

Computer Discovered Mathematics: Grebe Triangles

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Abstract. By using the computer program “Discoverer”, we study the Grebe Triangles.

Keywords. Grebe triangle, triangle geometry, remarkable point, computer-discovered mathematics, Euclidean geometry, Discoverer.

Mathematics Subject Classification (2010). 51-04, 68T01, 68T99.

1. INTRODUCTION

The computer program “Discoverer”, created by the authors, is the first computer program, able easily to discover new theorems in mathematics, and possibly, the first computer program, able easily to discover new knowledge in science. See [4].

In this paper, by using the “Discoverer”, we investigate the Grebe triangles. The Outer Grebe Triangle is studied by Yiu [13], Section 4.3.2, pages 47-48. The definition of the Outer Grebe triangle is as follows (see figure 2): Given triangle ABC . Consider the square BCX_cX_b erected externally on BC , the square CAY_aY_c erected externally on CA , and the square ABZ_bZ_a erected externally on AB . Denote by A_2 the intersection point of lines Y_cY_a and Z_aZ_b , by B_2 the intersection point of lines Z_aZ_b and X_bX_c , and by C_2 the intersection point of lines X_bX_c and Y_cY_a . Then $A_2B_2C_2$ is the Outer Grebe Triangle.

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If the squares are erected internally, we obtain the Inner Grebe triangle $A_1B_1C_1$ (See figure 1).

In 2003 Darij Grinberg [3] has studied the Inner and Outer Grebe triangles. In accordance with [3], the definition of the Inner Grebe triangle is pointed by Edward Brisse.

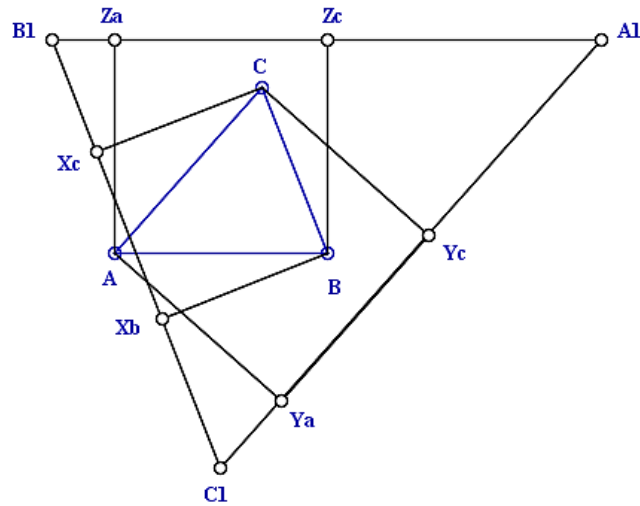


FIGURE 1. Inner Grebe Triangle $A_1B_1C_1$

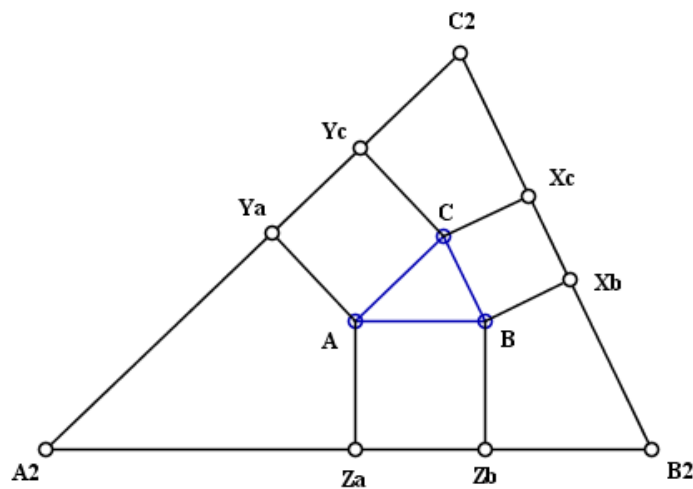


FIGURE 2. Outer Grebe Triangle $A_2B_2C_2$

We refer the reader for barycentric coordinates to [5], [4], [6], [7], [10], [11], [13], [1], [2], [9], [12]. We use the 10200 points version of 2016 of the Kimberling's ETC [8].

