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The extension from a circle to a conic having center: The creative method of new theorems

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Finding new theorems is the best target of keen mathematicians. If finding a new theorem is difficult then finding the method of discovering it is much more difficult. However, mathematics has such a lot of methods such as: The extension of theorems from Euclidean geometry to spherical geometry and Lobachevskian geometry, the extension of theorems from plane geometry to solid geometry, etc. In this paper, we give a new extensional method which is the extension from theorems for circle to theorems for conic having center

I. The basic concepts and properties

1. The affine transformation

1.1. The introduction on the affine transformation in a plane

Denote by P the set of points of plane P. Each subset of P is called a *figure*. A bijection $f : P \rightarrow P$ is called a *transformation* of plane P. For each point M of P, point $M' = f(M)$ is called *the image* of point M through the transformation f, point M is called *the pre-image* of point M'. We still say that: The transformation f maps point M into point M'.

If H is an arbitrary figure, the set $H' = f(H)$ including the images of all points H is called *the image* of figure H. We still say that: the transformation f maps the figure H to the figure H'.

The composition (or still called the product) of two transformations is a transformation, and each transformation f has the inverse transformation f^{-1} which is still a transformation. Observely, $f \circ f^{-1} = f^{-1} \circ f = e$ Thus, *the set of all transformations of plane P makes into a group for the composition of two transformations.*

1.2. The ratio of three collinear points and the definition of affine transformation

Given three distinct and collinear points *A, B, C*. Then there exists a number $k \neq 1$ such that $\overrightarrow{CA} = k\overrightarrow{CB}$. The number k is called *the ratio* of three points *A, B, C* and denoted by *(ABC)*.

Clearly: *C is the midpoint of segment AB if and only if (ABC) = -1.*

We know the following property:

Given three points A, B, C lying on a line d and three points A', B', C' lying on a line d' // d. Then the necessary and sufficient condition for (ABC) = (A'B'C') is three lines

AA', BB', CC' being parallel or concurrent.

Definition 1

The transformation $f : P \to P$ is called an affine transformation if f maps three collinear points into three collinear points and ratios of collinear points are preserved.

That is, if three points *A, B, C* are collinear, then their images *A' = f(A), B' = f(B), C' = f(C)* are still collinear, and *(ABC) = A'B'C')*.

The following are the examples of affine transformations: identical transformation, translation, homothety.

1.3. The properties of affine transformation

i) Affine transformation maps a line into a line.

ii) Affine transformation maps two intersecting lines into two intersecting lines, two parallel lines into two parallel lines.

iii) Affine transformation maps a segment into a segment, a ray into a ray, an angle into an angle, a half of plane into a half of plane, a triangle into a triangle, a region of triangle into a region of triangle, a parallelogram into a parallelogram.

iv) Suppose that an affine transformation maps four points *A, B, C, D* into four points *A', B', C', D'*, respectively. If $\overrightarrow{AB} = k\overrightarrow{CD}$ then $\overrightarrow{A'B'} = k\overrightarrow{C'D'}$.

2. The affine equivalent transformation

2.1. The affine and affine equivalent group

We consider the set $Af(P)$ including the affine transformations of plane P. Then $Af(P)$ makes into a group for the composition of two affine transformations, because

i) The composition of two affine transformations is an affine transformation.

ii) The inverse of an affine transformation is also an affine transforation

iii) The identical transformation e is an affine transformation.

Definition 2

Figure H is called the affine equivalent of figure H', if we have an affine transformation f mapping figure H into figure H', that is f(H) = H'.

By the definition, we follow that:

i') Each of figures H is affine equivalent to itself.

ii') If figure H is affine equivalent to figure H' then figure H' is affine equivalent to figure H. Thus, we can say that two figures H and H' are equivalent.

iii') If figure H is affine equivalent to H' and H' is affine equivalent to H'' then H is affine equivalent to H''.

The following are some examples:

- Two arbitrary triangles are affine equivalent.
- Two arbitrary parallelograms are affine equivalent.
- Trapezoid $ABCD$ (two bases are AB and CD) and trapezoid *A'B'C'D'* (two bases *A'B'* and *C'D'*) are equivalent if and only if $\frac{AB}{CD} = \frac{A'B'}{C'D'}$.
- Two arbitrary ellipses are affine equivalent.
- Two arbitrary hyperbolas are affine equivalent.
- Two arbitrary parabolas are affine equivalent.

2.2. The affine properties

Definition 3

A certain property of figure H is called affine property if figure H' is affine equivalent to H having this one.

The following are affine properties:

- The property of *"being trapezoid"* of quadrilateral is an affine one. However, the property of *"being isosceles trapezoid"* or *"being perpendicular trapezoid"* is not affine one.
- The *"isosceles", "equilateral", "perpendicular"* properties are not affine ones.
- The property of *"being the median of a triangle"* is an affine one, however, the properties of *"being an altitude", "being a bisector", "being a mid-perpendicular"* are not affine ones.
- The properties of *"being the center of an ellipse", "being the diameter of an ellipse",*

"being the tangent of an ellipse" (or hyperbola, parabola) are affine ones.
- The property of *"the centroid of a triangle"* is affine one, however the property of *"being the orthocenter of a triangle"* is not affine one.

2.3. The affine concept

Definition 4

A certain concept is called an affine one if it is not changed by an arbitrary affine transformation.

The following are affine concepts: Point, line, ray, segment, half of plane, triangle, region of triangle, centroid, quadrilateral, parallelogram, trapezoid, ratio of three collinear points, ellipse, hyperbola, parabola, tangent, conic, center of conic.

The following are not concepts: Length of segment, length of angle, isosceles triangle, equilateral triangle, perpendicular triangle, orthocenter of triangle, in-center, circum-center, rhombus, isosceles trapezoid, area of figure, circle, equilateral hyperbola, ect. [6, p.51-61]

II. The extension of geometric theorems

We state the method of generalizing from a circle to a conic having center when polygon is a triangle and circle is a circumcircle. The cases of arbitrary polygon or circle being an inscribed circle or escribed circle, ect, are for the readers.

The method of generalization from a circle to a conic having center

- Given a triangle ABC inscribed a circle (O).
- Draw the altitude passing through vertex A of triangle ABC by drawing the line passing through vertex A parallel to the line connecting the center O with the midpoint of segment BC.
- Draw the bisector passing through the vertex A of triangle ABC by letting the intersection point of the line connecting the center and the midpoint of segment BC with the circumcircle be A'(A, A' are the different sides of the line BC). AA' is the bisector of angle $\widehat{BAC}$.
etc.
- We obtain the problem containing affine invariants.
- Generalizing from circle to conic having center, we obtain the generalized problem of the original one.

