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Ways of predicting mathematics

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Abstract. By using new terminology of 'non-formal logic', we give new theorems from plane geometry.

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1. Introduction

Here we introduce category of statements which are almost everywhere true. We will use very basic concepts from category theory, see them here [7]. All theorems are discovered by Author and can be checked by computer.

2. Interpretation of logic

For any statements A, B we can draw arrow $A \to B$ if statement B follows from statement A. So we can interpret logic as category with such diagrams, where objects correspond to statements and arrows to "followness" of one statement from another. Name this category as **Log**. Our goal here is to introduce bigger category **NFLog** ('non-formal logic') and show how to use it to predict new facts in mathematics.

3. Non-formal logic

Here we will introduce class of "non-formal statements" which can be seen as statements which we can't prove in their full generality but "intuitively" they seems to be true.

Such kind of statements can be easily defined from next examples :

$(\mathbb{T}_1)$: If we have some sets A and B with property that "size" of B is no more than "size" of A and given some correspondence $f : A \to B$, then $B = f(A)$ in most cases of natural considerations of (A, B, f)

$(\mathbb{T}_2)$: To every statement in some sense there exists dual statement

So if we consider logic with such new class of statements, then we will get new category **NFLog** and we can see that category **Log** is it's subcategory. Throughout the rest of this paper "**NF** statement" or simply **NF** will mean element from category **NFLog**. We will use intuition that under any statement A there exists some **NF** statement $\widetilde{A}$ from which A follows.

$$\begin{array}{c}\widetilde{A}\\ \downarrow\\ A\end{array}$$

Next we will show how to produce more non-formal statements from "formal" ones and how to get formal interpretations of them.

4. First examples of using NF's

Here we will consider some natural examples from Euclidean plane geometry.

Consider statement $\mathbb{K}$: Let given two lines l_1, l_2 such that $\angle(l_1, l_2) = \pi/k$, consider two points $P_0 \in l_1, Q_0 \in l_2$. Construct point $Q_0 \neq Q_1 \in l_2$, such that $|P_0Q_1| = |P_0Q_0|$, then we can construct point $P_0 \neq P_1 \in l_1$, such that $|Q_1P_0| = |Q_1P_1|$. So from this constructions we see that from point P_0 on l_1 we can construct point P_1 on l_1, like the same if we start from P_1 we can construct P_2 and so on. Then statement says that $P_k = P_0$.

To find **NF** statement which corresponds to $\mathbb{K}$, we can easily see that non-formally statement $\mathbb{K}$ says that if we have something periodical which depends on angle value $2\pi/k$ for some natural k, then size of period is k. We can say that this is **NF** statement $\mathbb{T}_\angle$. In other words we can say that $\mathbb{T}_\angle$:

$$\boxed{\text{angle } 2\pi/k \rightsquigarrow \text{periodic with period } k}$$

So we can see that $\mathbb{K}$ is particular case of $\mathbb{T}_\angle$. Now we will show how to find another assertions of $\mathbb{T}_\angle$. Consider Poncelet's porism, it deals with cyclic polygons with period n, as we know that for $n = 4$ two diagonals of inner circle are orthogonal. So it is natural (from $\mathbb{T}_\angle$) to predict that for another angle $2\pi/k$ we get some periodical construction with period k. So from some computer experiment we get that this predictions are particularly true, see next statement $\widetilde{\mathbb{K}}$: Let given circle ω and point P, consider k lines $l_1, \ldots, l_k$, which goes through point P and such that $\angle(l_i, l_{i+1}) = 2\pi/k, i = 1, \ldots, k$. Then intersections of this lines with circle ω forms cyclic polygon $A_1, A_2, \ldots, A_{2k}$, consider intersection points of tangents to ω through points A_i, A_{i+1} and get points L_i. Then statement says that for $k = 3, 4, 6$ there exists some conic $\mathcal{K}(\omega, P, k)$, which depends on circle ω, point P and number k and doesn't depend on lines l_i, such that for every i, $A_i \in \mathcal{K}$.