We denote by S the twice area of the reference triangle ABC .

2. INNER GREBE TRIANGLE

2.1. Barycentric Coordinates.

Theorem 2.1. *The barycentric coordinates of the Inner Grebe Triangle $A_1B_1C_1$ are as follows:*

$$(1) \quad \begin{aligned} A_1 &= (S - b^2 - c^2, b^2, c^2), \\ B_1 &= (a^2, S - a^2 - c^2, c^2), \\ C_1 &= (a^2, b^2, S - a^2 - b^2). \end{aligned}$$

2.2. Kimberling's Points. We have investigated 195 remarkable points of the Inner Grebe Triangle. Of these 21 points are available in the Kimberling's ETC [8] and the rest of 174 points are not available in [8].

The Table 1 gives a few of the centers of the Inner Grebe triangle in terms of the centers of the reference triangle for Kimberling centers $X(n)$. The reader may find additional notable points of the Inner Grebe triangle in the Supplementary Material.

$X(n)$	Notable Points of the Inner Grebe Triangle	$X(n)$
X(1)	Incenter	X(3641)
X(2)	Centroid	X(5861)
X(3)	Circumcenter	X(1161)
X(4)	Orthocenter	X(5871)
X(5)	Nine-Point Center	X(5875)
X(6)	Symmedian Point	X(6) Symmedian Point
X(13)	Outer Fermat Point	X(6270)
X(14)	Inner Fermat Point	X(6271)
X(54)	Kosnita Point	X(6277).
X(64)	Isogonal Conjugate of de Longchamps Point	X(6267).
X(68)	Prasolov Point	X(9929).
X(74)	Ceva Product of First Isodynamic Point and Second Isodynamic Point	X(7725).
X(76)	Third Brocard Point	X(6273).
X(80)	Reflection of Incenter in Feuerbach Point	X(6263).

TABLE 1.

We can use the properties of the corresponding notable points given in Kimberling's ETC [8], but also we can use the Discoverer in order to discover new properties. For example, we find new properties of point X(5861), the Centroid of the Inner Grebe Triangle.

Theorem 2.2. *The Centroid of the Inner Grebe Triangle is the Perspector of the Inner Grebe Triangle and the Outer Kiepert $\arctan(2)$ Triangle.*

Theorem 2.3. *The Centroid of the Inner Grebe Triangle is the Harmonic Conjugate of the Homothetic Center of the Medial Triangle and the Outer Grebe Triangle with respect to the Malfatti-Moses Point and the Retrocenter. (the Homothetic Center of the Medial Triangle and the Outer Grebe Triangle is between the Malfatti-Moses Point and Retrocenter).*

2.3. New Notable Points. In our investigation we have found 174 new notable points of the Inner Grebe Triangle which are not available in Kimberling's ETC [8]. By using the Discoverer, we can investigate the properties of these new notable points. Below we give two theorems about new points. Clearly, by using the Discoverer we can find a number of additional new theorems.

Theorem 2.4. *The Gergonne Point of the Inner Grebe Triangle lies on the following lines:*

- (1) *Line through the Gergonne Point and the Symmedian Point.*
- (2) *The through the Mittenpunkt and the Perspector and Homothetic Center of the Inner Grebe Triangle and the Medial Triangle.*
- (3) *Line through the Perspector and Homothetic Center of the Inner Grebe Triangle and the Johnson Triangle and the Reflection of the Gergonne Point in the Nine-Point Center.*
- (4) *Line through the Midpoint of the Gergonne Point and the Orthocenter and the Perspector and Homothetic Center of the Euler Triangle and the Inner Grebe Triangle.*
- (5) *Line through the Mittenpunkt of the Medial Triangle and the Perspector and Homothetic Center of the Half-Median Triangle and the Inner Grebe Triangle.*
- (6) *Line through the Gergonne Point of the Antimedial Triangle and the Perspector and Homothetic Center of the Antimedial Triangle and the Inner Grebe Triangle.*

Theorem 2.5. *The Spieker Center of the Inner Grebe Triangle lies on the following lines:*

