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Computer Discovered Mathematics: Orthopoles

Sava Grozdev[a], Hiroshi Okumura[b] and Deko Dekov[c] [2]
[a] VUZF University of Finance, Business and Entrepreneurship,
Gusla Street 1, 1618 Sofia, Bulgaria
e-mail: sava.grozdev@gmail.com
[b] Department of Mathematics, Yamato University, Osaka, Japan
e-mail: okumura.hiroshi@yamato-u.ac.jp
[c]Zahari Knjazheski 81, 6000 Stara Zagora, Bulgaria
e-mail: ddekov@ddekov.eu
web: http://www.ddekov.eu/

Abstract. By using the computer program “Discoverer”, we present theorems about orthopoles of lines.

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1. Introduction

The computer program “Discoverer”, created by Grozdev and Dekov, with collaboration by Hiroshi Okumura, is the first computer program, able easily to discover new theorems in mathematics, and the first computer program, able easily to discover new knowledge in science. See [3].

In this paper, by using the "Discoverer", we investigate the orthopoles of lines. We expect that the majority of the theorems are new, discovered by a computer.

2. Orthopole of a Line

In the famous book by Hristo Hitov [6], the Problem 1014 is as follows (See also [12], Exercise 5, page 56, [10],Orthopole):

[2]Corresponding author

Problem 2.1. *Consider a triangle ABC and a line $\mathcal{L}$. Project vertices A, B and C on this line, to points Qa, Qb and Qc respectively. Denote by L_1 the line through point Qa and perpendicular to the line BC, by L_2 the line through point Qb and perpendicular to the line CA and by L_3 the line through point Qc and perpendicular to the line AB. Prove that the lines L_1, L_2 and L_3 concur in a point.*

Point of concurrence of the lines in the Problem 2.1 is called the *orthopole of line $\mathcal{L}$ with respect to triangle ABC.*

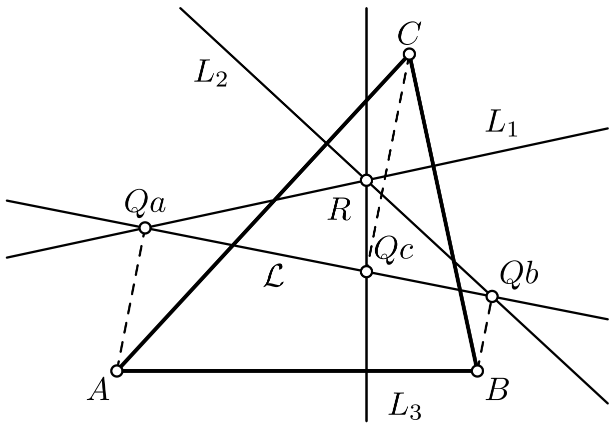

FIGURE 1.

Figure 1 illustrates the construction of the orthopole. In Figure 1, $\mathcal{L}$ is an arbitrary line, Qa is th intersection point of line $\mathcal{L}$ and the line through A and perpendicular to $\mathcal{L}$. Similarly we define points Qb and Qc. Line L_1 is the line through Qa and perpendicular to the line BC. Similarly we define the lines L_2 and L_3. Then the lines L_1, L_2 and L_3 concur in point R, the orthopole of line $\mathcal{L}$.

The orthopole of the line $\mathcal{L}$ through the Circumcenter lies on the Nine-Point Circle (See [10], Orthopole).

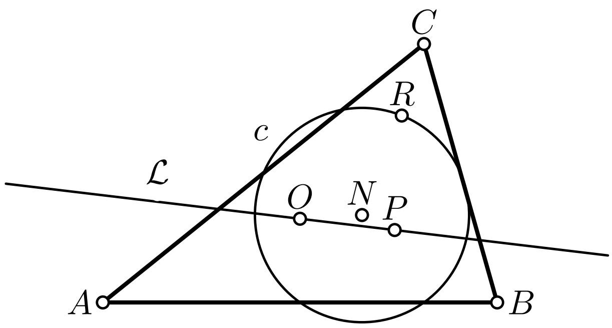

FIGURE 2.