Another case of $\mathbb{T}_\angle$: Consider two circles Ω_1, Ω_2 on plane and their two external tangent lines l_1, l_2. For any point $P_1 \in l_1$ we can construct unique point $Q_1 \in l_2$, such that line P_1Q_1 is tangent to Ω_1. Also we can construct point $P_2 \in l_1$, such that line Q_1P_2 is tangent to Ω_2. So from this constructions we see that from point P_1 on l_1 we can construct point P_2 on l_1, like the same if we start from P_2 we can construct P_3 and so on. Prove that if angle $\angle(\Omega_1, \Omega_2) = \pi/k$, then $P_{k+1} = P_1$.

And we finish this section with next precise assertion of $\mathbb{T}_\angle$: Consider rectangular hyperbola $\mathcal{H}$ with center at O. Let $\mathcal{H}'$ be rotation of hyperbola $\mathcal{H}$ wrt point O on angle π/k. From [8, Problem 11.4.3] we know that angle between hyperbolas $\mathcal{H}$ and $\mathcal{H}'$ is equivalent to $2\pi/k$. Consider two intersections P, Q of these conics, let $P_1 \in \mathcal{H}$ be given. Consider points $Q_1 = \mathcal{H}' \cap P_1P$, $P_2 = \mathcal{H} \cap Q_1Q$. Like the same if we start from point P_2 we can uniquely define next point P_3 and so on. Prove that $P_{k+1} = P_k$.

5. Next examples of using NF's

Here we will show precise assertions of statement $\mathbb{T}_1$ (see section 2).

Consider next statement $\mathbb{K}_0$: For any three points X, Y, Z on plane there exists some complex triangle ABC, such that X, Y, Z are its in-center, circumcenter and orthocenter respectively. Note that this statement follows from $\mathbb{T}_1$, because if we denote set of triangles as A, set of different triples of points on plane as B and correspondence $f : A \to B$ which sends every triangle to its in-center, circumcenter and orthocenter, then we know that A can be seen as algebraic variety of dimension 6 and B is algebraic variety with same dimension (dimension can be seen as "size") and A, B have same sizes, so from $\mathbb{T}_1$ we get that $f(A) = B$ (it is not very clear, but in most particular cases of (A, B, f) it is true). Like the same we can produce another precise assertions of $\mathbb{T}_1$, for example:

$(\mathbb{K}_1)$: For most pairwise different numbers i, j, k and any three points X, Y, Z on plane there exists complex triangle ABC, such that $X = K_i(ABC), Y = K_j(ABC), Z = K_k(ABC)$, where K_p is p -th Kimberling center of triangle ABC, see definitions and properties of Kimberling centers here [5]

$(\mathbb{K}_2)$: For any three pairs of points X, X', Y, Y', Z, Z' on plane there exists some complex triangle ABC, such that X' is isogonal conjugated to X, Y' is isogonal conjugated to Y and Z' is isogonal conjugated to Z.

Definition 5.1. *For points any points A, B, C, D, denote point $\mathcal{M}(AB, CD)$ as Miquel point of lines AC, DA, CB, DB.*

$(\mathbb{K}_3)$: Let given three segments AA', BB', CC'. Prove that next conditions are equivalent:

a) midpoints of segments AA', BB', CC' lie on same line.

b) points $\mathcal{M}(AA', BB')$, $\mathcal{M}(BB', CC')$, $\mathcal{M}(AA', CC')$ lie on same line.

Proof of statement $\mathbb{K}_3$: From statement $\mathbb{K}_2$ we get that for some triangle XYZ on complex plane we have that pairs of points A, A'; B, B' and C, C' are isogonal wrt triangle XYZ. Name midpoints of segments AA', BB', CC' as M_A, M_B, M_C.

b) $\Rightarrow$ a). Let given that points $\mathcal{M}(AA', BB')$, $\mathcal{M}(BB', CC')$, $\mathcal{M}(AA', CC')$ lie on same line l. From [2, lemma 1] we get that circle

$$(\mathcal{M}(AA', BB')\mathcal{M}(AA', CC')\mathcal{M}(BB', CC'))$$

is equivalent to line l and is circumcircle of triangle XYZ, so one of the point X, Y or Z should be infinite, let it be X. So from isogonality of pairs of points A, A'; B, B' and C, C' easy to see that then midpoints M_A, M_B, M_C lie on same line which is equal distant from lines XY, XZ.