The following are some illustrated examples. We first go to Dao's theorem. Dao's theorem is one of very difficult ones of elementary geometry which was proved. Up to now, they have found out two proofs. We recall the theorem:

Problem 1 (Dao's Theorem on Six Circumcenters associated with a Cyclic Hexagon). *Given a circle (O). Let $A_1, A_2, A_3, A_4, A_5, A_6$ be six points on this circle. Let $A_{12}, A_{23}, A_{34}, A_{45}$, A_{56}, A_{61} be the points of intersection of A_6A_1, A_2A_3; A_1A_2, A_3A_4; A_2A_3, A_4A_5; A_3A_4, A_5A_6; A_4A_5, A_6A_1; A_5A_6, A_1A_2. Let $B_1, B_2, B_3, B_4, B_5, B_6$ be the midpoints of sides A_1A_2, A_2A_3, $A_3A_4, A_4A_5, A_5A_6, A_6A_1$, respectively. The lines passing through the midpoints C_1, C_2, C_3, C_4, C_5, C_6 of segments $A_2A_{12}, A_3A_{23}, A_4A_{34}, A_5A_{45}, A_6A_{56}, A_1A_{61}$ and being parallel to $OB_2, OB_3, OB_4, OB_5, OB_6, OB_1$ meet $OB_1, OB_2, OB_3, OB_4, OB_5, OB_6$ at $O_{12}, O_{23}, O_{34}, O_{45}$, O_{56}, O_{61}, respectively. Prove that $O_{12}O_{45}, O_{23}O_{56}, O_{34}O_{61}$ are concurrent at a point.*

The solution can be found in [5], [12].
We state the problem 1 containing affine invariants as follows

Problem 2. *Given a circle (O) with center O. Let $A_1, A_2, A_3, A_4, A_5, A_6$ be six points on this circle (O). Let $A_{12}, A_{23}, A_{34}, A_{45}, A_{56}, A_{61}$ be the points of intersection of A_6A_1, A_2A_3; A_1A_2, A_3A_4; A_2A_3, A_4A_5; A_3A_4, A_5A_6; A_4A_5, A_6A_1; A_5A_6, A_1A_2. Let $B_1, B_2, B_3, B_4, B_5, B_6$ be the midpoints of sides $A_1A_2, A_2A_3, A_3A_4, A_4A_5, A_5A_6, A_6A_1$, respectively. The lines passing through the midpoints $C_1, C_2, C_3, C_4, C_5, C_6$ of segments $A_2A_{12}, A_3A_{23}, A_4A_{34}, A_5A_{45}$, A_6A_{56}, A_1A_{61} and being parallel to $OB_2, OB_3, OB_4, OB_5, OB_6, OB_1$ meet OB_1, OB_2, OB_3,*

OB_4, OB_5, OB_6 at $O_{12}, O_{23}, O_{34}, O_{45}, O_{56}, O_{61}$, respectively. Prove that $O_{12}O_{45}, O_{23}O_{56}, O_{34}O_{61}$ are concurrent at a point.

We now extend the circle to conic having center. We obtain the generalization of Dao's theorem as follows:

Problem 3 (The generalization of Dao's Theorem on Six Circumcenters associated with a Cyclic Hexagon)**.** *Given a conic (S) having center O. Let $A_1, A_2, A_3, A_4, A_5, A_6$ be six points on this conic (S). Let $A_{12}, A_{23}, A_{34}, A_{45}, A_{56}, A_{61}$ be the points of intersection of $A_6A_1, A_2A_3; A_1A_2, A_3A_4; A_2A_3, A_4A_5; A_3A_4, A_5A_6; A_4A_5, A_6A_1; A_5A_6, A_1A_2$. Let $B_1, B_2, B_3, B_4, B_5, B_6$ be the midpoints of sides $A_1A_2, A_2A_3, A_3A_4, A_4A_5, A_5A_6, A_6A_1$, respectively. The lines passing through the midpoints $C_1, C_2, C_3, C_4, C_5, C_6$ of segments $A_2A_{12}, A_3A_{23}, A_4A_{34}, A_5A_{45}, A_6A_{56}, A_1A_{61}$ and being parallel to $OB_2, OB_3, OB_4, OB_5, OB_6, OB_1$ meet $OB_1, OB_2, OB_3, OB_4, OB_5, OB_6$ at $O_{12}, O_{23}, O_{34}, O_{45}, O_{56}, O_{61}$, respectively. Prove that $O_{12}O_{45}, O_{23}O_{56}, O_{34}O_{61}$ are concurrent at a point.*

Remarks
If (S) is a circle then we obtain Dao's theorem.
Problem 3 is very difficult. Because conic having center only has two types which are ellipse and hyperbola, we only consider two particular cases: problem for ellipse and problem for hyperbola.

Problem 4. *Given an ellipse (E) with center O. Let $A_1, A_2, A_3, A_4, A_5, A_6$ be six points on this ellipse (E). Let $A_{12}, A_{23}, A_{34}, A_{45}, A_{56}, A_{61}$ be the points of intersection of $A_6A_1, A_2A_3; A_1A_2, A_3A_4; A_2A_3, A_4A_5; A_3A_4, A_5A_6; A_4A_5, A_6A_1; A_5A_6, A_1A_2$. Let $B_1, B_2, B_3, B_4, B_5, B_6$ be the midpoints of sides $A_1A_2, A_2A_3, A_3A_4, A_4A_5, A_5A_6, A_6A_1$, respectively. The lines passing through the midpoints $C_1, C_2, C_3, C_4, C_5, C_6$ of segments $A_2A_{12}, A_3A_{23}, A_4A_{34}, A_5A_{45}, A_6A_{56}, A_1A_{61}$ and being parallel to $OB_2, OB_3, OB_4, OB_5, OB_6, OB_1$ meet $OB_1, OB_2, OB_3, OB_4, OB_5, OB_6$ at $O_{12}, O_{23}, O_{34}, O_{45}, O_{56}, O_{61}$, respectively. Prove that $O_{12}O_{45}, O_{23}O_{56}, O_{34}O_{61}$ are concurrent at a point. (Figure 1)*

In order to prove problem 4, we do as follows
Step 1. The definition of affine concepts and properties
- Ellipse and center O of the ellipse are affine concepts.
- Lines, midpoints and points are affine concepts.
- Parallel, intersection and concurrence are affine concepts.
- Belonging concept is affine one.

