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Computer Discovered Mathematics: Triangles Associated with Triangulation Triangles

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Abstract. By using the computer program "Discoverer" we study notable points of triangles associated with the Triangulation triangles of a triangle.

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Mathematics Subject Classification (2010). 51-04, 68T01, 68T99.

1. Introduction

Given triangle ABC with sides $a = BC, b = CA, c = AB$. We denote the following notable points of triangle ABC:

- Incenter = I = X(1),
- Centroid = G = X(2),
- Circumcenter = O = X(3),
- Orthocenter = H = X(4),
- Nine-Point Centers = N = X(5).

Let X be a point not on the sidelines of triangle ABC. Consider the three triangles XBC, XCA and XAB formed by joining X to the vertices (see figure 1).

We say that X is the triangulation point of triangle ABC and that XBC, XCA and XAB are the *triangulation* triangles of X ([14]; *partition* triangles in [10]).

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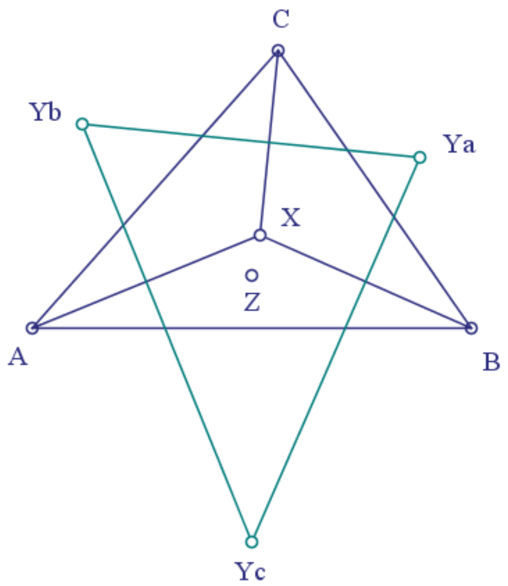

Figure 1.

The triangle composed on the centers Y of these triangles (denoted Ya, Yb and Yc, respectively) will be denoted by T(X,Y). We call this triangle the triangle *associated with the triangulation triangles of X*. For example, the vertices of triangle T(I,O) are the circumcenters of triangles IBC, ICA and IAB. Sometimes we denote T(X(i),X(j) as Tij, for short.

Gerry Leversha [10] has extensively studied the notable points of the triangles associated with the triangulation triangles.

Theorem A1. ([10], Theorem 24.9). The Centroid of triangle T(G,O) is the Circumcenter of triangle ABC.

Theorem A2. ([10], Theorem 24.10). The Centroid of triangle T(G,H) is the Orthocenter of triangle ABC.

Theorem A3. ([10], Theorem 2411). The Centroid of triangle T(G,N) is the Nine-Point Center of triangle ABC.

Theorem A4. ([10], Theorem 24.12). The Orthocenter and Circumcenter of triangle T(I,N) are the Incenter and Nine-Point Center of triangle ABC.

Gerry Leversha [10] has used methods of Euclidean geometry to prove the above theorems. In this paper we use the barycentric coordinates. For the barycentric coordinates see [1] - [17]. We use the computer program "Discoverer" created by the authors. The "Discoverer" has studied approximately 4400 notable points of triangles T(X,Y). The results include the above Leversha theorems. See the supplementary material.

2. Barycentric coordinates of triangles T(X,Y)

Theorem 2.1. *Let X =(u,v,w), and let $Pa = (p_1, q_1, r_1)$, $Pb = (p_2, q_2, r_2)$ and $Pc = (p_3, q_3, r_3)$ are the barycentric coordinates of point Y wrt triangles XBC, XCA and XAB respectively. Then the barycentric coordinates of triangle T(X,Y)* *are as follows:*

$$\begin{aligned} Ya &= (up_1, vp_1 + q_1u + q_1v + q_1w, wp_1 + r_1u + r_1v + r_1w), \\ Yb &= up_2 + r_2u + r_2v + r_2w, vp_2, wp_2 + q_2u + q_2v + q_2w(), \\ Yc &= up_3 + q_3u + q_3v + q_3w, vp_3 + r_3u + r_3v + r_3w, wp_3(). \end{aligned}$$

Proof. We use the change of coordinates formula (10) in [6]. □

We use the same approach to find the barycentric coordinates of a point Z if the barycentric coordinates of Z are given wrt triangle $YaYbYc$.