a) $\Rightarrow$ b). Let given that M_A, M_B, M_C lie on same line l. Consider point X_∞ — infinite point on line l. Then if we construct reflection of line $X_\infty Y$ wrt l and intersect it with reflection of line $X_\infty Y$ wrt angle bisector of $\angle AYA'$, then we get intersection point Z^*. And from isogonal conjugation theorem we get that pairs of points A, A'; B, B' and C, C' are isogonal wrt triangle $X_\infty YZ^*$. Circumcircle of this triangle is equivalent to line YZ^*, because point X_∞ is infinite point. So from [2, lemma 1] we get that circle

$$(\mathcal{M}(AA', BB')\mathcal{M}(AA', CC')\mathcal{M}(BB', CC'))$$

is equivalent to line YZ^*. $\square$

6. Examples related to orthocenter construction

First consider next construction $(ABCH)$: Triangle ABC with orthocenter H and heights AH_A, BH_B, CH_C.

Consider next **NF** statement $\mathbb{T}_{\text{orthocenter}}$: Most of facts with construction $(ABCH)$ also true if we rename H_B, H_C as $K_B \in BC$, $K_C \in AB$, where BCK_BK_C is cyclic and H can be replaced by $\widetilde{H} = BK_B \cap CK_C$.

Consider next theorem (discovered by Author) : Let given triangle ABC. Let triangle A', B', C' is midpoint triangle of orthic triangle of ABC, let triangle $A_1B_1C_1$ formed by midpoints of altitudes of ABC. Prove that tangent points A_2, B_2, C_2 of in-circle of triangle $A'B'C'$ with it's sides, lie on sides B_1C_1, A_1C_1, A_1B_1 respectively, and that lines A_1A_2, B_1B_2, C_1C_2 intersects at center of $(A_2B_2C_2)$.

So if we use **NF** statement $\mathbb{T}_{\text{orthocenter}}$, then we get next statement $\mathbb{P}$: Let given triangle ABC and points $K_B \in AC, K_C \in AB$, such that points B, C, K_B, K_C lie on same circle. Let circles $(ABK_B), (ACK_C)$ intersects at point L_A. Let triangle $A'B'C'$ is midpoint triangle of triangle $L_AK_BK_C$. Let triangle $A_1B_1C_1$ formed by midpoints of segments AL_A, BK_B, CK_C. Prove that tangent points B_2, C_2 of in-circle of triangle $A'B'C'$ with it's sides $A'C', A'B'$ lie on sides A_1C_1, A_1B_1 of triangle $A_1B_1C_1$. Also prove that lines B_1B_2, C_1C_2 intersects at point I, such that $|IB_1| = |IC_1|$.

For another example consider statement [8, Problem 4.2.6], so if we use $\mathbb{T}_{\text{orthocenter}}$ then we can get next statement : Let given triangle ABC and two points $K_B \in AC, K_C \in AB$, such that points B, C, K_B, K_C lie on same circle. Let O - circumcenter of triangle ABC. Let circles $(ABK_B), (ACK_C)$ intersects at point L_A. Consider point $H = BK_B \cap CK_C$. Let reflections of lines HB, HC wrt line HA intersects with lines AC, AB at points X_B, X_C respectively. Then we get that line X_BX_C is orthogonal to line OK_C.

7. Example related to conics

From [8, Problem 11.1.19] we can naturally define next **NF** : In most statements we can replace some segment by conic.

So if we use this **NF** to statement $\mathbb{K}_3$ from section 4 then we get next statement : Consider any three conics α, β, γ. For conics α, β we can consider all their 4 tangents and name Miquel point of these 4 lines as $M_{\alpha,\beta}$, like the same define other points $M_{\beta,\gamma}, M_{\gamma,\alpha}$. Then next conditions are equivalent :

a) Centers of conics α, β, γ lie on same line

b) Points $M_{\alpha,\beta}, M_{\beta,\gamma}, M_{\gamma,\alpha}$ lie on same line

For another example consider next statement : Let given segments $\{X_iY_i\}_{i=1}^4$ and segments $\{A_jB_j\}_{j=1}^4$, such that for every $(i,j) \neq (1,1)$ we have that points X_i, Y_i, A_j, B_j lie on circle. Then we can prove that points X_1, Y_1, A_1, B_1 lie on circle.

So if we use **NF** to this construction then we get next statement : Let given conics $\{\alpha_i\}_{i=1}^4$ and another set of conics $\{\beta_j\}_{j=1}^4$, such that for every $(i,j) \neq (1,1)$ we have that there exists some circle which is tangent to each of conics α_i, β_j at two points. Then we can prove that there exists some circle which is tangent to each of conics α_1, β_1 at two points.

