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Computer Discovered Mathematics: Excenters-Incenter Reflections Triangle

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Abstract. By using the computer program "Discoverer" we study the Excenters-Incenter Reflections triangle.

Keywords. triangle geometry, notable point, Euclidean geometry, computer discovered mathematics, "Discoverer".

Mathematics Subject Classification (2010). 51-04, 68T01, 68T99.

1. Introduction

Given a point P in the plane of triangle ABC. Let $JaJbJc$ be the Anticevian triangle of P. Define Ta as the reflection point of point Ja in point P, and define points Jb and Jc similarly. Then $T = TaTbTc$ is the Triangle of Reflections of the Anticevian triangle of point P in point P.

Triangle $TaTbTc$ is the homothetic image of triangle $JaJbJc$ under homothety with center point P and ratio $k = -1$.

If P = Incenter, then triangle T is the known Excenters-Incenter Reflections (EIR for short) triangle (see [1]).

If P = Centroid, then triangle T is the homothetic image of the Centroid under the homothety with Center the Centroid and ratio $k = 2$.

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We study the special case when P is the Incenter. We encourage the reader to investigate other special cases, e.g. when P is the Circumcenter, Orthocenter, Nine-point Center, Symmedian point.

Figure 1 illustrates the triangle T. In figure 1,

- P is an arbitrary point,
- $JaJbJc$ is the Anticevian triangle of point P,
- Ta is the reflection of Ja in P,
- Tb is the reflection of Jb in P,
- Tc is the reflection of Jc in P, and
- $T = TaTbTc$ is the Triangle of Reflections of the Anticevian triangle of point P in point P.

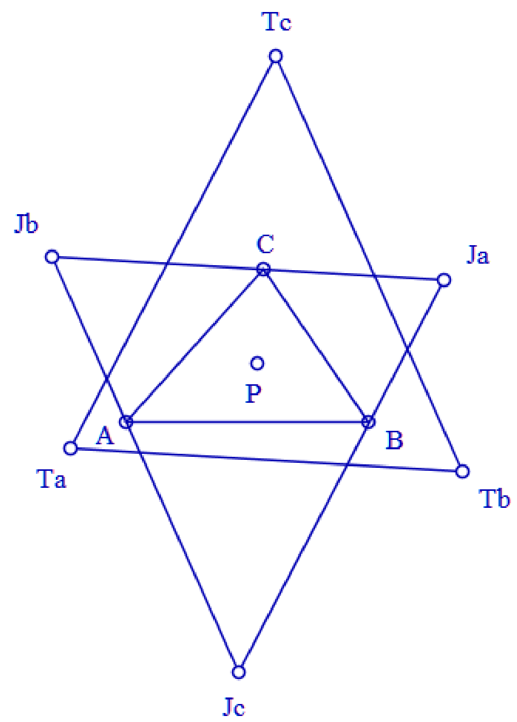

Figure 1.

We use the computer program "Discoverer", created by the authors.

We use barycentric coordinates. See [1] - [16].

The Kimberling points are denoted by $X(n)$. The side lengths of the Reference triangle ABC are $a = BC, b = CA, c = AB$. The area of triangle ABC is denoted by Δ.

2. General Case

In this section we denote by $T = TaTbTc$ the triangle of the Reflections of the Anticevian triangle of point P in point P.

Theorem 2.1. *The barycentric coordinates of triangle T are as follows:*

$$Ta = (u(u-3v-3w), v(3u-v-w), w(3u-v-w)), \tag{1}$$

$$Tb = (u(u-3v+w), v(3u-v+3w), w(u-3v+w)), \tag{2}$$

$$Tc = (u(u+v-3w), v(u+v-3w), w(3u+3v-w)). \tag{3}$$

Proof. We use the midpoint formula (14),[5] □

Theorem 2.2. *The area of triangle T is*

$$area = \frac{4uvw\Delta}{(v+w-u)(w+u-v)(u+v-w)}. \tag{4}$$

Proof. We use the area formula (2),[5]. □

Theorem 2.3. *The side lines of triangle T are as follows:*

$$a_T = \frac{2u\sqrt{a^2vw - b^2wv + b^2w^2 + c^2v^2 - c^2vw}}{(w+u-v)(u+v-w)}, \tag{5}$$

$$b_T = \frac{2v\sqrt{u^2c^2 + b^2wu - ua^2w - uc^2w + a^2w^2}}{(u+v-w)(v+w-u)}, \tag{6}$$

$$c_T = \frac{2w\sqrt{u^2b^2 + c^2uv - ua^2v - ub^2v + a^2v^2}}{(v+w-u)(w+u-v)}. \tag{7}$$

