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Leversha Triangles and Leversha Points

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Abstract. By using the computer program "Discoverer" we study the Leversha triangles and Leversha points.

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1. Introduction

Let X be a point not on the sidelines of triangle ABC. The three triangles XBC, XCA and XAB are the *triangulation* triangles [13] (*partition* triangles in [9]) of X. Denote by $GaGbGc$ the triangle whose vertices are the centroids of triangles XBC, XCA and XAB respectively. See Figure 1.

Gerry Leversha has published the following theorem:

Theorem A. ([9], Theorem 24.1) Triangles ABC and $GaGbGc$ are homothetic. The center of homothety W is the point dividing segment XG in ratio 3:1, and the scale factor is $-\frac{1}{3}$.

We call triangle $GaGbGc$ *Leversha triangle of* X and the center of homothety W *Leversha point of* X.

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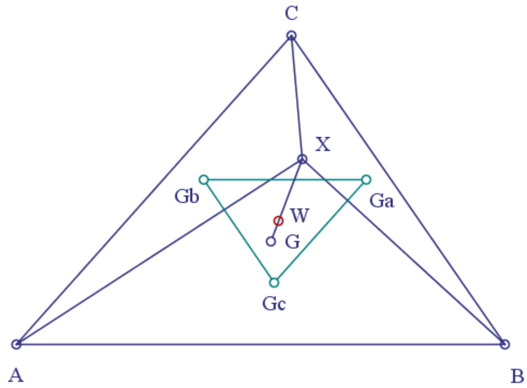

FIGURE 1.

In this paper we investigate the Leversha triangles and Leversha points. We have studied 194 Leversha points. Of these 95 are Kimberling points and the rest of 99 points are new points, not available in Kimberling [8].

We use barycentric coordinates. See [1] - [16]. The side lengths of triangle ABC are denoted $a = BC, b = CA$ and $c = AB$. The Kimberling points are denoted X(n)

2. BARYCENTRIC COORDINATES

Given point $X = (u, v, w)$ in barycentric coordinates.

Theorem 2.1. *The Leversha triangle of X has barycentric coordinates*

$$\begin{aligned} Ga &= (u, 2v + u + w, 2w + u + v) \\ Gb &= (2u + v + w, v, 2w + u + v) \\ Gc &= (2u + v + w, 2v + u + w, w) \end{aligned}$$

Proof. We use barycentric coordinates of the Centroid and change of coordinates formula (10) in [5]. □

Theorem 2.2. *The Leversha point of point X has barycentric coordinates*

$$W = (2u + v + w, u + 2v + w, u + v + 2w).$$

Proof. We use theorem 24.1 in [9] and the division of segment formula (12) in [5]. □

Theorem 2.3. *The Leversha point of point X is the complement of complement of point X.*

Proof. We use Theorem 24.1, [9], and the definition of complement. □

Theorem 2.4. *Given point Z in barycentric coordinates (p, q, r) wrt triangle $GaGbGc$. Then the barycentric coordinates (p_1, q_1, r_1) of Z wrt triangle ABC are as follows:*

$$\begin{aligned} p_1 &= up + 2qu + qv + qw + 2ru + rv + rw, \\ q_1 &= 2pv + up + pw + qv + 2rv + ru + rw, \\ r_1 &= 2pw + up + pv + 2qw + qu + qv + rw. \end{aligned}$$

Proof. We use barycentric coordinates of triangle *GaGbGc* and change of coordinates formula (10) in [5]. □

3. Leversha Points

We have investigated 194 Leversha points. Of these 95 are Kimberling points and the rest of 99 points are new points, not available in Kimberling [8]. See the Supporting material,

Table 1 gives a few of the Leversha points of X which are Kimberling points. The reader can see the complete list in the Supplementary material.

	Leversha Point of X	**Point of Triangle ABC**
1	X(1) Incenter	X(1125)
2	X(2) Centroid	X(2)
3	X(3) Circumcenter	X(140)
4	X(4) Orthocenter	X(5)
5	X(5) Nine-Point Center	X(3628)
6	X(6) Symmedian Point	X(3589)
7	X(7) Gergonne Point	X(142)
8	X(8) Nagel Point	X(10)
9	X(9) Mittenpunkt	X(6666)
10	X(10) Spieker Center	X(3634)
11	X(11) Feuerbach Point	X(6667)
12	X(12) Feuerbach Perspector	X(6668)
13	X(13) Outer Fermat Point	X(6669)
14	X(14) Inner Fermat Point	X(6670)
15	X(15) First Isodynamic Point	X(6671)
16	X(16) Second Isodynamic Point	X(6672)
17	X(17) Outer Napoleon Point	X(6673)
18	X(18) Inner Napoleon Point	X(6674)

Table 1.

The above table is an addition to the results available in Kimberling [8]. See also the Supplementary material.

By using Theorem 2.2 we can easily find the barycentric coordinates of the new Leversha points. For example.