8. Some more non-formal facts

Here we define some more **NF**'s which "naturally" can be related to triangle geometry.

Main **NF** : If in particular case of equilateral triangle some points or circles of this triangle are equivalent, then in general case they are connected. So in fact this **NF** says that the triangle geometry can be interpreted as deformation of equilateral triangle geometry.

More precise here we will use next two statements $\mathbf{NF}_{\triangle}^{pt}$: If in particular case of equilateral triangle some points are equivalent, then in general case circle which goes through this points has many "good" properties with respect to base triangle.

$\mathbf{NF}_{\triangle}^{\omega}$: If in particular case of equilateral triangle some circles are equivalent, then in general case radical line of this circles has many "good" properties with respect to base triangle.

Now we present some examples of facts which we can get if we use this two principles. Rename Ex^{pt} if this example is related to $\mathbf{NF}_{\triangle}^{pt}$ and Ex^{ω} if it is related to $\mathbf{NF}_{\triangle}^{\omega}$.

Ex 1^{pt}: Let given triangle ABC with in-center I. Let N_a, N_b, N_c be nine point centers of triangles IBC, IAC, IAB. Prove that

(a) Circumcenter of triangle $N_aN_bN_c$ is same with nine point center of ABC

(b) Point I is orthocenter of triangle $N_aN_bN_c$

(c) Reflections of midpoints of arcs AB, BC, CA of circle (ABC) wrt lines AB, BC, CA lie on lines IN_c, IN_a, IN_b.

(d) If AH_a, BH_b, CH_c be altitudes of triangle ABC. Reflect H_a, H_b, H_c wrt sides of triangle I_a, I_b, I_c formed by midpoints of segments IA, IB, IC and get triangle

XYZ. Prove that circle (XYZ) goes through point I and that center of this circle lie on circle $(N_aN_bN_c)$.

(e) If M_a, M_b, M_c be midpoints of BC, CA, AB. Reflect M_a, M_b, M_c wrt sides of triangle formed by midpoints of segments IA, IB, IC and get triangle $X'Y'Z'$. Prove that circle $(X'Y'Z')$ goes through I and that radius's of circles $(N_aN_bN_c)$, $(X'Y'Z')$ are same.

(f) Let P_a, P_b, P_c be midpoints of sides of triangle $I_aI_bI_c$. Prove that reflections of points M_a, M_b, M_c, wrt P_a, P_b, P_c lie on circle which goes through point I.

(g) Prove that reflections of points H_a, H_b, H_c, wrt P_a, P_b, P_c lie on circle which goes through point I.

(h) Let K_a, K_b, K_c be tangent points of in-circle of of triangle $I_aI_bI_c$ with it's sides. Prove that reflections of points M_a, M_b, M_c, wrt K_a, K_b, K_c lie on circle which goes through point I.

(j) Let K_a, K_b, K_c be tangent points of in-circle of of triangle $I_aI_bI_c$ with it's sides. Prove that reflections of points H_a, H_b, H_c, wrt K_a, K_b, K_c lie on circle which goes through point I.

Ex 1^{ω}: Let given triangle ABC and any point P. Let $A'B'C'$ be circumchevian triangle of P wrt triangle ABC, let $A'A_1, B'B_1, C'C_1$ be perpendiculars from A', B', C' on sides of ABC. Then pedal circle of P wrt ABC, 9 - point circle of ABC and circle $A_1B_1C_1$ intersects at same point.

Ex 2^{ω}: Let given triangle ABC and its centroid M. Let circumcenters of triangles ABM, CBM, AMC form triangle with circumcircle ω. Prove that circumcenter of pedal triangle of point M wrt triangle ABC lie on radical line of circles $(ABC), \omega$.

Ex 3^{ω}: For any triangle ABC with circumcenter O. Let circle ω goes through circumcenters of triangles AOB, BOC, AOC. Prove that nine-point center of triangle ABC, lies on radical line of circles $(ABC), \omega$.

Also note that we can use this **NF**'s not only to triangle but to other figures.

