International Journal of Computer Discovered Mathematics (IJCDM)
ISSN 2367-7775 ©IJCDM
Volume 2, 2017, pp.90-96
Received 1 May 2017. Published on-line 10 May 2017
web: http://www.journal-1.eu/

Triangles Homothetic with Triangle ABC. Part 2

Sava Grozdev[a], Hiroshi Okumura[b] and Deko Dekov[c] [2]
[a] VUZF University of Finance, Business and Entrepreneurship,
Gusla Street 1, 1618 Sofia, Bulgaria
e-mail: sava.grozdev@gmail.com
[b] Maebashi Gunma, 371-0123, Japan
e-mail: okmr@protonmail.com
[c]Zahari Knjazheski 81, 6000 Stara Zagora, Bulgaria
e-mail: ddekov@ddekov.eu
web: http://www.ddekov.eu/

Abstract. By using the computer program "Discoverer" we study triangles homothetic with the reference triangle ABC.

Keywords. homothety, triangle geometry, remarkable point, computer discovered mathematics, Euclidean geometry, "Discoverer".

Mathematics Subject Classification (2010). 51-04, 68T01, 68T99.

1. Introduction

We continue the investigation of triangles homothetic with the reference triangle ABC. For the first part of this papers see [7].

Theorems in this papers are discovered by the computer program "Discoverer" created by the authors.

We use barycentric coordinates. See [1]-[17]. The Kimberling points are denoted by X(n).

We present a few problems related the topic. We encourage th students and researchers to solve them.

[2]Corresponding author

2. Homothetic Triangles

Theorem 1. *Triangle ABC is homothetic with the Triangle T_1 of Reflections of the Nine-Point Center in the Sidelines of the Medial triangle. The center of the homothety is the Circumcenter. The ratio of the homothety is $\frac{1}{2}$.*

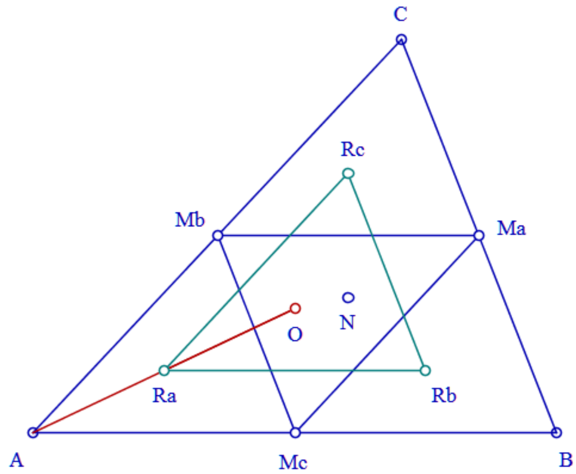

Figure 1.

Figure 1 illustrates Theorem 1. In figure 1,

- $MaMbMc$ is the Medial triangle,
- N is the Nine-Point Center,
- Ra is the reflection of point N in the line $MbMc$,
- Rb is the reflection of point N in the line $McMa$,
- Rc is the reflection of point N in the line $MaMb$,
- $RaRbRc$ is the triangle of reflections of point N in the sidelines of triangle $MaMbMc$,
- O is the Circumcenter.

Triangles ABC and $RaRbRc$ are homothetic and the center of the homothety is the Circumcenter.

Proof. We leave to the reader the proof that triangles ABC and $RaRbRc$ are homothetic with the Circumcenter as the center of the homothety. We will find the ratio of the homothety.

We use barycentric coordinates. The Medial triangle is the cevian triangle of the Centroid. By using formula (3) in [5] we find the barycentric equation of the line $MbMc$ as the line through points Mb and Mc as follows: $-x+y+z=0$. By using formula (8) in [5] we find the equation of the line L through the Nine-point center N and perpendicular to line $MbMc$, as follows:

$$L : (b^2-c^2)x+(c^2+2a^2-b^2)y+(c^2-2a^2-b^2)z=0.$$

By using formula (5) in [5] we find the intersection Q of the lines $MbMc$ and L as follows: $Q=(2,1,1)$.

By using formula (15) in [5] we find the reflection Ra of point N in point Q, as follows:

$$Ra = (b^4-3b^2a^2-2b^2c^2+c^4-3c^2a^2+2a^4, \\ -b^2(-b^2+c^2+a^2), -c^2(-c^2+a^2+b^2)).$$

By using the distance formula (9) in [5], we find the segments ORa and OA, and finally we obtain for the ratio:

$$k = \frac{ORa}{OA} = \frac{1}{2}.$$

This completes the proof. □

We see that the triangle T_1 in fact is the Euler triangle of the Circumcenter.