- (1) *Line through the Spieker Center and the Symmedian Point.*
- (2) *Line through the Incenter and the Perspector and Homothetic Center of the Antimedial Triangle and the Inner Grebe Triangle.*
- (3) *Line through the Perspector and Homothetic Center of the Inner Grebe Triangle and the Medial Triangle and the Spieker Center of the Medial Triangle.*
- (4) *Line through the Perspector and Homothetic Center of the Euler Triangle and the Inner Grebe Triangle and the Spieker Center of the Euler Triangle.*
- (5) *Line through the Midpoint of the Incenter and the Orthocenter and the Perspector and Homothetic Center of the Inner Grebe Triangle and the Johnson Triangle.*

2.4. Homothetic Triangles.

Theorem 2.6. *The Homothetic Center of the Inner Grebe Triangle and the*

- (1) *Triangle ABC is the X(6) Symmedian Point.*
- (2) *Medial Triangle is the X(5591).*
- (3) *Antimedial Triangle is the X(1271).*
- (4) *Euler Triangle is the X(6202).*
- (5) *Outer Grebe Triangle is the X(6) Symmedian Point.*
- (6) *Johnson Triangle is the X(6215).*
- (7) *Inner Yff Triangle is the X(10040).*
- (8) *Outer Yff Triangle is the X(10048).*

2.5. Perspective Triangles.

Theorem 2.7. *The Perspector of the Inner Grebe Triangle and the*

- (1) *Orthic Triangle is the $X(1163)$.*
- (2) *Symmedianal Triangle is the $X(6)$ Symmedian Point.*
- (3) *Tangential Triangle is the $X(6)$ Symmedian Point.*
- (4) *Triangle of Reflections is the $X(6220)$.*
- (5) *Second Brocard Triangle is the $X(6)$ Symmedian Point.*
- (6) *Half-Altitude Triangle is the $X(6218)$.*
- (7) *Half-Symmedian Triangle is the $X(6)$ Symmedian Point.*

2.6. Internal Similitude Centers.

Theorem 2.8. *The Internal Center of Similitude of the*

- (1) *Circumcircle of the Inner Grebe Triangle and the Nine-Point Circle of the Inner Grebe Triangle is the $X(5861)$.*
- (2) *Circumcircle of the Inner Grebe Triangle and the Second Brocard Circle of the Inner Grebe Triangle is the $X(1161)$.*
- (3) *Incircle of the Inner Grebe Triangle and the Spieker Circle of the Inner Grebe Triangle is the $X(5861)$.*
- (4) *Nine-Point Circle of the Inner Grebe Triangle and the Sine-Triple-Angle Circle of the Inner Grebe Triangle is the $X(7732)$.*
- (5) *Antimedial Circle of the Inner Grebe Triangle and the Circumcircle of the Inner Grebe Triangle is the $X(5861)$.*
- (6) *Adams Circle of the Inner Grebe Triangle and the Incircle of the Inner Grebe Triangle is the $X(3641)$.*
- (7) *Adams Circle of the Inner Grebe Triangle and the Conway Circle of the Inner Grebe Triangle is the $X(3641)$.*
- (8) *Conway Circle of the Inner Grebe Triangle and the Incircle of the Inner Grebe Triangle is the $X(3641)$.*
- (9) *Inner Johnson-Yff Circle of the Inner Grebe Triangle and the Outer Johnson-Yff Circle of the Inner Grebe Triangle is the $X(3641)$.*
- (10) *Cosine Circle of the Inner Grebe Triangle and the Nine-Point Circle of the Inner Grebe Triangle is the $X(6279)$.*

2.7. External Similitude Centers.

Theorem 2.9. *The External Center of Similitude of the*

- (1) *Circumcircle of the Inner Grebe Triangle and the Nine-Point Circle of the Inner Grebe Triangle is the $X(5871)$.*
- (2) *Circumcircle of the Inner Grebe Triangle and the Excentral Circle of the Inner Grebe Triangle is the $X(3641)$.*
- (3) *Circumcircle of the Inner Grebe Triangle and the Half-Moses Circle of the Inner Grebe Triangle is the $X(6)$.*
- (4) *Incircle of the Inner Grebe Triangle and the Inner Johnson-Yff Circle of the Inner Grebe Triangle is the $X(5871)$.*
- (5) *Incircle of the Inner Grebe Triangle and the Outer Johnson-Yff Circle of the Inner Grebe Triangle is the $X(5871)$.*
- (6) *Nine-Point Circle of the Inner Grebe Triangle and the Tangential Circle of the Inner Grebe Triangle is the $X(1161)$.*