Ex 4^{ω}: Consider quadrilateral $ABCD$ which is inscribed in circle ω_1 and circumscribed around ω_2. Let ω_2 is tangent to sides AB, BC, CD, DA at points $T_{ab}, T_{bc}, T_{cd}, T_{ad}$ respectively. Let $E = T_{ab}T_{cd} \cap T_{bc}T_{ad}$. Let EH_A, EH_B, EH_C, EH_D be perpendiculars to segments $T_{ab}T_{ad}$, $T_{bc}T_{ab}$, $T_{bc}T_{cd}$, $T_{cd}T_{ad}$. Let points O_A, O_B, O_C, O_D be circumcenters of triangles $AT_{ab}T_{ad}$, $BT_{bc}T_{ab}$, $CT_{bc}T_{cd}$, $DT_{cd}T_{ad}$. Prove that quadrilateral $O_AO_BO_CO_D$ cyclic and radical line of circles $(O_AO_BO_CO_D)$, $(H_AH_BH_CH_D)$ goes through center of $(H_AH_BH_CH_D)$.

Ex 2^{pt}: Consider quadrilateral $ABCD$ which is inscribed in circle ω_1 and circumscribed around ω_2. Let ω_2 is tangent to sides AB, BC, CD, DA at points $T_{ab}, T_{bc}, T_{cd}, T_{ad}$ respectively. Let $E = T_{ab}T_{cd} \cap T_{bc}T_{ad}$. Let EH_A, EH_B, EH_C, EH_D be perpendiculars to segments $T_{ab}T_{ad}$, $T_{bc}T_{ab}$, $T_{bc}T_{cd}$, $T_{cd}T_{ad}$.

(a) Prove that reflections of points A, B, C, D wrt lines $T_{ab}T_{ad}, T_{bc}T_{ab}, T_{bc}T_{cd}, T_{cd}T_{ad}$ respectively lie on same line which goes through point E.

(b) Prove that of points A, B, C, D wrt points H_A, H_B, H_C, H_D respectively lie on same circle with center P. Also prove that points P, centers of circles ω_1, ω_2 and point E lie on same line.

(c) Let points O_A, O_B, O_C, O_D be circumcenters of triangles $AT_{ab}T_{ad}$, $BT_{bc}T_{ab}$, $CT_{bc}T_{cd}$, $DT_{cd}T_{ad}$. Prove that quadrilateral $O_AO_BO_CO_D$ cyclic and it's circumcenter lies on line PE.

Ex 5^ω: Let given points A, B, C, D, E, such that AB, BC, CD, DE, EA are tangent circle ω at points K_1, K_2, K_3, K_4, K_5 respectively. Let given that $P = AB \cap DE, Q = BC \cap AE, R = AB \cap CD, T = BC \cap DE, K = DC \cap EA$. Let $X_1 = (AEP) \cap (AQB)$, $X_2 = (AQB) \cap (BRC)$, $X_3 = (BRC) \cap (CTD)$, $X_4 = (CTD) \cap (DEK)$, $X_5 = (DEK) \cap (AEP)$. Let I be center of ω and O be center of $(X_1 \dots X_5)$. Let $\mathcal{H}_X^2(\ -)$ denote as homotety with center at X and coefficient 2. Prove that circles $(X_1 \dots X_5)$, $\mathcal{H}_I^2(\omega)$, $\mathcal{H}_O^2(\omega)$ have same radical line.

Ex 6^ω: Let given triangle ABC with it's inc-enter I and Euler line l. Consider intersections A', B, C' of line l with angle bisectors of triangle ABC. Let O_A, O_B, O_C be circumcenters of triangles $A'BC, AB'C, ABC'$ respectively. Is it true that circle $(O_AO_BO_C)$ always intersect circle (ABC)?

Ex 7^ω: Consider triangle ABC, let F_1 – first Fermat point of ABC. Prove that second Fermat points of triangles ABC, AF_1B, AF_1C, BF_1C lie on same circle.

9. MEANING OF IMO2011 PROBLEM

Well known that for any triangle ABC and point P on it's circumcircle there exists Simson line of point P wrt triangle ABC. So we can ask next question : What is dual Simson line of line wrt triangle? We can look on Simson line wrt another point of view : Reflections P_a, P_b, P_c of point P wrt sides of triangle ABC lie on same line which goes through orthocenter of ABC (1).