Theorem 2. *Triangle ABC is homothetic with the Triangle T_2 of Reflections of the Orthocenter in the Sidelines of the Orthic triangle. The center of the homothety is the point X(24). The ratio of the homothety is*

$$k = \frac{(b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2)}{2a^2b^2c^2}$$

If triangle ABC is acute, then $k > 0$, if it is obtuse, then $k < 0$. □

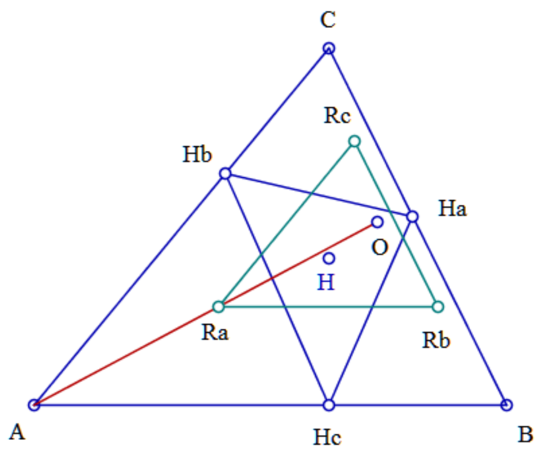

FIGURE 2.

Figure 2 illustrates Theorem 2. In figure 2,

- H is the Orthocenter,
- $HaHbHc$ is the Orthic triangle,
- Ra is the reflection of H in the line $HbHc$,
- Rb is the reflection of H in the line $HcHa$,
- Rc is the reflection of H in the line $HaHb$,
- $RaRbRc$ is the Triangle of Reflections of point H in the side lines of triangle $HaHbHc$, O is the point $X(24)$.

Triangles ABC and $RaRbRc$ are homothetic and the center of the homothety is the point X(24).

Theorem 3. *Triangle ABC is homothetic with the Triangle T_3 of Reflections of the Circumcenter in the Sidelines of the Tangential triangle. The center of the homothety is the Circumcenter. The retio of the homothety is* 2. □

Figure 3 illustrates Theorem 3. In figure 3,

- K is the Symmedian Point,
- $KaKbKv$ is the Tangential triangle,
- Ra is the Reflection of the Circumcenter in the side line $KbKc$,

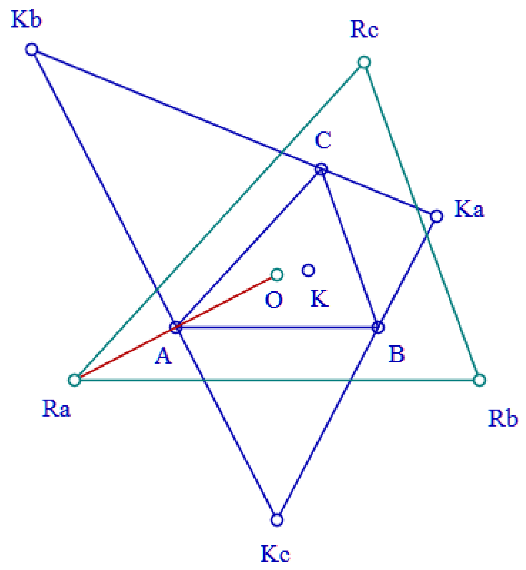

Figure 3.

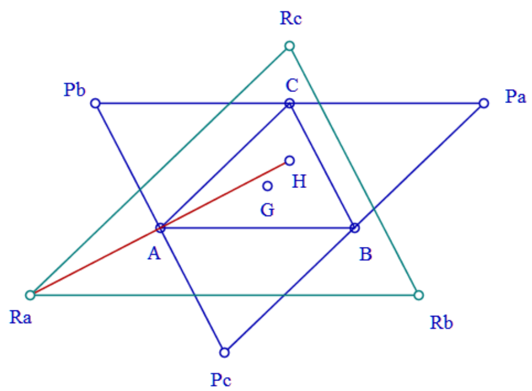

Figure 4.

- Rb is the Reflection of the Circumcenter in the side line $KcKa$,
- Rc is the Reflection of the Circumcenter in the side line $KaKb$,
- O is the Circumcenter.

Triangles ABC and $RaRbRc$ are homothetic and the center of the homothety is the Circumcenter.

Theorem 4. *Triangle ABC is homothetic with the Triangle T_4 of Reflections of the Orthocenter in the Sidelines of the Antimedial Triangle. The center of the homothety is the Orthocenter. The ratio is* 2.

Figure 4 illustrates Theorem 4. In figure 4,

- H is the Orthocenter,
- G is the Centroid,
- $PaPbPc$ is the Antimedial triangle,
- Ra is the Reflection of H in the side line $PbPc$,
- Rb is the Reflection of H in the side line $PcPa$,
- Rc is the Reflection of H in the side line $PaPb$,

- *RaRbRc* is the Triangle of Reflections of H in the side lines of the Antimedial triangle.