Thus, problem contains all affine invariants.
Step 2. The performing of affine equivalent transformation
Because ellipse and circle are two affine equivalent concepts in the affine plane, we can solve problem for circle.

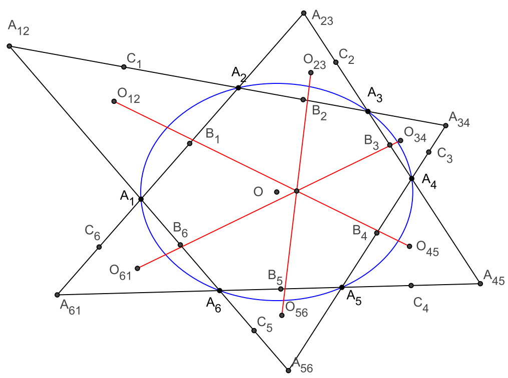

FIGURE 1. The extension of Dao's theorem from a circle to an ellipse

Take the circle for the equivalent figure to the given ellipse (E) by an affine transformation f. We give problem 4 to problem 2 (problem 1) *(Dao's Theorem on Six Circumcenters associated with a Cyclic Hexagon)*. Because problem 1 was proved, we perform the affine transformation f^{-1} mapping circle into ellipse, we obtain the desired result for the ellipse.

Problem 5. *Given a hyperbola (H) with center O. Let $A_1, A_2, A_3, A_4, A_5, A_6$ be six points on this hyperbola (H). Let $A_{12}, A_{23}, A_{34}, A_{45}, A_{56}, A_{61}$ be the points of intersection of $A_6A_1, A_2A_3; A_1A_2, A_3A_4; A_2A_3, A_4A_5; A_3A_4, A_5A_6; A_4A_5, A_6A_1; A_5A_6, A_1A_2$. Let $B_1, B_2, B_3, B_4, B_5, B_6$ be the midpoints of sides $A_1A_2, A_2A_3, A_3A_4, A_4A_5, A_5A_6, A_6A_1$, respectively. The lines passing through the midpoints $C_1, C_2, C_3, C_4, C_5, C_6$ of segments $A_2A_{12}, A_3A_{23}, A_4A_{34}, A_5A_{45}, A_6A_{56}, A_1A_{61}$ and being parallel to $OB_2, OB_3, OB_4, OB_5, OB_6, OB_1$ meet $OB_1, OB_2, OB_3, OB_4, OB_5, OB_6$ at $O_{12}, O_{23}, O_{34}, O_{45}, O_{56}, O_{61}$, respectively. Prove that $O_{12}O_{45}, O_{23}O_{56}, O_{34}O_{61}$ are concurrent at a point.*

In order to prove problem 5, we do as follows
Step 1. The definition of affine concepts and properties
- Hyperbola and center O of the hyperbola are affine concepts.
- Lines, midpoints and points are affine concepts.
- Parallel, intersection and concurrence are affine concepts.
- Belonging concept is affine one.

Thus, problem contains all affine invariants.
Step 2. The performing of affine equivalent transformation
Because arbitrary hyperbola and equilateral hyperbola are two affine equivalent concepts in the affine plane, we can solve problem for equilateral hyperbola.
Take the equilateral hyperbola for the equivalent figure to the given hyperbola (H) by an affine transformation f. We give problem 5 to the following one

Problem 6. *Given an equilateral hyperbola (H) with center O. Let $A_1, A_2, A_3, A_4, A_5, A_6$ be six points on this hyperbola (H). Let $A_{12}, A_{23}, A_{34}, A_{45}, A_{56}, A_{61}$ be the points of intersection of $A_6A_1, A_2A_3; A_1A_2, A_3A_4; A_2A_3, A_4A_5; A_3A_4, A_5A_6; A_4A_5, A_6A_1; A_5A_6, A_1A_2$. Let $B_1, B_2, B_3, B_4, B_5, B_6$ be the midpoints of sides $A_1A_2, A_2A_3, A_3A_4, A_4A_5, A_5A_6, A_6A_1$, respectively. The lines passing through the midpoints $C_1, C_2, C_3, C_4, C_5, C_6$ of segments $A_2A_{12}, A_3A_{23}, A_4A_{34}, A_5A_{45}, A_6A_{56}, A_1A_{61}$ and being parallel to $OB_2, OB_3, OB_4, OB_5, OB_6, OB_1$ meet $OB_1, OB_2, OB_3, OB_4, OB_5, OB_6$ at $O_{12}, O_{23}, O_{34}, O_{45}, O_{56}, O_{61}$, respectively. Prove that $O_{12}O_{45}, O_{23}O_{56}, O_{34}O_{61}$ are concurrent at a point.*

Because the equation of equilateral hyperbola is easily to calculate in Cartesian coordinate system, equilateral hyperbola and arbitrary hyperbola are affine equivalent. We know that, if we choose two asymptotes of equilateral hyperbola containing two axes then the equation of equilateral hyperbola is $xy = k$.
The following calculations are helped by the software of Maple 2015.

Because $A_1, A_2, A_3, A_4, A_5, A_6$ belong to (H), we have

$$A_1(a_1; \tfrac{k}{a_1}), A_2(a_2; \tfrac{k}{a_2}), A_3(a_3; \tfrac{k}{a_3}), A_4(a_4; \tfrac{k}{a_4}), A_5(a_5; \tfrac{k}{a_5}), A_6(a_6; \tfrac{k}{a_6}).$$

The equation of line A_1A_2 is of the form

$$(x - a_1)(\tfrac{k}{a_2} - \tfrac{k}{a_1}) - (y - \tfrac{k}{a_1})(a_2 - a_1) = 0.$$

The equation of line A_2A_3 is of the form

$$(x - a_2)(\tfrac{k}{a_3} - \tfrac{k}{a_2}) - (y - \tfrac{k}{a_2})(a_3 - a_2) = 0.$$