- (7) *Nine-Point Circle of the Inner Grebe Triangle and the Sine-Triple-Angle Circle of the Inner Grebe Triangle is the X(6277).*
- (8) *Antimedial Circle of the Inner Grebe Triangle and the Nine-Point Circle of the Inner Grebe Triangle is the X(5861).*
- (9) *Antimedial Circle of the Inner Grebe Triangle and the Cosine Circle of the Inner Grebe Triangle is the X(4).*
- (10) *Inner Johnson-Yff Circle of the Inner Grebe Triangle and the Outer Johnson-Yff Circle of the Inner Grebe Triangle is the X(5871).*
- (11) *Cosine Circle of the Inner Grebe Triangle and the Nine-Point Circle of the Inner Grebe Triangle is the X(6281).*
- (12) *Cosine Circle of the Inner Grebe Triangle and the Excentral Circle of the Inner Grebe Triangle is the X(40).*

3. OUTER GREBE TRIANGLE

3.1. Barycentric Coordinates. The barycentric coordinates of the Outer Grebe triangle are given in Yiu [13], Section 4.3.2, pages 47-48.

Theorem 3.1. *The barycentric coordinates of the Outer Grebe Triangle $A_1B_1C_1$ are as follows:*

$$(2) \quad \begin{aligned} A_2 &= (-(S + b^2 + c^2), b^2, c^2), \\ B_2 &= (a^2, -(S + a^2 + c^2), c^2), \\ C_2 &= (a^2, b^2, -(S + a^2 + b^2)). \end{aligned}$$

3.2. Kimberling's Points. We have investigated 195 remarkable points of the Outer Grebe Triangle. Of these 22 points are available in the Kimberling's ETC [8] and the rest of 173 points are not available in [8].

The Table 2 gives a few of the centers of the Inner Grebe triangle in terms of the centers of the reference triangle for Kimberling centers X(n). The reader may find additional notable points of the Outer Grebe Triangle in the Supplementary Material.

3.3. New Notable Points. In our investigation we have found 173 new notable points of the Inner Grebe Triangle which are not available in Kimberling's ETC [8]. By using the Discoverer, we can investigate the properties of these new notable points. Below we give two theorems about new points. Clearly, by using the Discoverer we can find a number of additional new theorems.

Theorem 3.2. *The Gergonne Point of the Outer Grebe Triangle lies on the following lines:*

- (1) *Line through the Gergonne Point and the Symmedian Point.*
- (2) *Line through the Mittenpunkt and the Perspector and Homothetic Center of the Medial Triangle and the Outer Grebe Triangle.*
- (3) *Line through the Perspector and Homothetic Center of the Johnson Triangle and the Outer Grebe Triangle and the Reflection of the Gergonne Point in the Nine-Point Center.*
- (4) *Line through the Midpoint of the Gergonne Point and the Orthocenter and the Perspector and Homothetic Center of the Euler Triangle and the Outer Grebe Triangle.*

X(n)	Notable Points of the Outer Grebe Triangle	X(n)
X(1)	Incenter	X(3640).
X(2)	Centroid	X(5860).
X(3)	Circumcenter	X(1160).
X(4)	Orthocenter	X(5870).
X(5)	Nine-Point Center	X(5874).
X(6)	Symmedian Point	X(6).
X(13)	Outer Fermat Point	X(6268).
X(14)	Inner Fermat Point	X(6269).
X(54)	Kosnita Point	X(6276).
X(64)	ogonal Conjugate of de Longchamps Point	X(6266).
X(68)	Prasolov Point	X(9930).
X(74)	Ceva Product of First odynamic Point and Second odynamic Point	X(7726).
X(76)	Third Brocard Point	X(6272).
X(80)	Reflection of Incenter in Feuerbach Point	X(6262).

TABLE 2.

