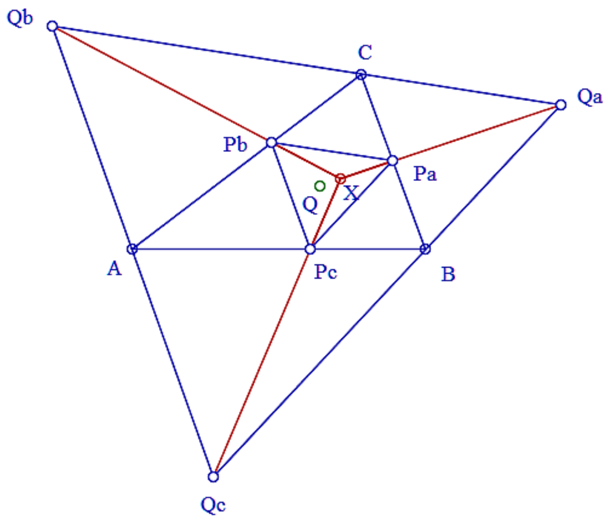

FIGURE 1.

Figure 1 illustrates Theorem 4.1. In figure 1,$PaPbPc$ is the Incentral triangle, Q is the Grinberg point, $QaQbQc$ is the Anticevian triangle of Q. The lines $PaQa, PbQb$ and $PcQc$ concur in the point $X = X(42)$.

Proof. The barycentric coordinates of the Anticevian triangle of the Grinberg point X(37) $= (a(b+c), b(c+a), (c(a+b))$ are as follows:

$$Qa = (-a(b+c), b(c+a), c(a+b)),$$
$$Qb = (a(b+c), -b(c+a), c(a+b)),$$
$$Qc = (a(b+c), b(c+a), -c(a+b)).$$

It is known that any Cevian triangle and any Anticevian triangle are always perspective. By using formulas (3) and (5), [4], we find the perspector X of the Incentral triangle $PaPbPc$ and the Anticevian triangle of the Grinberg point $QaQbQc$ as the intersection of the lines $L_1 = PaQa$ and $L_2 = PbQb$. We obtain

$$L1: \quad bc(-c+b)x - ac(b+c)y + ab(b+c)z = 0$$
$$L2: \quad bc(c+a)x - ac(-c+a)y - ab(c+a)z = 0$$
$$X = (a^2(b+c), b^2(c+a), c^2(a+b)).$$

It is easy to see that point X is the Kimberling point X(42) $=$ Product od the Incenter and the Grinberg point $=$ Product of the Symmedian point and the Spieker center.

Now we prove that triangles $PaPbPc$ and $QaQbQc$ are homothetic. By using formula (9), [4] we calculate the distances $|XPa|, |XPb|, |XPc|, |XQa|, |XQb|, |XQc|$, we define

$$k_a = \frac{|XQa|}{|XPa|}, \quad k_b = \frac{|XQb|}{|XPb|}, \quad k_c = \frac{|XQc|}{|XPc|}.$$

and we see that

$$k_a = k_b = k_c = \frac{(a+b)(b+c)(c+a)}{2abc}.$$

Hence, the triangles are homothetic with center X(42) and ratio of the homothety is $k = \frac{(a+b)(b+c)(c+a)}{2abc}$. □

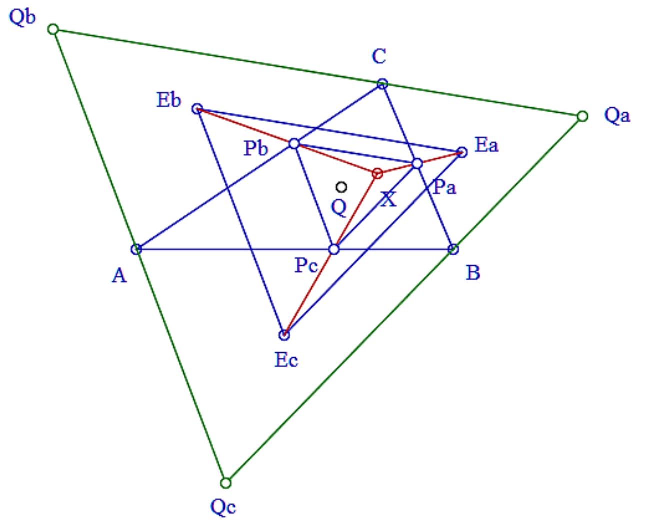

FIGURE 2.