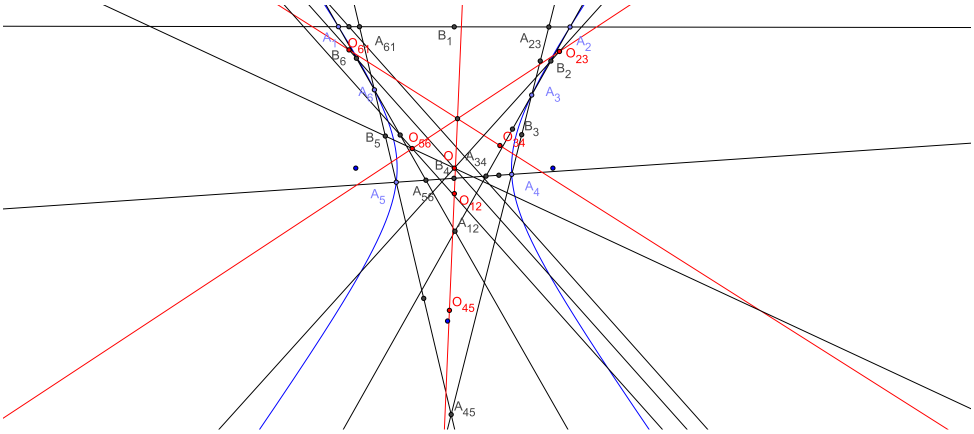

FIGURE 2. The extension of Dao's theorem from a circle to a hyperbola

The equation of line A_3A_4 is of the form

$$(x-a_3)(\tfrac{k}{a_4}-\tfrac{k}{a_3})-(y-\tfrac{k}{a_3})(a_4-a_3)=0.$$

The equation of line A_4A_5 is of the form

$$(x-a_4)(\tfrac{k}{a_5}-\tfrac{k}{a_4})-(y-\tfrac{k}{a_4})(a_5-a_4)=0.$$

The equation of line A_5A_6 is of the form

$$(x-a_5)(\tfrac{k}{a_6}-\tfrac{k}{a_5})-(y-\tfrac{k}{a_5})(a_6-a_5)=0.$$

The equation of line A_6A_1 is of the form

$$(x-a_6)(\tfrac{k}{a_1}-\tfrac{k}{a_6})-(y-\tfrac{k}{a_6})(a_1-a_6)=0.$$

Since A_{12} is the point of intersection of A_6A_1 and A_2A_3, the coordinates of point A_{12} satisfy the system of equations

$$\begin{cases}(x-a_6)(\tfrac{k}{a_1}-\tfrac{k}{a_6})-(y-\tfrac{k}{a_6})(a_1-a_6)=0.\\(x-a_2)(\tfrac{k}{a_3}-\tfrac{k}{a_2})-(y-\tfrac{k}{a_2})(a_3-a_2)=0.\end{cases}$$

Solving the system, we have

$$A_{12}=(-\tfrac{a_1a_2a_3-a_1a_2a_6-a_1a_3a_6+a_2a_3a_6}{a_1a_6-a_2a_3};\tfrac{k(a_1-a_2-a_3+a_6)}{a_1a_6-a_2a_3}).$$

Similarly, since A_{23} is the point of intersection of A_1A_2 and A_3A_4, the coordinates of point A_{23} are of the form

$$A_{23}=(\tfrac{a_1a_2a_3+a_1a_2a_4-a_1a_3a_4-a_2a_3a_4}{a_1a_2-a_3a_4};\tfrac{k(a_1+a_2-a_3-a_4)}{a_1a_2-a_3a_4}).$$

A_{34} is the point of intersection of A_2A_3 and A_4A_5, so the coordinates of point A_{34} are of the form

$$A_{34}=(\tfrac{a_2a_3a_4+a_2a_3a_5-a_2a_4a_5-a_3a_4a_5}{a_2a_3-a_4a_5};\tfrac{k(a_2+a_3-a_4-a_5)}{a_2a_3-a_4a_5}).$$

A_{45} is the point of intersection of A_3A_4 and A_5A_6, so the coordinates of point A_{45} are of the form

$$A_{45}=(\tfrac{a_3a_4a_5+a_3a_4a_6-a_3a_5a_6-a_4a_5a_6}{a_3a_4-a_5a_6};\tfrac{k(a_3+a_4-a_5-a_6)}{a_3a_4-a_5a_6}).$$

A_{56} is the point of intersection of A_4A_5 and A_6A_1, so the coordinates of point A_{56} are of the form

$$A_{56}=(-\frac{a_1a_4a_5-a_1a_4a_6-a_1a_5a_6+a_4a_5a_6}{a_1a_6-a_4a_5};\frac{k(a_1-a_4-a_5+a_6)}{a_1a_2-a_5a_6}).$$

A_{61} is the point of intersection of A_5A_6 and A_1A_2, so the coordinates of point A_{61} are of the form

$$A_{61}=(\frac{a_1a_2a_5+a_1a_2a_6-a_1a_5a_6-a_2a_5a_6}{a_1a_2-a_5a_6};\frac{k(a_1+a_2-a_5-a_6)}{a_1a_2-a_5a_6}).$$

B_1 is the midpoint of A_1A_2, so the coordinates of point B_1 are of the form

$$x_{B_1}=\frac{x_{A_1}+x_{A_2}}{2}=\frac{1}{2}a_1+\frac{1}{2}a_2;$$
$$y_{B_1}=\frac{y_{A_1}+y_{A_2}}{2}=\frac{1}{2}\frac{k}{a_1}+\frac{1}{2}\frac{k}{a_2}.$$

The equation of line OB_1 is of the form

$$(OB_1):y-\frac{k}{a_2a_1}x=0.$$

C_1 is the midpoint of $A_{12}A_2$, so the coordinates of point C_1 are of the form

$$x_{C_1}=\frac{x_{A_{12}}+x_{A_2}}{2}=-\frac{1}{2}\frac{a_1a_2a_3-a_1a_2a_6-a_1a_3a_6+a_2a_3a_6}{a_1a_6-a_2a_3}+\frac{1}{2}a_2;$$
$$y_{C_1}=\frac{y_{A_{12}}+y_{A_2}}{2}=\frac{1}{2}\frac{k(a_1-a_2-a_3+a_6)}{a_1a_6-a_2a_3}+\frac{1}{2}\frac{k}{a_2}.$$

B_2 is the midpoint of A_2A_3, so the coordinates of point B_2 are of the form

$$x_{B_2}=\frac{x_{A_2}+x_{A_3}}{2}=\frac{1}{2}a_2+\frac{1}{2}a_3;$$
$$y_{B_2}=\frac{y_{A_2}+y_{A_3}}{2}=\frac{1}{2}\frac{k}{a_2}+\frac{1}{2}\frac{k}{a_3}.$$

The equation of line OB_2 is of the form

$$(OB_2):y-\frac{k}{a_2a_1}x=0.$$

C_2 is the midpoint of $A_{23}A_3$, so the coordinates of point C_2 are of the form

$$x_{C_2}=\frac{x_{A_{23}}+x_{A_3}}{2}=\frac{1}{2}\frac{a_1a_2a_3+a_1a_2a_4-a_1a_3a_4-a_2a_3a_4}{a_1a_2-a_3a_4}+\frac{1}{2}a_3;$$
$$y_{C_2}=\frac{y_{A_{23}}+y_{A_3}}{2}=\frac{1}{2}\frac{k(a_1+a_2-a_3-a_4)}{a_1a_2-a_3a_4}+\frac{1}{2}\frac{k}{a_3}.$$