- (5) *Line through the Mittenpunkt of the Medial Triangle and the Perspector and Homothetic Center of the Half-Median Triangle and the Outer Grebe Triangle.*
- (6) *Line through the Gergonne Point of the Antimedial Triangle and the Perspector and Homothetic Center of the Antimedial Triangle and the Outer Grebe Triangle.*

Theorem 3.3. *The Nagel Point of the Outer Grebe triangle lies on the following lines:*

- (1) *Line through the Nagel Point and the Symmedian Point.*
- (2) *Line through the Incenter and the Perspector and Homothetic Center of the Medial Triangle and the Outer Grebe Triangle.*
- (3) *Line through the Spieker Center and the Perspector and Homothetic Center of the Half-Median Triangle and the Outer Grebe Triangle.*
- (4) *Line through the Center of the Fuhrmann Circle and the Perspector and Homothetic Center of the Euler Triangle and the Outer Grebe Triangle.*
- (5) *Line through the Perspector and Homothetic Center of the Antimedial Triangle and the Outer Grebe Triangle and the Reflection of the Nagel Point in the Incenter.*
- (6) *Line through the Perspector and Homothetic Center of the Johnson Triangle and the Outer Grebe Triangle and the Reflection of the Circumcenter in the Incenter.*

3.4. Homothetic Triangles.

Theorem 3.4. *The Homothetic Center of the Outer Grebe Triangle and the*

- (1) *Triangle ABC is the X(6) Symmedian Point.*

- (2) *Medial Triangle is the X(5590).*
- (3) *Antimedial Triangle is the X(1270).*
- (4) *Euler Triangle is the X(6201).*
- (5) *Inner Grebe Triangle is the X(6) Symmedian Point.*
- (6) *Johnson Triangle is the X(6214).*
- (7) *Inner Yff Triangle is the X(10041).*
- (8) *Outer Yff Triangle is the X(10049).*

3.5. Perspective Triangles.

Theorem 3.5. *The Perspector of the Outer Grebe Triangle and the*

- (1) *Orthic Triangle is the X(1162).*
- (2) *Symmedian Triangle is the X(6) Symmedian Point.*
- (3) *Tangential Triangle is the X(6) Symmedian Point.*
- (4) *Triangle of Reflections is the X(6219).*
- (5) *Second Brocard Triangle is the X(6) Symmedian Point.*
- (6) *Half-Altitude Triangle is the X(6217).*
- (7) *Half-Symmedian Triangle is the X(6) Symmedian Point.*

3.6. Internal Similitude Centers.

Theorem 3.6. *The Internal Center of Similitude of the*

- (1) *Circumcircle of the Outer Grebe Triangle and the Nine-Point Circle of the Outer Grebe Triangle is the X(5860).*
- (2) *Circumcircle of the Outer Grebe Triangle and the Second Brocard Circle of the Outer Grebe Triangle is the X(1160).*
- (3) *Incircle of the Outer Grebe Triangle and the Spieker Circle of the Outer Grebe Triangle is the X(5860).*
- (4) *Nine-Point Circle of the Outer Grebe Triangle and the Sine-Triple-Angle Circle of the Outer Grebe Triangle is the X(7733).*
- (5) *Antimedial Circle of the Outer Grebe Triangle and the Circumcircle of the Outer Grebe Triangle is the X(5860).*
- (6) *Antimedial Circle of the Outer Grebe Triangle and the Cosine Circle of the Outer Grebe Triangle is the X(4).*
- (7) *Adams Circle of the Outer Grebe Triangle and the Incircle of the Outer Grebe Triangle is the X(3640). Inner Johnson-Yff Circle of the Outer Grebe Triangle and the Outer Johnson-Yff Circle of the Outer Grebe Triangle is the X(3640).*
- (8) *Cosine Circle of the Outer Grebe Triangle and the Nine-Point Circle of the Outer Grebe Triangle is the X(6278).*
- (9) *Cosine Circle of the Outer Grebe Triangle and the Excentral Circle of the Outer Grebe Triangle is the X(40).*