Proof. We use the distance formula (9), [5]. □

3. Special Case P = Incenter

3.1. **Barycentric Coordinates, Area, Side Lengths.** The barycentric coordinates of the EIR triangle are given in [1], excenters-incenter reflections.

Theorem 3.1. *The barycentric coordinates of the EIR triangle are as follows:*

$$Ta = (a(a-3b-3c), b(3a-b-c), c(3a-b-c)), \tag{8}$$

$$Tb = (a(a-3b+c), b(3a-b+3c), c(a-3b+c)), \tag{9}$$

$$Tc = (a(a+b-3c), b(a+b-3c), c(3a+3b-c)). \tag{10}$$

Proof. See [1]. Also, we can use Theorem 2.1 for the special case when P is the Incenter $I(a,b,c)$. □

The EIR triangle and the Excentral triangle are congruent, so that their areas and the side lengths are the same.

Theorem 3.2. *The area of the EIR triangle is*

$$area = \frac{4abc\Delta}{(b+c-a)(c+a-b)(a+b-c)}.$$

Proof. For the area of the Excentral triangle see ([14]),Excentral triangle. Also, we can use Theorem 2.2 for the special case when P is the Incenter $I(a,b,c)$. □

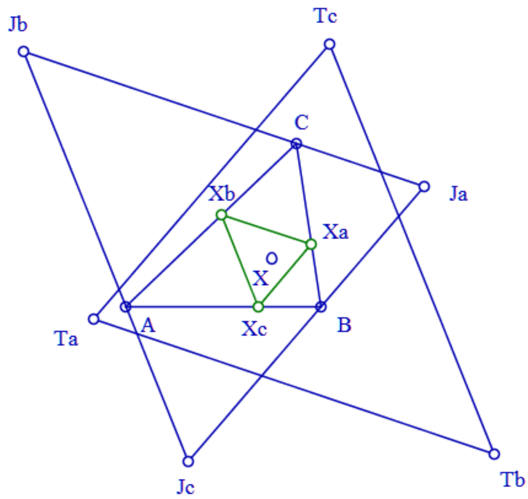

FIGURE 2.

Theorem 3.3. *The side lengths of the EIR triangle are as follows:*

$$a_T = \frac{2a\sqrt{bc}}{\sqrt{(c+a-b)(a+b-c)}}, \tag{11}$$

$$b_T = \frac{2b\sqrt{ca}}{\sqrt{(a+b-c)(b+c-a)}}, \tag{12}$$

$$c_T = \frac{2c\sqrt{ab}}{\sqrt{(b+c-a)(c+a-b)}}. \tag{13}$$

Proof. For the side lengths of the Excentral triangle see ([14]),Excentral triangle. Also, we can use Theorem 2.3 for the special case when P is the Incenter $I(a, b, c)$. □

3.2. **Homothetic Triangles.** The EIR triangle is homothetic with the Intouch triangle and the Center of the homothety is the point X(3340) (see [8]).