Theorem 4.2. *The Incentral triangle is homothetic with the Euler Anticevian triangle of the Grinberg point. The Homothetic Center is the Kimberling point X(872).*

Figure 2 illustrates Theorem 4.2. In figure 2, $PaPbPc$ is the Incentral triangle, Q is the Grinberg point, $QaQbQc$ is the Anticevian triangle of Q, and $EaEbEc$ is the Euler Anticevian triangle of Q. The lines $PaEa, PbEb$ and $PcEc$ concur in the point $X = X(872)$.

Problem 1. *Find the ratio of homothety.*

Theorem 4.3. *The Incentral triangle is homothetic with the Half-Cevian Triangle of the Isotomic Conjugate of the Incenter. The Center of homothety is the Centroid of triangle ABC.*

Figure 3 illustrates Theorem 4.3. In figure 3, $PaPbPc$ is the Incentral triangle and $QaQbQc$ is the Half-Cevian Triangle of the Isotomic Conjugate of the Incenter. The Center of homothety is the Centroid G and the ratio of homothety is $k = -\frac{1}{2}$. Note that the Isotomic conjugate of the Incenter is the Moses point X(75) with barycentric coordinates (bc, ca, ab), so that the Cevian triangle of X(75) is the triangle

$$T = ((0, ca, ab), (bc, 0, ab), (bc, ca, 0)),$$

so that the Half-Cevian triangle of X(75) is the triangle

$$QaQbQc = ((c + b, c, b).(c, c + a, a), (b, a, b + a)).$$

Theorem 4.4. *The Incentral triangle is homothetic with the the Monge Triangle of the Miquel Pedal Circles of the Second Isodynamic Point. The Center of homothety is the Second Isodynamic point.*

Figure 4 illustrates Theorem 4.4. In figure 4,

- $PaPbPc$ is the Incentral triangle,
- Q is the Second isodynamic point,

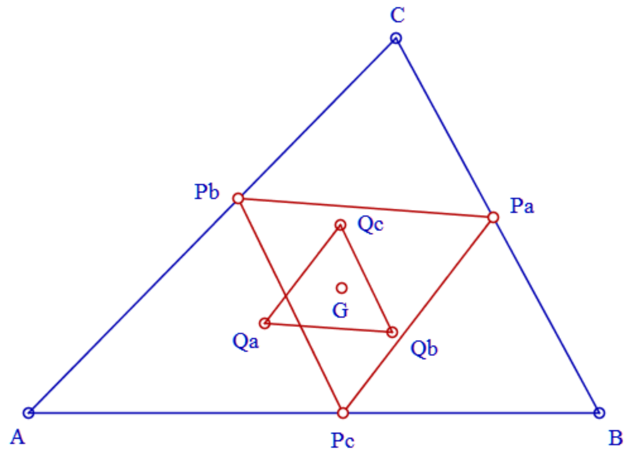

FIGURE 3.

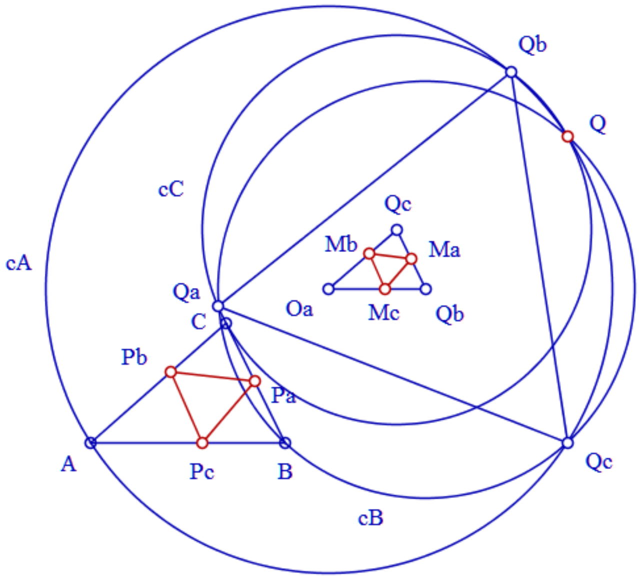

FIGURE 4.

- $QaQbQc$ is the Pedal triangle of Q,
- cA is the Miquel circle through points A, Qb, Qc with center Oa,
- cB is the Miquel circle through points B, Qc, Qa with center Ob,
- cC is the Miquel circle through points C, Qa, Qb with center Oc.