B_3 is the midpoint of A_3A_4, so the coordinates of point B_3 are of the form

$$x_{B_3}=\frac{x_{A_3}+x_{A_4}}{2}=\frac{1}{2}a_3+\frac{1}{2}a_4;$$
$$y_{B_3}=\frac{y_{A_3}+y_{A_4}}{2}=\frac{1}{2}\frac{k}{a_3}+\frac{1}{2}\frac{k}{a_4}.$$

The equation of line OB_3 is of the form

$$(OB_3):y-\frac{k}{a_4a_3}x=0.$$

C_3 is the midpoint of $A_{34}A_4$, so the coordinates of point C_3 are of the form

$$x_{C_3}=\frac{x_{A_{34}}+x_{A_4}}{2}=\frac{1}{2}\frac{a_2a_3a_4+a_2a_3a_5-a_2a_4a_5-a_3a_4a_5}{a_2a_3-a_4a_5}+\frac{1}{2}a_4;$$
$$y_{C_3}=\frac{y_{A_{34}}+y_{A_4}}{2}=\frac{1}{2}\frac{k(a_2+a_3-a_4-a_5)}{a_2a_3-a_4a_5}+\frac{1}{2}\frac{k}{a_4}.$$

B_4 is the midpoint of A_4A_5, so the coordinates of point B_4 are of the form

$$x_{B_4} = \frac{x_{A_4}+x_{A_5}}{2} = \frac{1}{2}a_4 + \frac{1}{2}a_5;$$
$$y_{B_4} = \frac{y_{A_4}+y_{A_5}}{2} = \frac{1}{2}\frac{k}{a_4} + \frac{1}{2}\frac{k}{a_5}.$$

The equation of line OB_4 is of the form

$$(OB_4) : y - \frac{k}{a_5a_4}x = 0.$$

C_4 is the midpoint of $A_{45}A_5$, so the coordinates of point C_4 are of the form

$$x_{C_4} = \frac{x_{A_{45}}+x_{A_5}}{2} = \frac{1}{2}\frac{a_3a_4a_5+a_3a_4a_6-a_3a_5a_6-a_4a_5a_6}{a_3a_4-a_5a_6} + \frac{1}{2}a_5;$$
$$y_{C_4} = \frac{y_{A_{45}}+y_{A_5}}{2} = \frac{1}{2}\frac{k(a_3+a_4-a_5-a_6)}{a_3a_4-a_5a_6} + \frac{1}{2}\frac{k}{a_5}.$$

B_5 is the midpoint of A_5A_6, so the coordinates of point B_5 are of the form

$$x_{B_5} = \frac{x_{A_5}+x_{A_6}}{2} = \frac{1}{2}a_5 + \frac{1}{2}a_6;$$
$$y_{B_5} = \frac{y_{A_5}+y_{A_6}}{2} = \frac{1}{2}\frac{k}{a_5} + \frac{1}{2}\frac{k}{a_6}.$$

The equation of line OB_5 is of the form

$$(OB_5) : y - \frac{k}{a_6a_5}x = 0.$$

C_5 is the midpoint of $A_{56}A_6$, so the coordinates of point C_5 are of the form

$$x_{C_5} = \frac{x_{A_{56}}+x_{A_6}}{2} = -\frac{1}{2}\frac{a_1a_4a_5-a_1a_4a_6-a_1a_5a_6+a_4a_5a_6}{a_1a_6-a_4a_5} + \frac{1}{2}a_6;$$
$$y_{C_5} = \frac{y_{A_{56}}+y_{A_6}}{2} = \frac{1}{2}\frac{k(a_1-a_4-a_5+a_6)}{a_1a_6-a_4a_5} + \frac{1}{2}\frac{k}{a_6}.$$

B_6 is the midpoint of A_6A_1, so the coordinates of point B_6 are of the form

$$x_{B_6} = \frac{x_{A_6}+x_{A_1}}{2} = \frac{1}{2}a_6 + \frac{1}{2}a_1;$$
$$y_{B_6} = \frac{y_{A_6}+y_{A_1}}{2} = \frac{1}{2}\frac{k}{a_6} + \frac{1}{2}\frac{k}{a_1}.$$

The equation of line OB_6 is of the form

$$(OB_6) : y - \frac{k}{a_1a_6}x = 0.$$

C_6 is the midpoint of $A_{61}A_1$, so the coordinates of point C_6 are of the form

$$x_{C_6} = \frac{x_{A_{61}}+x_{A_1}}{2} = \frac{1}{2}\frac{a_1a_2a_5+a_1a_2a_6-a_1a_5a_6-a_2a_5a_6}{a_1a_2-a_5a_6} + \frac{1}{2}a_1;$$
$$y_{C_6} = \frac{y_{A_{61}}+y_{A_1}}{2} = \frac{1}{2}\frac{k(a_1+a_2-a_5-a_6)}{a_1a_2-a_5a_6} + \frac{1}{2}\frac{k}{a_1}.$$

The equation of line d_{C_1} passing through point C_1 parallel to OB_2 is of the form

$$(d_{C_1}) : \frac{x+\frac{1}{2}\frac{a_1a_2a_3-a_1a_2a_6-a_1a_3a_6+a_2a_3a_6}{a_1a_6-a_2a_3}-\frac{1}{2}a_2}{\frac{1}{2}a_2+\frac{1}{2}a_3} - \frac{y-\frac{1}{2}\frac{k(a_1-a_2-a_3+a_6)}{a_1a_6-a_2a_3}-\frac{1}{2}\frac{k}{a_2}}{\frac{1}{2}\frac{k}{a_2}+\frac{1}{2}\frac{k}{a_3}} = 0$$

$O_{12} = d_{C_1} \bigcap OB_1$, so the coordinates of point O_{12} are of the form

$$O_{12} = \left(-\frac{a_2a_1(a_3-a_6)}{a_1a_6-a_2a_3}; -\frac{k(a_3-a_6)}{a_1a_6-a_2a_3}\right)$$