Theorem 3.4. *The ratio of homothety of the EIR triangle and the Intouch triangle is*

$$k = \frac{-(b+c-a)(c+a-b)(a+b-c)}{4abc}. \tag{14}$$

Proof. Denote by O the center of the homothety $X(3340)$. The barycentric coordinates of the EIR triangle $TaTbTc$ are given in Theorem 3.1. The barycentric coordinates of the Intouch triangle $XaXbXc$ are given in [14], Contact triangle. By using the distance formula (9), [5], we calculate the distances OTa and OXa. The ratio is negative, $k < 0$, so that we obtain

$$k = \frac{-OXa}{OTa} = \frac{-(b+c-a)(c+a-b)(a+b-c)}{4abc}. \tag{15}$$

□

Figure 2 illustrates Theorem 3.4. In figure 2,

- $JaJbJc$ is the Anticevian triangle of the Incenter, that is, the Excentral triangle,
- $TaTbTc$ is the Excenters-Incenter Reflections triangle,
- $XaXbXc$ is the Intiuch Triangle, and
- X is point $X(3340)$, the Center of the homothety.

3.3. **Kimberling Points.** We have investigated 195 notable points of the EIR triangle. Of these 20 are Kimberling points and the rest of 175 points are new points. See the Complementary material.

Table 1 gives the notable points of the EIR triangle T in terms of the notable points of the Reference Triangle ABC that are Kimberling centers $X(n)$.

	Notable Points of EIR triangle	**Notable Point of Triangle *ABC***
X(2)	Centroid	X(11224)
X(3)	Circumcenter	X(7982)
X(4)	Orthocenter	X(1)
X(5)	Nine-Point Center	X(1482)
X(6)	Symmedian Point	X(3243)
X(20)	de Longchamps Point	X(11531)
X(25)	Product of the Orthocenter and the Symmedian Point	X(7962)
X(33)	Product of the Orthocenter and the Mittenpunkt	X(11534)
X(40)	Bevan Point	X(11528)
X(51)	Centroid of the Orthic Triangle	X(3241)
X(52)	Orthocenter of the Orthic Triangle	X(944)
X(53)	Symmedian Point of the Orthic Triangle	X(3242)
X(64)	Isogonal Conjugate of the de Longchamps Point	X(3680)
X(84)	Perspector of Triangle ABC and the Hexyl Triangle	X(11527)
X(113)	Jerabek Antipode	X(10698)
X(114)	Kiepert Antipode	X(10697)
X(115)	Kiepert Center	X(10695)
X(125)	Center of the Jerabek Hyperbola	X(1320)
X(389)	Center of the Taylor Circle	X(3244)
X(647)	Perspector of the Jerabek Hyperbola	X(2516)

TABLE 1.

3.4. **Centers of Circles.** We have investigated the centers of 29 notable circles of the EIR triangle. Of these 7 are Kimberling points, and the rest of 22 centers are new points. See the Complementary material.

Table 2 gives notable circles of the EIR triangle whose centers are Kimberling points.

	Circle C of EIR triangle	**Center of Circle C**
1	Circumcircle	X(7982)
2	Nine-Point Circle	X(1482)
3	Excentral Circle	X(11528)
4	Antimedial Circle	X(1)
5	Second Brocard Circle	X(7982)
6	Cosine Circle	X(3243)
7	Taylor Circle	X(3244)

Table 2.

3.5. **Similitude Centers of Circles.** We have investigated 841 internal similitude centers (ISC) of the EIR triangle. Of these 24 are Kimberling points, and the rest of 817 centers are new points. See the Complementary material.

Table 3 gives notable ISC of circles of triangle ABC and circles of EIR triangle which are Kimberling points.

	Circle of triangle ABC	**Circle of EIR triangle**	ISC
1	Circumcircle	Circumcircle	X(1)
2	Circumcircle	Nine-Point Circle	X(1)
3	Circumcircle	Antimedial Circle	X(7987)
4	Circumcircle	Tangential Circle	X(3340)
5	Incircle	Circumcircle	X(3340)
6	Incircle	Incircle	X(11534)
7	Incircle	Nine-Point Circle	X(2099)
8	Incircle	Antimedial Circle	X(1)
9	Nine-Point Circle	Circumcircle	X(11522)
10	Nine-Point Circle	Nine-Point Circle	X(5603)
11	Nine-Point Circle	Antimedial Circle	X(7988)
12	Nine-Point Circle	Tangential Circle	X(3577)
13	Excentral Circle	Circumcircle	X(1)
14	Excentral Circle	Nine-Point Circle	X(1)
15	Excentral Circle	Antimedial Circle	X(165)
16	Antimedial Circle	Circumcircle	X(4301)
17	Antimedial Circle	Antimedial Circle	X(1699)
18	Adams Circle	Antimedial Circle	X(1)
19	Adams Circle	Lemoine Circle	X(11526)
20	Conway Circle	Antimedial Circle	X(1)
21	Conway Circle	Taylor Circle	X(3241)
22	Moses Circle	Antimedial Circle	X(9592)
23	Outer Johnson-Yff Circle	Circumcircle	X(5727)
24	Outer Johnson-Yff Circle	Nine-Point Circle	X(10950)