The Miquel circles cA, cB and cC intersect in the Miquel point Q.

- Point Ma is the Internal Center of similitude of circles cB and cC,
- point Mb is the Internal Center of similitude of circles cC and cA,
- point Mc is the Internal Center of similitude of circles cA and cB, and
- triangle $MaMbMc$ is the Monge triangle of the Miquel Pedal circles.

Triangles $PaPbPc$ and $MaMbMc$ are homothetic with Center of homothety the point Q. Note that the ratio of homothety k is equal to 2.

Theorem 4.5. *The Incentral triangle is homothetic with the the Monge Triangle of the Miquel Pedal Circles of the First Isodynamic Point. The Center of homothety is the First Isodynamic point.*

We recommend the reader to draw the figure.

Problem 2. *Find the ratio of homothety.*

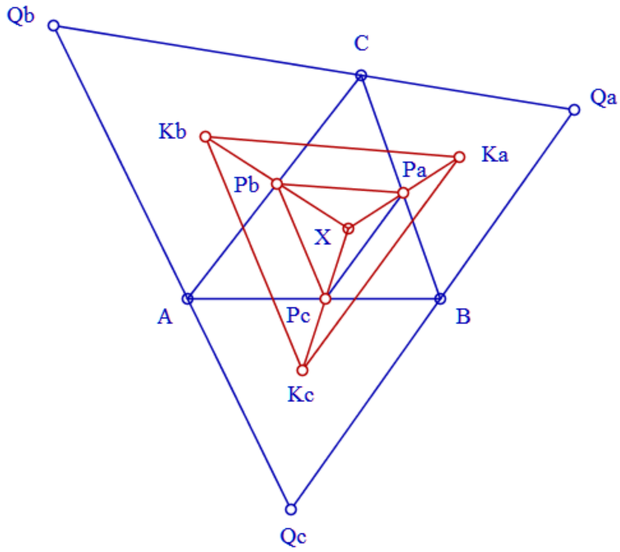

FIGURE 5.

Theorem 4.6. *The Incentral triangle is homothetic with the Triangle of the Kosnita Points of the Anticevian Corner Triangles of the Incenter. The Center of homothety is the Kimberling point X(58) = Isogonal Conjugate of the Spieker Center.*

Figure 5 illustrates Theorem 4.6. In figure 5, $PaPbPc$ is the Incentral triangle, $QaQbQc$ is the Excentral triangle, that is, the Anticevian triangle of the Incenter, Ka is the Kosnita point of triangle $BCQa$, Kb is the Kosnita point of triangle $CAQb$, and Kc is the Kosnita point of triangle $ABQc$. Then triangles $PaPbPc$ and $KaKbKc$ are homothetic. The center of homothety is the point $X =$ X(58) Isogonal Conjugate of the Spieker Center.

Problem 3. *Find the ratio of homothety.*

5. SIMILAR TRIANGLES

Theorem 5.1. *The Incentral triangle and the Triangle of the Euler Reflection Points of the Triangulation Triangles of the Incenter are similar (but not homothetic).*

Figure 6 illustrates Theorem 5.1. In figure 6, P is the Incenter, $PaPbPc$ is the Incentral triangle, Qa is the Euler Reflection point (point X(110) in [7]) of triangle PBC, Qb is the Euler Reflection point of triangle PCA, and Qc is the Euler Reflection point of triangle PAB. Triangle $PaPbPc$ is similar with triangle $QaQbQc$.

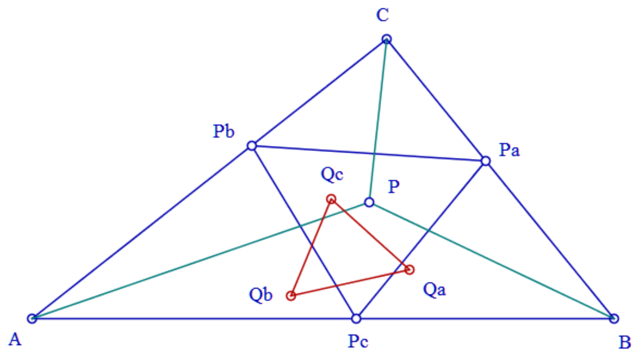

Figure 6.