One of the natural answer gives us IMO 2011 Problem 6 see it here [9, Problem G8]. Here we consider tangent line l to (ABC) instead of point P. Then instead of reflections of P wrt sides of ABC we can consider reflections l_a, l_b, l_c of l to sides of ABC. And IMO problem 6 says informally that (1) transport into next statement : Curcumcircle of triangle formed by lines l_a, l_b, l_c is connected to base triangle (it is tangent to (ABC)). So we can state next **NF**: Dual analog of Simpson line is circumcircle of reflections lines of tangent line to (ABC) wrt it's sides.

Definition 9.1. *For any triangle ABC and point $P \in (ABC)$, denote line $\mathcal{L}(ABC, P)$ as line which goes through reflections of point P wrt sides of ABC.*

Definition 9.2. *For any triangle ABC and point $P \in (ABC)$, denote circle $\otimes(ABC, P)$ as circumsircle of triangle formed by reflections of tangent line through P to (ABC) wrt sides of ABC.*

Consider next simple fact : For every quadrilateral $ABCD$ lines $\mathcal{L}(ABC, M)$, $\mathcal{L}(ACD, M)$, $\mathcal{L}(BCD, M)$, $\mathcal{L}(ABD, M)$ are equivalent, where $M = \mathcal{M}(AC, BD)$ - Miquel point of lines AB, BC, CD, DA.

Informally it says that this lines are connected, so we can ask next question : What are connections of circles $\otimes(ABC, M)$, $\otimes(ABD, M)$, $\otimes(ACD, M)$, $\otimes(CBD, M)$ for quadrilateral $ABCD$ and Miquel point M.

Answer is next : All tangent points of this circles with circles (ABC), (ABD), (ACD), (CBD) respectively lie on same circle.

10. Example from number theory

Consider next statement : for every integer i and prime number p, sum $1^i + 2^i + \ldots + (p-1)^i \equiv 0 \pmod{p}$. So it's naturally to state next **NF** conjecture $\mathbb{T}_{\mathbb{N}}$: For most sequences of functions f_i: $\mathbb{Z}_p \to \mathbb{Z}_p$, $i = 1, \ldots, p-1$ if we define sums $S_a^{[j]} := \sum_{i=1}^{p-1} f_i(a)^j$ and if sums $S_a^{[1]}$ have non-trivial relations, then sums $S_a^{[j]}$ have non-trivial relations in $\mathbb{Z}_p$ for general considerations of parameter j.

For first example consider $f_i = i$, then $S_a^{[j]} \equiv 0$, for every j. Consider next fact [9, Problem N7] : If $f_i(a) = \frac{a^i}{i}$, then we get relation $S_3^{[1]} + S_4^{[1]} - 3S_2^{[1]} \equiv -1 \pmod{p}$. So from $\mathbb{T}_{\mathbb{N}}$ it's natural to search relations in sequence $\{S_a^{[j]}\}_{a \in \mathbb{Z}_p}$. One can get next relations :

(1) $S_2^{[-1]} \equiv 1 \pmod{p}$
(2) $S_{-1}^{[-1]} \equiv S_3^{[-1]} - 1 \pmod{p}$
(3) $S_4^{[-1]} - 2S_{-2}^{[-1]} \equiv 1 \pmod{p}$
(4) $2S_2^{[-1]} + 2S_{-1}^{[-1]} \equiv -1 \pmod{p}$
(5) $2S_3^{[-1]} \equiv 1 \pmod{p}$
(6) $3S_4^{[-1]} \equiv 2S_3^{[-1]} \pmod{p}$
(7) $S_1^{[-2]} \equiv S_{-1}^{[-2]} \equiv 0 \pmod{p}$
(8) $4S_{-1}^{[-3]} \equiv 1 \pmod{p}$

For another example consider next sum $S_a^{[1]} := \sum_{i=a}^{p-1} C_i^a \pmod{p}$. Easy to see that $S_a^{[1]} \equiv (-1)^a \pmod{p}$, so from $\mathbb{T}_{\mathbb{N}}$ it's natural to search relations between sums $S_a^{[j]}$.

Some of these relations are :

(1) $S_{p-1}^{[-1]} \equiv 1 \pmod{p}$
(2) $S_{2k+1}^{[-1]} \equiv 0 \pmod{p}$
(3) $S_2^{[-1]} \equiv 2 \pmod{p}$
(4) $S_{p-5}^{[-1]} \equiv S_4^{[-1]} - 1 \pmod{p}$
(5) $2S_{p-3}^{[-1]} \equiv 3 \pmod{p}$

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