Triangles ABC and $RaRbRc$ are homothetic and the center of the homothety is the Orthocenter.

3. Barycentric Coordinates via Homothety

Now we are in position to find the barycentric coordinates of homothetic triangles. If triangles ABC and $RaRbRc$ are homothetic under the homothety $h(O,k)$ with center O and ratio k, then $Ra = h(A), Rb = h(B)$ and $Rc = h(C)$. We use the homothety formula (17) in [5].

Theorem 5. *The barycentric coordinates of the Triangle T_1 of the Reflections of the Nine-Point Center in the Sidelines of the Medial triangle are as follows:*

$$Ra = (3a^2b^2 + 3a^2c^2 - 2a^4 + 2b^2c^2 - b^4 - c^4, b^2(c^2 + a^2 - b^2), c^2(a^2 + b^2 - c^2)),$$
$$Rb = (a^2, (b^2 + c^2 - a^2), 3b^2c^2 + 3a^2b^2 - 2b^4 + 2a^2c^2 - a^4 - c^4, c^2(a^2 + b^2 - c^2)),$$
$$Rc = (a^2, (b^2 + c^2 - a^2), b^2, (c^2 + a^2 - b^2), 3a^2c^2 + 3b^2c^2 - 2c^4 + 2a^2b^2 - a^4 - b^4).$$

Note that the same barycentric coordinates are given in [6].

Problem 3.1. *Find the barycentric coordinates of triangles T_2 to T_4 in Theorems 2 to 4.*

Now we are also in position to find the barycentric coordinates of notable points of triangles homothetic with triangle ABC. We use the homothety formula (17) in [5].

Theorem 6. *The barycentric coordinates of the Centroid G_T of the Triangle T_1 of the Reflections of the Nine-Point Center in the Sidelines of the Medial triangle are as follows:*

$$uG_{T1} = 5a^2b^2 + 5a^2c^2 - 4a^4 + 2b^2c^2 - b^4 - c^4$$
$$vG_{T1} = 5b^2c^2 + 5a^2b^2 - 4b^4 + 2a^2c^2 - a^4 - c^4$$
$$wG_{T1} = 5a^2c^2 + 5b^2c^2 - 4c^4 + 2a^2b^2 - a^4 - b^4$$

Problem 3.2. *Find the barycentric coordinates of the following notable points of triangle T_1 in Theorems 1: Centroid, Incenter, Circumcenter, Orthocenter.*

Problem 3.3. *Find the barycentric coordinates of the following notable points of triangles T_2–T_4 in Theorems 2-4: Centroid, Incenter, Circumcenter, Orthocenter.*

4. Kimberling Points of Triangle T_1

We have investigated 195 notable points of triangle T_1. Of these 42 are Kimberling points and the rest of 153 points ane new points, not available in Kimberling [10]. Below is a part of the Kimberlin points. See also the Supplementary material.

Table 1 gives notable points of Triangle T_1 in terms of the notable points of the Reference Triangle ABC that are Kimberling points $X(n)$.

The "Disciverer" gives us the opportunity to add a number of new properties to the properties available in [10]. For example:

	Notable Points of triangle T_1	Notable Points of Triangle ABC
1	Incenter	X(1385)
2	Centroid	X(549)
3	Circumcenter	X(3)
4	Orthocenter	X(5)
5	Nine-Point Center	X(140)
6	Symmedian Point	X(182)

Table 1.

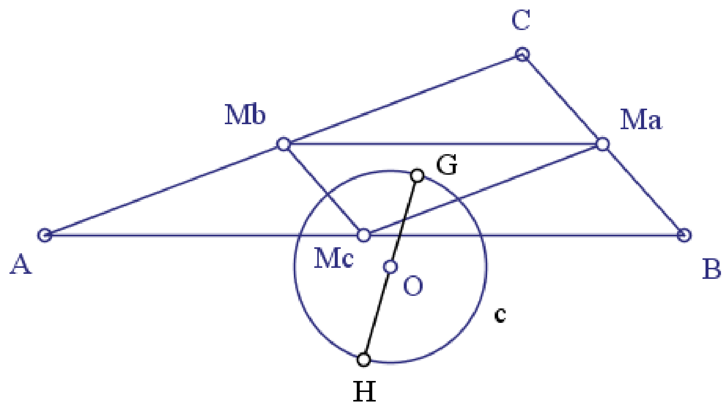

Figure 5.