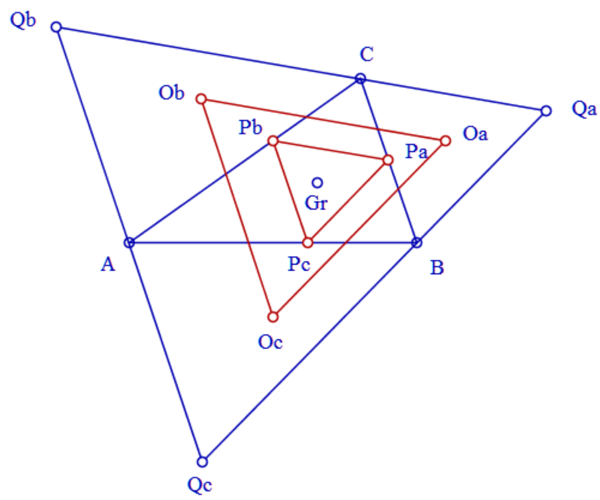

Figure 7.

Problem 4. *Find the ratio of similitude.*

Theorem 5.2. *The Incentral triangle and the Triangle of the Circumcenters of the Anticevian Corner Triangles of the Grinberg Point are similar (but not homothetic).*

Figure 7 illustrates Theorem 5.2. In figure 7, $PaPbPc$ is the Incentral triangle, Gr is the Grinberg point, $QaQbQc$ is the Anticevian triangle of the Grinberg point, Oa is the Circumcenter of triangle $QaBC$, Ob is the Circumcenter of triangle $QbCA$ and Oc is the Circumcenter of triangle $QcAB$. Triangle $PaPbPc$ is similar with triangle $OaObOc$.

Problem 5. *Find the ratio of similitude.*

6. Kimberling Points

We have investigated 195 notable points of the Incentral triangle. Of these 3 are Kimberlng point and the rest of 192 points are not available in [7]. See the Supplementary material, folder Points.

Table 1 gives the centers of the Incentral triangle in terms of the centers of the reference triangle that are Kimberling points $X(n)$.

	Notable Point of Incentral triangle	**Notable Point of Triangle** ABC
1	Centroid	X(1962)
2	Circumcenter	X(8143)
3	Orthocenter	X(500)

TABLE 1.

7. NEW NOTABLE POINTS

In the our investigation the "Discoverer" has found 192 notable points of the Incentral triangle that are not included in [7]. These points are new points, discovered by the "Discoverer". See the Supplementary material, folder Points.

Below we give the barycentric coordinates of seven new points of the Incentral triangle. The barycentric coordinates of the Incenter of the Incentral triangle are also available, but they are too complicated.

In order to find the barycentric coordinates of a point X with respect to the Incentral triangle, we use the formulas (1) for the side lengths of the Incentral triangle. Then we use formula (10),[4] in order to change the coordinates of X to coordinates with respect to reference triangle ABC.

Theorem 7.1. *The Nine-Point Center of the Incentral triangle has barycentric coordinates*

$$[a(-5a^3bc^2 - 5ca^3b^2 - 6a^3b^3 - 6c^3a^3 - 6a^2c^3b - 6a^2b^2c^2 + 2c^5b + 2b^5c - b^4c^2 - b^2c^4 - 4b^3c^3 + 3a^5c + a^4c^2 - 2c^4a^2 + 3c^5a + b^6 + c^6 + 4a^4bc - 6a^2b^3c - 5ac^3b^2 + 2acb^4 - 5ab^3c^2 + 2abc^4 + 3a^5b + a^4b^2 - 2a^2b^4 + 3ab^5)]$$

Theorem 7.2. *The Symmedian point of the Incentral triangle has barycentric coordinates*

$$[a(-4a^2bc - 4bc^2a - 3a^2b^2 - 4ab^2c - 3c^2a^2 + 2a^4 + a^3b - b^3a - 2c^2b^2 + c^4 + b^4 + ca^3 - ac^3)].$$

Theorem 7.3. *The de Longchamps point of the Incentral triangle has barycentric coordinates*

$$[a(-2bca^4 + a^4b^2 + a^4c^2 - 2ca^3b^2 - 2a^3bc^2 - 2a^2c^4 - 2a^2b^4 - 2b^3c^2a - 2b^2c^3a + 2bc^4a + 2b^4ca - b^4c^2 + c^6 + 2b^5c + b^6 - b^2c^4 - 4b^3c^3 + 2bc^5)].$$