Table 3.

We have investigated 835 external similitude centers (ESC) of the EIR triangle. Of these 18 are Kimberling points, and the rest of 817 centers are new points. See the Complementary material.

Table 4 gives notable ESC of circles of triangle ABC and circles of EIR triangle which are Kimberling points.

	Circle of ABC triangle	**Circle of EIR triangle**	ESC
1	Circumcircle	Circumcircle	X(7991)
2	Circumcircle	Antimedial Circle	X(165)
3	Incircle	Circumcircle	X(7962)
4	Incircle	Nine-Point Circle	X(2098)
5	Nine-Point Circle	Circumcircle	X(3679)
6	Nine-Point Circle	Nine-Point Circle	X(8)
7	Nine-Point Circle	Antimedial Circle	X(7989)
8	Excentral Circle	Nine-Point Circle	X(11531)
9	Excentral Circle	Antimedial Circle	X(7991)
10	Antimedial Circle	Nine-Point Circle	X(145)
11	Antimedial Circle	Antimedial Circle	X(5691)
12	Antimedial Circle	Tangential Circle	X(11682)
13	Spieker Circle	Nine-Point Circle	X(5289)
14	Spieker Circle	Antimedial Circle	X(8580)
15	Conway Circle	Taylor Circle	X(145)
16	Moses Circle	Antimedial Circle	X(9593)
17	Inner Johnson-Yff Circle	Nine-Point Circle	X(10944)
18	Cosine Circle	Cosine Circle	X(1)

TABLE 4.

3.6. **Euler line.**

Theorem 3.5. *The Euler line of the EIR triangle is the Line 1-3, that is, the line through the Incenter and Circumcenter of triangle* ABC.

Figure 3 illustrates Theorem 3.5. In figure 3,

- $TaTbTc$ is the EIR triangle,
- I is the Incenter of triangle ABC,
- O is the Circumcenter of triangle ABC,
- Gx is the Centroid of triangle $TaTbTc$, and
- Ox is the Circumcenter of triangle $TaTbTc$.

Points Gx and Ox lie on Line 1-3, so that it is the Euler line of triangle $TaTbTc$.

SUPPLEMENTARY MATERIAL

The enclosed supplementary material contains theorems related to the topic.

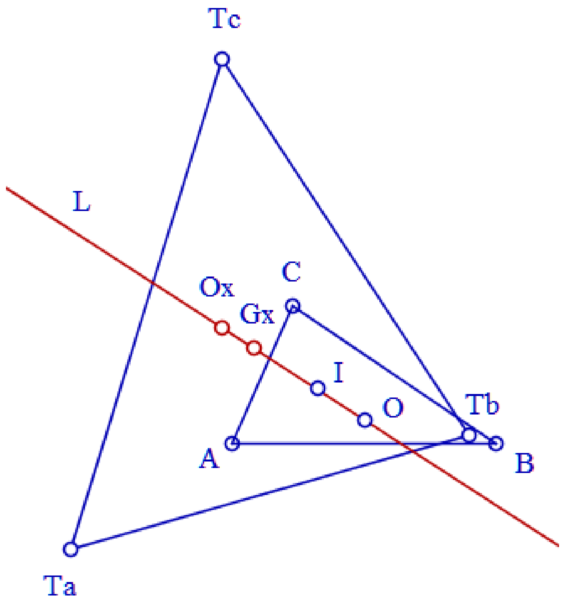

Figure 3.

Acknowledgement

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