The equation of line d_{C_2} passing through point C_2 parallel to OB_3 is of the form

$$(d_{C_2}): \frac{x-\frac{1}{2}\frac{a_1a_2a_3+a_1a_2a_4-a_1a_3a_4-a_2a_3a_4}{a_1a_2-a_3a_4}-\frac{1}{2}a_3}{\frac{1}{2}a_3+\frac{1}{2}a_4} - \frac{y-\frac{1}{2}\frac{k(a_1+a_2-a_3-a_4)}{a_1a_2-a_3a_4}-\frac{1}{2}\frac{k}{a_3}}{\frac{1}{2}\frac{k}{a_3}+\frac{1}{2}\frac{k}{a_4}} = 0$$

$O_{23} = d_{C_2} \bigcap OB_2$, so the coordinates of point O_{23} are of the form

$$O_{23} = (\frac{a_2a_3(a_1-a_4)}{a_1a_2-a_3a_4}; \frac{k(a_1-a_4)}{a_1a_2-a_3a_4})$$

The equation of line d_{C_3} passing through point C_3 parallel to OB_4 is of the form

$$(d_{C_3}): \frac{x-\frac{1}{2}\frac{a_2a_3a_4+a_2a_3a_5-a_2a_4a_5-a_3a_4a_5}{a_2a_3-a_4a_5}-\frac{1}{2}a_4}{\frac{1}{2}a_4+\frac{1}{2}a_5} - \frac{y-\frac{1}{2}\frac{k(a_2+a_3-a_4-a_5)}{a_2a_3-a_4a_5}-\frac{1}{2}\frac{k}{a_4}}{\frac{1}{2}\frac{k}{a_4}+\frac{1}{2}\frac{k}{a_5}} = 0$$

$O_{34} = d_{C_3} \bigcap OB_3$, so the coordinates of point O_{34} are of the form

$$O_{34} = (\frac{a_4a_3(a_2-a_5)}{a_2a_3-a_4a_5}; \frac{k(a_2-a_5)}{a_2a_3-a_4a_5})$$

The equation of line d_{C_4} passing through point C_4 parallel to OB_5 is of the form

$$(d_{C_4}): \frac{x-\frac{1}{2}\frac{a_3a_4a_5+a_3a_4a_6-a_3a_5a_6-a_4a_5a_6}{a_3a_4-a_5a_6}-\frac{1}{2}a_5}{\frac{1}{2}a_5+\frac{1}{2}a_6} - \frac{y-\frac{1}{2}\frac{k(a_3+a_4-a_5-a_6)}{a_3a_4-a_5a_6}-\frac{1}{2}\frac{k}{a_5}}{\frac{1}{2}\frac{k}{a_5}+\frac{1}{2}\frac{k}{a_6}} = 0$$

$O_{45} = d_{C_4} \bigcap OB_4$, so the coordinates of point O_{45} are of the form

$$O_{45} = (\frac{(a_3-a_6)a_5a_4}{a_3a_4-a_5a_6}; \frac{k(a_3-a_6)}{a_3a_4-a_5a_6})$$

The equation of line d_{C_5} passing through point C_5 parallel to OB_6 is of the form

$$(d_{C_5}): \frac{x+\frac{1}{2}\frac{a_1a_4a_5-a_1a_4a_6-a_1a_5a_6+a_4a_5a_6}{a_1a_6-a_4a_5}-\frac{1}{2}a_6}{\frac{1}{2}a_6+\frac{1}{2}a_1} - \frac{y-\frac{1}{2}\frac{k(a_1-a_4-a_5+a_6)}{a_1a_6-a_4a_5}-\frac{1}{2}\frac{k}{a_6}}{\frac{1}{2}\frac{k}{a_6}+\frac{1}{2}\frac{k}{a_1}} = 0$$

$O_{56} = d_{C_5} \bigcap OB_5$, so the coordinates of point O_{56} are of the form

$$O_{56} = (\frac{(a_1-a_4)a_6a_5}{a_1a_6-a_4a_5}; \frac{k(a_1-a_4)}{a_1a_6-a_4a_5})$$

The equation of line d_{C_6} passing through point C_6 parallel to OB_1 is of the form

$$(d_{C_6}): \frac{x-\frac{1}{2}\frac{a_1a_2a_5+a_1a_2a_6-a_1a_5a_6-a_2a_5a_6}{a_1a_2-a_5a_6}-\frac{1}{2}a_1}{\frac{1}{2}a_1+\frac{1}{2}a_2} - \frac{y-\frac{1}{2}\frac{k(a_1+a_2-a_5-a_6)}{a_1a_2-a_5a_6}-\frac{1}{2}\frac{k}{a_1}}{\frac{1}{2}\frac{k}{a_1}+\frac{1}{2}\frac{k}{a_2}} = 0$$

$O_{61} = d_{C_6} \bigcap OB_6$, so the coordinates of point O_{61} are of the form

$$O_{61} = (\frac{a_1a_6(a_2-a_5)}{a_1a_2-a_5a_6}; \frac{k(a_2-a_5)}{a_1a_2-a_5a_6})$$

The equation of line $O_{12}O_{45}$ is of the form

$$(O_{12}O_{45}): (x+\frac{a_2a_1(a_3-a_6)}{a_1a_6-a_2a_3})(\frac{k(a_3-a_6)}{a_3a_4-a_5a_6}+\frac{k(a_3-a_6)}{a_1a_6-a_2a_3})-(y+\frac{k(a_3-a_6)}{a_1a_6-a_2a_3})(\frac{(a_3-a_6)a_5a_4}{a_3a_4-a_5a_6}+\frac{a_2a_1(a_3-a_6)}{a_1a_6-a_2a_3} = 0$$

The equation of line $O_{23}O_{56}$ is of the form

$$(O_{23}O_{56}): (x-\frac{a_2a_3(a_1-a_4)}{a_1a_2-a_3a_4})(\frac{k(a_1-a_4)}{a_1a_6-a_4a_5}-\frac{k(a_1-a_4)}{a_1a_2-a_3a_4})-(y-\frac{k(a_1-a_4)}{a_1a_2-a_3a_4})(\frac{(a_1-a_4)a_6a_5}{a_1a_6-a_4a_5}-\frac{a_2a_3(a_1-a_4)}{a_1a_2-a_3a_4}) = 0$$