Theorem 7.4. *The Exeter point of the Incentral triangle has barycentric coordinates*

$$[a(-11b^8c^3 - 6b^7c^4 + 14b^6c^5 - 6b^4c^7 - 11b^3c^8 + 14b^5c^6 + 3b^{10}c - b^9c^2 - c^9b^2 + 3c^{10}b + b^{11} + c^{11} + a^9b^2 - 2a^9bc + a^9c^2 + a^8b^3 - 5a^8b^2c - 5a^8bc^2 + a^8c^3 - 4a^7b^4 - 3a^7b^3c - 10a^7b^2c^2 - 3a^7bc^3 - 4a^7c^4 - 4a^6b^5 + 4a^6b^4c + 8a^6b^3c^2 + 8a^6b^2c^3 + 4a^6bc^4 - 4a^6c^5 + 6a^5b^6 + 13a^5b^5c + 25a^5b^4c^2 + 24a^5b^3c^3 + 25a^5b^2c^4 + 13a^5bc^5 + 6a^5c^6 + 6a^4b^7 + 10a^4b^6c + 19a^4b^5c^2 + 25a^4b^4c^3 + 25a^4b^3c^4 + 19a^4b^2c^5 + 10a^4bc^6 + 6a^4c^7 - 4a^3b^8 - 9a^3b^7c - 3a^3b^6c^2 + 20a^3b^5c^3 + 32a^3b^4c^4 + 20a^3b^3c^5 - 3a^3b^2c^6 - 9a^3bc^7 - 4a^3c^8 - 4a^2b^9 - 12a^2b^8c - 21a^2b^7c^2 - 3a^2b^6c^3 + 40a^2b^5c^4 + 40a^2b^4c^5 - 3a^2b^3c^6 - 21a^2b^2c^7 - 12a^2bc^8 - 4a^2c^9 + ab^{10} + ab^9c - 13ab^8c^2 - 24ab^7c^3 + 12ab^6c^4 + 46ab^5c^5 + 12ab^4c^6 - 24ab^3c^7 - 13ab^2c^8 + abc^9 + ac^{10})].$$

Theorem 7.5. *The Brocard Midpoint of the Incentral triangle has barycentric coordinates*

$[a(16a^4b^5 - 12a^6b^3 - 7a^7b^2 + 7a^5b^4 + 3a^8c - a^3b^6 - 8a^2b^7 + 7a^5c^4 + 16a^4c^5 - a^3c^6 - 8a^2c^7 - 7a^7c^2 - 12a^6c^3 + 3a^8b + 11b^4c^5 + 11b^5c^4 - 5b^3c^6 - 7b^2c^7 - 5b^6c^3 - 7b^7c^2 - b^8a - c^8a + 20a^3b^5c - 29a^6b^2c + 45a^4b^4c - 8b^2c^2a^5 + 60b^2c^3a^4 + 60b^3c^2a^4 + 45bc^4a^4 + 96b^3c^3a^3 + 68b^4c^2a^3 + 64b^4c^3a^2 + 68b^2c^4a^3 + 64b^3c^4a^2 + 8b^2c^5a^2 + 8b^5c^2a^2 + 42b^4c^4a - 29bc^2a^6 + 12b^5c^3a + 20bc^5a^3 + 12b^3c^5a - 19bc^6a^2 - 20b^2c^6a - 8bca^7 - 19b^6ca^2 - 20b^6c^2a - 12bc^7a - 12ab^7c + 2a^9 + c^9 + b^9)]$.

Theorem 7.6. *The Euler Reflection Point of the Incentral triangle has barycentric coordinates* $[-a(-b^3 - ab^2 - cb^2 + ba^2 - bca - bc^2 + a^3 + ca^2 - ac^2 - c^3)(8bac^4 - 4a^3b^3 - 4c^3a^3 - a^2b^2c^2 - c^4a^2 + 2a^5c - a^4c^2 + 2ac^5 - 4b^3c^3 - b^2c^4 + 2bc^5 - b^4c^2 + 2b^5c + a^6 + c^6 + b^6 - a^4b^2 + 8b^4ca + 14b^3ac^2 - 2b^3ca^2 + 14b^2ac^3 - 2ba^2c^3 + 2b^5a - b^4a^2 + 2ba^5 - 10a^3bc^2 - 10ca^3b^2)]$.