3.7. External Similitude Centers.

Theorem 3.7. *The External Center of Similitude of the*

- (1) *Circumcircle of the Outer Grebe Triangle and the Nine-Point Circle of the Outer Grebe Triangle is the X(5870).*
- (2) *Circumcircle of the Outer Grebe Triangle and the Excentral Circle of the Outer Grebe Triangle is the X(3640).*

- (3) *Circumcircle of the Outer Grebe Triangle and the Half-Moses Circle of the Outer Grebe Triangle is the X(6).*
- (4) *Incircle of the Outer Grebe Triangle and the Inner Johnson-Yff Circle of the Outer Grebe Triangle is the X(5870).*
- (5) *Incircle of the Outer Grebe Triangle and the Outer Johnson-Yff Circle of the Outer Grebe Triangle is the X(5870).*
- (6) *Nine-Point Circle of the Outer Grebe Triangle and the Tangential Circle of the Outer Grebe Triangle is the X(1160).*
- (7) *Nine-Point Circle of the Outer Grebe Triangle and the Sine-Triple-Angle Circle of the Outer Grebe Triangle is the X(6276).*
- (8) *Antimedial Circle of the Outer Grebe Triangle and the Nine-Point Circle of the Outer Grebe Triangle is the X(5860).*
- (9) *Inner Johnson-Yff Circle of the Outer Grebe Triangle and the Outer Johnson-Yff Circle of the Outer Grebe Triangle is the X(5870).*
- (10) *Cosine Circle of the Outer Grebe Triangle and the Nine-Point Circle of the Outer Grebe Triangle is the X(6280).*

SUPPLEMENTARY MATERIAL

The enclosed supplementary material contains theorems related to the topic.

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REFERENCES

- [1] P. Douillet, *Translation of the Kimberling's Glossary into barycentrics*, 2012, <http://www.ddekov.eu/e2/htm/links/Douillet.pdf>.
- [2] Francisco Javier García Capitán. *Barycentric Coordinates*, International Journal of Computer Discovered Mathematics, 2015, vol. 0, no 0, 32-48. <http://www.journal-1.eu/2015/01/Francisco-Javier-Barycentric-Coordinates-pp.32-48.pdf>.
- [3] Darij Grinberg, Grebe Triangle, <http://mathforum.org/kb/message.jspa?messageID=1071652>.
- [4] S. Grozdev and D. Dekov, *A Survey of Mathematics Discovered by Computers*, International Journal of Computer Discovered Mathematics, 2015, vol.0, no.0, 3-20. <http://www.journal-1.eu/2015/01/Grozdev-Dekov-A-Survey-pp.3-20.pdf>.
- [5] S. Grozdev and D. Dekov, *Barycentric Coordinates: Formula Sheet*, International Journal of Computer Discovered Mathematics, vol.1, 2016, no 2, 75-82. <http://www.journal-1.eu/2016-2/Grozdev-Dekov-Barycentric-Coordinates-pp.75-82.pdf>.
- [6] S. Grozdev and V. Nenkov, *Three Remarkable Points on the Medians of a Triangle* (Bulgarian), Sofia, Archimedes, 2012.
- [7] S. Grozdev and V. Nenkov, *On the Orthocenter in the Plane and in the Space* (Bulgarian), Sofia, Archimedes, 2012.
- [8] C. Kimberling, *Encyclopedia of Triangle Centers - ETC*, <http://faculty.evansville.edu/ck6/encyclopedia/ETC.html>.
- [9] Gerry Leversha, *The Geometry of the Triangle*, The United Kingdom Mathematical Trust, The Pathways Series no.2, 2013.
- [10] G. Paskalev and I. Tchobanov, *Remarkable Points in the Triangle* (in Bulgarian), Sofia, Narodna Prosveta, 1985.
- [11] G. Paskalev, *With coordinates in Geometry* (in Bulgarian), Sofia, Modul-96, 2000.

- [12] M. Schindler and K.Chen, Barycentric Coordinates in Olympiad Geometry, 2012, <http://www.mit.edu/~evanchen/handouts/bary/bary-full.pdf>.
- [13] P. Yiu, *Introduction to the Geometry of the Triangle*, 2001, new version of 2013, <http://math.fau.edu/Yiu/YIUIntroductionToTriangleGeometry130411.pdf>.