$Z = O_{12}O_{45} \bigcap O_{23}O_{56}$, so the coordinates of point $Z = (x_Z; y_Z)$ are of the form

$$x_Z = (a_1^2a_2^2a_3^2a_4 - a_1^2a_2^2a_3^2a_6 + a_1^2a_2^2a_3a_6^2 - a_1^2a_2^2a_5a_6^2 + a_1^2a_2a_5^2a_6^2 - a_1^2a_4a_5^2a_6^2 - a_1a_2^2a_3^2a_4^2 + a_1a_4^2a_5^2a_6^2 + a_2^2a_3^2a_4^2a_5 - a_2a_3^2a_4^2a_5^2 + a_3^2a_4^2a_5^2a_6 - a_3a_4^2a_5^2a_6^2)/(a_1^2a_2^2a_3a_4 - a_1^2a_2^2a_5a_6 - a_1^2a_2a_3a_4a_6 + a_1^2a_2a_3a_6^2 + a_1^2a_2a_4a_5a_6 - a_1^2a_4a_5a_6^2 - a_1a_2^2a_3^2a_6 - a_1a_2^2a_3a_4a_5 + a_1a_2^2a_3a_5a_6 - a_1a_2a_3^2a_4^2 + a_1a_2a_3^2a_4a_6 + a_1a_2a_3a_4^2a_5 - a_1a_2a_3a_5a_6^2 - a_1a_2a_4a_5^2a_6 + a_1a_2a_5^2a_6^2 - a_1a_3a_4^2a_5a_6 + a_1a_3a_4a_5a_6^2 + a_1a_4^2a_5^2a_6 + a_2^2a_3^2a_4a_5 - a_2a_3^2a_4a_5a_6 - a_2a_3a_4^2a_5^2 + a_2a_3a_4a_5^2a_6 + a_3^2a_4^2a_5a_6 - a_3a_4a_5^2a_6^2);$$

$$y_Z = (k(a_1^2a_2^2a_3 - a_1^2a_2^2a_6 + a_1^2a_2a_6^2 - a_1^2a_5a_6^2 - a_1a_2^2a_3^2 + a_1a_5^2a_6^2 + a_2^2a_3^2a_4 - a_2a_3^2a_4^2 + a_3^2a_4^2a_5 - a_3a_4^2a_5^2 + a_4^2a_5^2a_6 - a_4a_5^2a_6^2))/(a_1^2a_2^2a_3a_4 - a_1^2a_2^2a_5a_6 - a_1^2a_2a_3a_4a_6 + a_1^2a_2a_3a_6^2 + a_1^2a_2a_4a_5a_6 - a_1^2a_4a_5a_6^2 - a_1a_2^2a_3^2a_6 - a_1a_2^2a_3a_4a_5 + a_1a_2^2a_3a_5a_6 - a_1a_2a_3^2a_4^2 + a_1a_2a_3^2a_4a_6 + a_1a_2a_3a_4^2a_5 - a_1a_2a_3a_5a_6^2 - a_1a_2a_4a_5^2a_6 + a_1a_2a_5^2a_6^2 - a_1a_3a_4^2a_5a_6 + a_1a_3a_4a_5a_6^2 + a_1a_4^2a_5^2a_6 + a_2^2a_3^2a_4a_5 - a_2a_3^2a_4a_5a_6 - a_2a_3a_4^2a_5^2 + a_2a_3a_4a_5^2a_6 + a_3^2a_4^2a_5a_6 - a_3a_4a_5^2a_6^2).$$

Consider the determinant

$$\Delta = (x_Z - x_{O_{56}})(y_{O_{23}} - y_{O_{56}}) - (y_Z - y_{O_{56}})(x_{O_{23}} - x_{O_{56}}).$$

We have

$$\Delta = ((a_1^2a_2^2a_3^2a_4 - a_1^2a_2^2a_3^2a_6 + a_1^2a_2^2a_3a_6^2 - a_1^2a_2^2a_5a_6^2 + a_1^2a_2a_5^2a_6^2 - a_1^2a_4a_5^2a_6^2 - a_1a_2^2a_3^2a_4^2 + a_1a_4^2a_5^2a_6^2 + a_2^2a_3^2a_4^2a_5 - a_2a_3^2a_4^2a_5^2 + a_3^2a_4^2a_5^2a_6 - a_3a_4^2a_5^2a_6^2)/(a_1^2a_2^2a_3a_4 - a_1^2a_2^2a_5a_6 - a_1^2a_2a_3a_4a_6 + a_1^2a_2a_3a_6^2 + a_1^2a_2a_4a_5a_6 - a_1^2a_4a_5a_6^2 - a_1a_2^2a_3^2a_6 - a_1a_2^2a_3a_4a_5 + a_1a_2^2a_3a_5a_6 - a_1a_2a_3^2a_4^2 + a_1a_2a_3^2a_4a_6 + a_1a_2a_3a_4^2a_5 - a_1a_2a_3a_5a_6^2 - a_1a_2a_4a_5^2a_6 + a_1a_2a_5^2a_6^2 - a_1a_3a_4^2a_5a_6 + a_1a_3a_4a_5a_6^2 + a_1a_4^2a_5^2a_6 + a_2^2a_3^2a_4a_5 - a_2a_3^2a_4a_5a_6 - a_2a_3a_4^2a_5^2 + a_2a_3a_4a_5^2a_6 + a_3^2a_4^2a_5a_6 - a_3a_4a_5^2a_6^2) - \frac{(a_1-a_4)a_6a_5}{a_1a_6-a_4a_5})(\frac{k(a_1-a_4)}{a_1a_2-a_3a_4} - \frac{k(a_1-a_4)}{a_1a_6-a_4a_5}) - ((k(a_1^2a_2^2a_3 - a_1^2a_2^2a_6 + a_1^2a_2a_6^2 - a_1^2a_5a_6^2 - a_1a_2^2a_3^2 + a_1a_5^2a_6^2 + a_2^2a_3^2a_4 - a_2a_3^2a_4^2 + a_3^2a_4^2a_5 - a_3a_4^2a_5^2 + a_4^2a_5^2a_6 - a_4a_5^2a_6^2))/(a_1^2a_2^2a_3a_4 - a_1^2a_2^2a_5a_6 - a_1^2a_2a_3a_4a_6 + a_1^2a_2a_3a_6^2 + a_1^2a_2a_4a_5a_6 - a_1^2a_4a_5a_6^2 - a_1a_2^2a_3^2a_6 - a_1a_2^2a_3a_4a_5 + a_1a_2^2a_3a_5a_6 - a_1a_2a_3^2a_4^2 + a_1a_2a_3^2a_4a_6 + a_1a_2a_3a_4^2a_5 - a_1a_2a_3a_5a_6^2 - a_1a_2a_4a_5^2a_6 + a_1a_2a_5^2a_6^2 - a_1a_3a_4^2a_5a_6 + a_1a_3a_4a_5a_6^2 + a_1a_4^2a_5^2a_6 + a_2^2a_3^2a_4a_5 - a_2a_3^2a_4a_5a_6 - a_2a_3a_4^2a_5^2 + a_2a_3a_4a_5^2a_6 + a_3^2a_4^2a_5a_6 - a_3a_4a_5^2a_6^2) - \frac{k(a_1-a_4)}{a_1a_6-a_4a_5})(\frac{a_2a_3(a_1-a_4)}{a_1a_2-a_3a_4} - \frac{(a_1-a_4)a_6a_5}{a_1a_6-a_4a_5}).$$