Theorem 7.7. *The Parry Reflection Point of the Incentral triangle has barycentric coordinates*

$[a(-3a^5b^4 - 3a^7b^2 - 7a^6b^3 + 3a^4b^5 + 7a^3b^6 + 3a^2b^7 - 3c^4a^5 - 3c^2a^7 + 3c^5a^4 + 7c^6a^3 - 7c^3a^6 + 3c^7a^2 + 3a^8c + 3a^8b - b^2c^7 + 3b^4c^5 + 5b^3c^6 + 3b^5c^4 - 5bc^8 + 5b^6c^3 - 3b^8a - b^7c^2 - 5b^8c - 3c^8a + 2a^9 - 2c^9 - 2b^9 - 4ab^7c - 4a^5b^2c^2 - 8a^5b^3c - 8a^5bc^3 + 2a^4bc^4 + 12a^3bc^5 - 10a^6bc^2 + 4a^2b^2c^5 - 10a^6b^2c + 2a^4b^4c - 12a^2b^4c^3 - 12a^2b^3c^4 + 12a^3b^5c + 4a^2b^5c^2 + 4a^4b^2c^3 + 4a^4b^3c^2 - 8a^3b^3c^3 + 10a^2bc^6 + 10a^2b^6c + 2b^2c^6a + 2b^4c^4a + 4b^3c^5a + 4b^5c^3a - 4bc^7a + 2b^6c^2a)(2a^3bc - a^3b^2 - a^3c^2 + 2a^5 - a^2c^3 - ac^4 + 2a^4c - bc^4 + 2ba^4 - b^3a^2 + 2b^3c^2 + 2c^3b^2 - ab^4 - cb^4 - bc^2a^2 - bc^3a - a^2b^2c + 2ab^2c^2 - b^3ca - c^5 - b^5)]$.

8. Centers of Circles

We have investigated 29 centers of notable circles of the Incentral triangle. Of these 3 centers are Kimberling points and the rest of 26 centers are new points. See the Supplementary material, folder Centers of Circles.

Table 2 gives the centers of circles of the Incentral triangle in terms of the points of the reference triangle ABC that are Kimberling points $X(n)$.

	Center of Circle wrt Incentral triangle	**Notable Point of Triangle** ABC
1	Center of Circumcircle	X(8143)
2	Center of Antimedial Circle	X(500)
3	Center of Second Brocard Circle	X(8143)

Table 2.

9. Similitude Centers

We have investigated 771 centers of similitude of notable circles of triangle ABC and notable circles of the Incentral triangle. Of these 3 centers of similitude are Kimberling points and the rest of 768 centers are new points. See the Supplementary material, folder Similitude Centers Part 1.

Table 3 gives the centers of similitude in terms of the points of the reference triangle that are Kimberling points $X(n)$.

	Center of Similitude	**Notable Point of Triangle** ABC
1	Internal Center of Similitude of the Incentral Circle of triangle ABC and the Nine-Point Circle of the Incentral Triangle	X(1962)
2	External Center of Similitude of the Incentral Circle of triangle ABC and the Nine-Point Circle of the Incentral Triangle	X(500)
3	Internal Center of Similitude of the Incentral Circle of triangle ABC and the Second Brocard Circle of the Incentral Triangle	X(8143)

TABLE 3.

Also, we have investigated 803 centers of similitude of notable circles of the Incentral triangle. Of these 10 centers of similitude are Kimberling points and the rest of 793 are new points. See the Supplementary material, folder Similitude Centers Part 2.

Table 4 gives a few of the centers of similitude in terms of the points of the reference triangle that are Kimberling points $X(n)$. For the rest of Table 4 see the Supplementary material.

	Center of Similitude	**Notable Point of Triangle** $\boldsymbol{ABC}$
1	Internal Center of Similitude of the Circumcircle wrt the Incentral Triangle and the Nine-Point Circle wrt the Incentral Triangle	X(1962)
2	External Center of Similitude of the Circumcircle wrt the Incentral Triangle and the Nine-Point Circle wrt the Incentral Triangle	X(500)
3	Internal Center of Similitude of the Circumcircle wrt the Incentral Triangle and the Second Brocard Circle wrt the Incentral Triangle	X(8143)

TABLE 4.

SUPPLEMENTARY MATERIAL

The enclosed supplementary material contains theorems related to the topic.

ACKNOWLEDGEMENT

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