Theorem 3.1. *The first barycentric coordinate of Leversha point of the Clawson point X(19) is*

$$2a^5 - 2ab^4 + 4ab^2c^2 - 2ac^4 + b^5 - bc^4 + 2bc^2a^2 - ba^4 + c^5 - ca^4 + 2ca^2b^2 - cb^4.$$

By using the “Discoverer” we easily find properties of the new point:

Theorem 3.2. *The Leversha Point of the Clawson Point is the Clawson Point of the Half-Median Triangle.*

Theorem 3.3. *The Leversha Point of the Clawson Point is the Homothetic Center of the Medial Triangle and the Triangle of the Circumcenters of the Pedal Corner Triangles of the Clawson Point.*

Theorem 3.4. *The Leversha Point of the Clawson Point lies on the Nine-Point Circle of the Triangle of the Nine-Point Centers of the Triangulation Triangles of the Clawson Point*

We encourage the reader to find the barycentric coordinates of the new points available in the Supplementary material.

4. LEVERSHA TRIANGLES

We denote by T(2,2) the Leversha triangle of the Centroid. Triangle ABC is homothetic from triangle T(2,2) with center of homothety the Centroid and scale factor -$\frac{1}{3}$. See Figure 2.

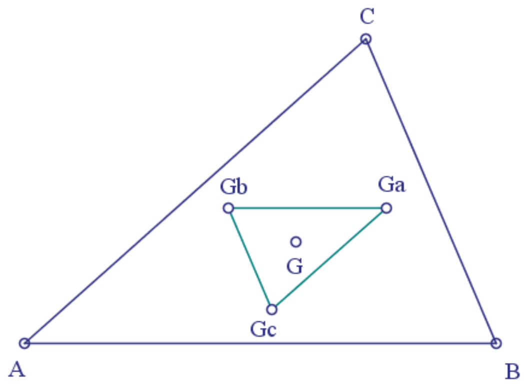

FIGURE 2.

Consider triangle T(2,2) $= GaGbGc$ as an example of Leversha triangle.

Theorem 4.1. *The barycentric coordinates of triangle T(2,2) are as follows:*

$$Ga = (1, 4, 4), \quad Gb = (4, 1, 4), \quad Gc = (4, 4, 1).$$

The above theorem is a special case of Theorem 2.1.

The theorem means that if Δ_1, Δ_2 and Δ_3 are the area of triangles $GaBC, GbCA$ and $GcAB$ respectively, then $\Delta_2 = \Delta_3 = 4\Delta_1$. Denote by Δ the area of triangle ABC. Then the area of triangle $GaGbGc$ is $\dfrac{\Delta}{9}$.

Theorem 4.2. *Given point Z in barycentric coordinates (p, q, r) wrt triangle T(2,2). Then the barycentric coordinates of Z wrt triangle ABC are as follows:*

$$(p + 4q + 4r, 4p + q + 4r, 4p + 4q + r)$$

The above theorem is a special case of Theorem 2.4.

By using Theorem 4.2 we can easily calculate the barycentric coordinates wrt triangle ABC of notable points of triangle T(2,2). For example, the Centroid of triangle T(2,2) is the Centroid of triangle ABC.

Another example. If $a_1 = GbGc, b_1 = GcGa$ and $c_1 = GaGb$ are the side lengths of triangle T(2,2), then the first barycentric coordinate of the Circumcenter of triangle T(2,2) is $a_1^2(b_1^2+c_1^2-a_1^2)$. Then the first barycentric coordinate wrt triangle ABC of the same point is

$$5a^2b^2+5a^2c^2+8b^2c^2-a^4-4b^4-4c^4,$$

that is, this is point X(5055) of triangle ABC.

We have studied 195 notable points of triangle $T(2,2)$. Of these 10 points are Kimberling points and the rest of 185 points are new points, not available in Kimberling [8]. See the Supplementary material.

Table 2 lists 10 notable points of triangle $GaGbGc$ which are Kimberling points, as well as their Kimberling numbers.

	Point of triangle T(2,2)	**Point of Triangle ABC**
1	X(2) Centroid	X(2)
2	X(3) Circumcenter	X(5055)
3	X(4) Orthocenter	X(3524)
4	X(5) Nine-Point Center	X(11539)
5	X(20) de Longchamps Point	X(3839)
6	X(51) Centroid of Orthic Triangle	X(5650)
7	X(99) Steiner Point	X(9166)
8	X(115) Kiepert Center	X(9167)
9	X(355) Center of Fuhrmann Circle	X(3653)
10	X(381) Center of Orthocentroidal Circle	X(5054)

Table 2.

Table 2 is an addition to results in [8].

By the same way we can calculate the barycentric coordinates of the new points of triangle T(2,2). For example, the first barycentric coordinate wrt triangle ABC of the Symmedian point of triangle T(2,2) is $a^2+4b^2+4c^2$. We encourage the reader to find the barycentric coordinates of the new points of triangle T(2,2) available in the Supplementary material.

Supplementary Material

The enclosed supplementary material contains theorems related to the topic.

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