Using the software of Maple 2015 and simplifying Δ, we obtain $\Delta = 0$. Thus $O_{12}O_{45}, O_{23}O_{56}$, $O_{34}O_{61}$ are concurrent at a point (Q. E. D).
By the extensional method from a circle to a conic having center as above, we find out and prove many theorems and problems. For example:

Problem 7 (The generalization of Simson theorem)**.** *Given a conic* (S) *having center* O*.* A, B, C, M *are points belonging to* (S)*. Let* A', B', C' *be the midpoints of segments* BC, CA, AB*, respectively. Through point* M *draw straight lines parallel to* OA', OB', OC' *and meet* BC, CA, AB *at* A'', B'', C''*, respectively. Prove that* A'', B'', C'' *are collinear.*

Problem 8 (The extension of Dao's generalization of the Simson line theorem)**.** *Given a triangle* ABC *inscribed in a conic* (S) *having center* O*, let* P *be an arbitrary point on* (S)*. Let* d *be an arbitrary line passing through* O *and meeting* PA, PB, PC *at* D, E, F*, respectively. Let* A', B', C' *be the midpoints of* BC, CA, AB*, respectively. The lines passing through* D, E, F *parallel to* OA', OB', OC' *meet* BC, CA, AB *at* A'', B'', C''*, respectively. Prove that* A'', B'', C'' *are collinear (figure 3).*

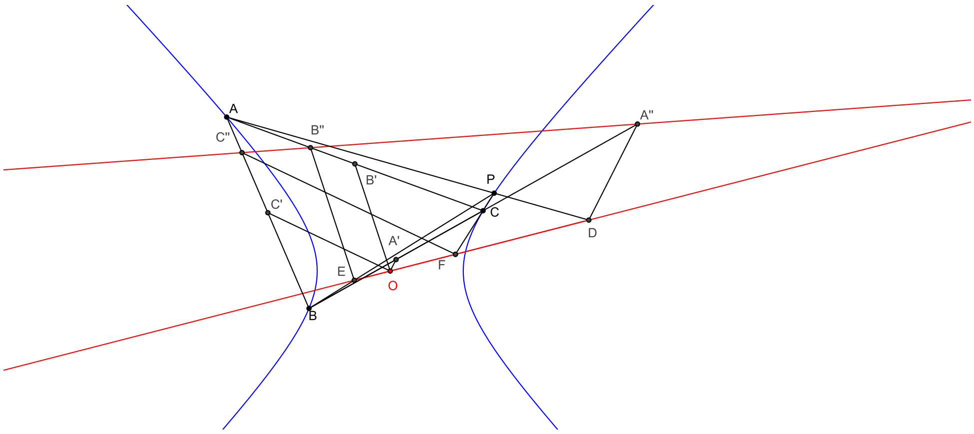

FIGURE 3. The extension of Dao's generalization of the Simson line theorem

Problem 9 (The extension of the angle bisector theorem)**.** *Given a hyperbola (H) with two branches are (H_1) and (H_2). Let A, B, C be three points on the branch (H_1), O be the center of the hyperbola. Let A', B', C' be the midpoints of sides BC, CA, AB. $OA' \bigcap (H_2) = A''$. Similarly to B'', C". Prove that AA'', BB'', CC'' are concurrent at a point (Figure 4).*

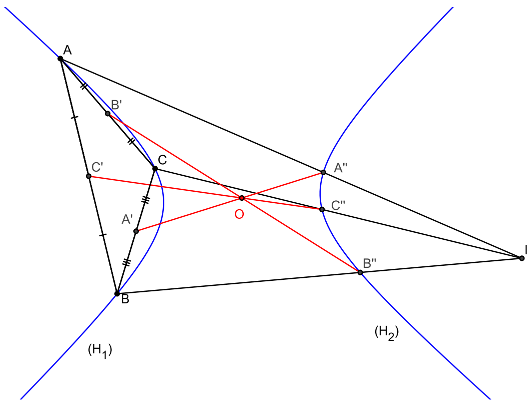

FIGURE 4. The extension of the angle bisector theorem

Remarks
1. The method of generalizing from a circle to a conic having center is a creative method of new theorems. This method should apply to the problem that its hypothesis only has a circle and its related objects in the hypothesis are lines, segments, rays, intersection points, ect.
2. When problem satisfies the remark 1, we can generalize the problem from a circle to an ellipse. However, if we want to extend the problem from a circle to a hyperbola, then we need to consider that does the problem for hyperbola exist? The illustrated example for

this thing is the problem 9 as above, A, B are on the branch (H_1), C is on the branch (H_2) then the problem for hyperbola does not exist.
The following is a new result that does not prove.

Problem 10 (The generalization of The Taylor circle 's theorem)**.** *Given a triangle ABC. Let M, N, P be the midpoints of segments BC, CA, AB, respectively. Let O be the arbitrary point that does not lie on lines containing sides of the triangle. Let A' be the point of intersection of the line passing through point A parallel to OM and BC. Similarly to B', C'. Through point A' draw lines parallel to ON, OP and meet AC, AB at A_2, A_1, respectively. Through point B' draw lines parallel to OP, OM and meet BA, BC at B_2, B_1 respectively. Through point C' draw lines parallel to OM, ON and meet CB, CA at C_2, C_1. Prove that 6 points $A_1, C_2, B_1, A_2, C_1, B_2$ lie on a conic (S) (figure 5).*

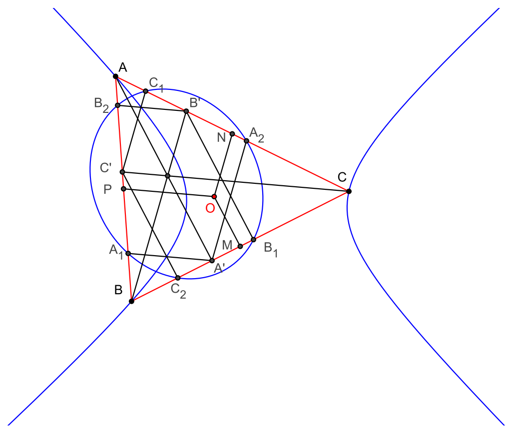

FIGURE 5. The generalization of The Taylor circle's theorem

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