Theorem 7. *The Centroid of the Triangle of Reflections of the Nine-Point Center in the Sidelines of the Medial Triangle (Point X(549 in* [10]*)) is the Center of the Orthocentroidal Circle of the Medial Triangle.*

Figure 5 illustrates Theorem 7. In figure 5

- $MaMbMc$ is the Medial triangle,
- G is the Centroid of the Medial triangle,
- H is the Orthocenter of the Medial triangle,
- c is the Orthocentroidal circle of the Medial triangle,
- O is the center of circle c, that is, the point X(549) = Centroid of triangle T_1.

5. New Points of Triangle T_1

We have found 153 new notable points of Triangle T_1. By using the homothety h_1 we can fing the barycentric coordinates of these points and by using the "Discoverer" we can find a number of properties of these points. For example:

Theorem 8. *The Gergonne Point of the Triangle T_1 is the Midpoint of the Circumcenter and the Gergonne Point.*

Problem 5.1. *Find the barycentric coordinates of the Gergonne Point of Triangle T_1.*

Supplementary Material

The enclosed supplementary material contains theorems related to the topic.

Acknowledgement

The authors are grateful to Professor René Grothmann for his wonderful computer program *C.a.R.* `http://car.rene-grothmann.de/doc_en/index.html`. See also `http://www.journal-1.eu/2016-1/Grothmann-CaR-pp.45-61.pdf`.

References

[1] César Lozada, Index of triangles referenced in ETC. http://faculty.evansville.edu/ck6/encyclopedia/IndexOfTrianglesReferencedInETC.html.

[2] Francisco Javier García Capitán, *Barycentric Coordinates*, International Journal of Computer Discovered Mathematics, 2015, vol.0, no.0, 32-48. http://www.journal-1.eu/2015/01/Francisco-Javier-Barycentric-Coordinates-pp.32-48.pdf.

[3] Pierre Douillet, *Translation of the Kimberling's Glossary into barycentrics*, 2012, v48, http://www.douillet.info/~douillet/triangle/Glossary.pdf.

[4] S. Grozdev and D. Dekov, *A Survey of Mathematics Discovered by Computers*, International Journal of Computer Discovered Mathematics, 2015, vol.0, no.0, 3-20. http://www.journal-1.eu/2015/01/Grozdev-Dekov-A-Survey-pp.3-20.pdf.

[5] S. Grozdev and D. Dekov, Barycentric Coordinates: Formula Sheet, International Journal of Computer Discovered Mathematics, vol.1, 2016, no 2, 75-82. http://www.journal-1.eu/2016-2/Grozdev-Dekov-Barycentric-Coordinates-pp.75-82.pdf.

[6] S. Grozdev and D. Dekov, Computer Discovered Mathematics: Euler Triangles, International Journal of Computer Discovered Mathematics, vol.1, 2016, no 1, 1-10. http://www.journal-1.eu/2016-1/Grozdev-Dekov-Euler-Triangles-pp.1-10.pdf.

[7] S. Grozdev, H.Okumura and D. Dekov, Triangles Homothetic with Triangle ABC. Part 1, International Journal of Computer Discovered Mathematics, vol.2, 2017, 86-89. http://www.journal-1.eu/2017/Grozdev-Okumura-Dekov-HT-pp.86-89.pdf.

[8] S. Grozdev and V. Nenkov, *Three Remarkable Points on the Medians of a Triangle* (Bulgarian), Sofia, Archimedes, 2012.

[9] S. Grozdev and V. Nenkov, *On the Orthocenter in the Plane and in the Space* (Bulgarian), Sofia, Archimedes, 2012.

[10] C. Kimberling, *Encyclopedia of Triangle Centers - ETC*, http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

[11] Gerry Leversha, *The Geometry of the Triangle*, The United Kingdom Mathematical Trust, The Pathways Series no.2, 2013.

[12] G. Paskalev and I. Tchobanov, *Remarkable Points in the Triangle* (in Bulgarian), Sofia, Narodna Prosveta, 1985.

[13] G. Paskalev, *With coordinates in Geometry* (in Bulgarian), Sofia, Modul-96, 2000.

[14] M. Schindler and K.Cheny, Barycentric Coordinates in Olympiad Geometry, 2012, http://www.mit.edu/~evanchen/handouts/bary/bary-full.pdf.

[15] E. W. Weisstein, *MathWorld - A Wolfram Web Resource*, http://mathworld.wolfram.com/.

[16] *Wikipedia*, https://en.wikipedia.org/wiki/.

[17] P. Yiu, *Introduction to the Geometry of the Triangle*, 2001, new version of 2013, http://math.fau.edu/Yiu/YIUIntroductionToTriangleGeometry130411.pdf.

[18] P. Yiu, *The uses of homogeneous barycentric coordinates in plane euclidean geometry*, Int. J. Math. Educ. Sci. Technol. 2000, vol.31, 569-578.