Lemma 2.1. *Let $X = (u, v, w)$. Then the lengths of segments XA, XB, XC are as follows:*

$$\begin{aligned} XA &= \frac{\sqrt{-a^2vw + b^2wv + b^2w^2 + c^2v^2 + c^2vw}}{u+v+w}, \\ XB &= \frac{\sqrt{a^2wu + a^2w^2 - b^2wu + c^2u^2 + c^2uw}}{u+v+w}, \\ XC &= \frac{\sqrt{a^2vu + a^2v^2 + b^2u^2 + b^2uv - c^2uv}}{u+v+w}. \end{aligned}$$

Proof. We use the distance formula (9) in [6]). □

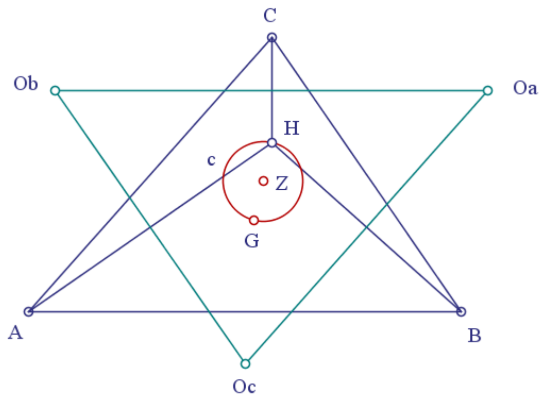

FIGURE 2.

Note that for the case T(X,G) where G is the Centroid of ABC, a nice theorem is due to Leversha [10], Theorem 24.1.

We illustrate Theorem 2.1 by the special case of triangle T(H,O) = T43 (The Johnson triangle [14], [5]). By using the barycentric coordinates of the Orthocenter and Lemma 2.1, we find the side lengths of triangle HBC as follows:

$$\begin{aligned} a_{HBC} = BC &= a, \\ b_{HBC} = CH &= \frac{c(a^2+b^2-c^2)}{\sqrt{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}}, \\ c_{HBC} = HB &= \frac{b(c^2+a^2-b^2)}{\sqrt{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}}. \end{aligned}$$

Then we use formula (10) in [6] and we obtain the barycentric coordinates of the Circumcenter Oa of triangle XBC as follows:

$$Oa = (a^2(b^2+c^2-a^2), a^4 - 2c^2a^2 - a^2b^2 + c^4 - c^2b^2, a^4 - c^2a^2 - 2a^2b^2 + b^4 - c^2b^2).$$

Similarly, we obtain the barycentric coordinates of points Ob and Oc.

Now we can obtain the side lengths of triangle $OaObOc$ by using the distance formula (9), [6]. (in this example the side lengths of triangle $OaObOc$ are a,b,c). By using the side lengths of $OaObOc$, we find the barycentric coordinates of a point Z wrt this triangle. Then by using the change of coordinates formula (10), [6], we obtain the barycentric coordinates of Z wrt triangle ABC.

For example, if Z is the Symmedian point of triangle $OaObOc$, then the first barycentric coordinate of Z wrt triangle ABC is as follows:

$$a^4b^2 + c^2a^4 - a^6 - b^4a^2 - 2c^2a^2b^2 - c^4b^2 - c^2b^4 + b^6 - c^4a^2 + c^6.$$

This is the reflection of the Symmedian Point in Nine-Point Center, point X(1352) in [9].

3. Kimberling Points

We have investigated notable points of triangles Tij = T(X(i),X(j)), i,j=1, ...5. Triangle T45 is degenerate and triangle T44 is triangle ABC (see Leversha [10], Chap. 24), so that in fact we have investigated notable points of 23 triangles.

We have investigated 195 notable points of each triangle. The results are summarized in the Supplementary material.

We have found a number of Kimberling points. For example, in triangle T43, of 195 investigated notable points, 65 are Kimberling points, and the rest of 130 points are new points, not available in Kimberling [9]. In Table 1 we list a few of the Kimberling points of triangle T43.

	Notable Points of triangle T43	**Notable Points of triangle ABC**
1	Incenter	X(355)
2	Centroid	X(381)
3	Circumcenter	X(4)
4	Orthocenter	X(3)
5	Nine-Point Center	X(5)
6	Symmedian Point	X(1352)
7	Gergonne Point	X(5779)
8	Nagel Point	X(1482)
9	Mittenpunkt	X(5805)
10	Spieker Center	X(946)
11	Feuerbach Point	X(119)

TABLE 1.

We encourage the reader to investigate the new points, available in the Supplementary material.

Supplementary Material

The enclosed supplementary material contains theorems related to the topic.

Acknowledgement

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