Figure 2 illustrates the position of the orthopole of a line through the Circumcenter. In Figure 2, O is the Circumcenter of triangle ABC, P is an arbitrary point, $\mathcal{L}$ is the line through points O and P, point N is the center of the Nine-Point Circle and c is the Nine-Point Circle. Then the orthopole R of line $\mathcal{L}$ lies on the Nine-Point Circle.

3. Preliminaries

In this section we review some basic facts about barycentric coordinates. We refer the reader to [4],[5],[9],[8],[12],[1].

The labeling of triangle centers follows Kimberling's ETC [7]. For example, X(1) denotes the Incenter, X(2) denotes the Centroid, X(37) is the Grinberg Point, X(75) is the Moses point (Note that in the prototype of the "Discoverer" the Moses point is the X(35)), etc.

The reader may find definitions in [10],[11],[2].

The reference triangle ABC has vertices $A = (1, 0, 0), B(0, 1, 0)$ and C(0,0,1). The side lengths of $\triangle ABC$ are denoted by $a = BC, b = CA$ and $c = AB$. A point is an element of $\mathbb{R}^3$, defined up to a proportionality factor, that is,

For all $k \in \mathbb{R} - \{0\}$: $P = (u, v, w)$ means that $P = (u, v, w) = (ku, kv, kw)$.

The equation of the line joining two points with coordinates u_1, v_1, w_1 and u_2, v_2, w_2 is

$$\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ x & y & z \end{vmatrix} = 0. \tag{3.1}$$

The infinite point of a line $L : px + qy + rz = 0$ is the point (f, g, h), where $f = q - r$, $g = r - p$ and $h = p - q$. The equation of the line through point $P(u, v, w)$ and perpendicular to the line $L : px + qy + rz = 0$ is as follows (The method is discovered by Floor van Lamoen):

$$\begin{vmatrix} F & G & H \\ u & v & w \\ x & y & z \end{vmatrix} = 0 \tag{3.2}$$

where $F = S_B g - S_C h$, $G = S_C h - S_A f$, and $H = S_A f - S_B g$.

The intersection of two lines $L_1 : p_1x + q_1y + r_1z = 0$ and $L_2 : p_2x + q_2y + r_2z = 0$ is the point

$$(q_1r_2 - q_2r_1, r_1p_2 - r_2p_1, p_1q_2 - p_2q_1) \tag{3.3}$$

Three lines $p_ix + q_iy + r_iz = 0$, $i = 1, 2, 3$ are concurrent if and only if

$$\begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix} = 0 \tag{3.4}$$

4. Theorems

The computer program "Discoverer" has discovered theorems about orthopoles. A few of the results discovered by the "Discoverer" are given below:

Theorem 4.1. *Given triangle ABC with side lengths $a = BC$, $b = CA$ and $c = AB$. Let $\mathcal{L}$ be the line through points $P = (u, v, w)$ and $Q = (p, q, r)$. Denote the barycentric coordinate of the orthopole R of the line $(\mathcal{L})$ as follows: $R = (uR, vR, wR)$. Then:*

$$uR = (2b^4ru + c^4rv + b^4rv + 2c^2qb^2u - 2c^2rb^2u - 2pc^2vb^2 + 2b^2pwa^2 - 2b^2ura^2$$
$$+2c^2uqa^2 - 2c^2pva^2 - 2pb^4w + 2pb^2c^2w - vra^4 + qwa^4 - 2c^4qu + 2pc^4v$$

$$-2c^2rb^2v - c^4qw - b^4qw + 2c^2qb^2w)(b^2uq - b^2pv - c^2uq + c^2pv - a^2uq$$
$$+a^2pv + 2a^2vr - 2a^2qw + c^2pw - c^2ur - a^2pw + a^2ur - b^2pw + b^2ur).$$

To obtain vR, in uR substitute $a, b, c, u, v, w, p, q, r$ for $b, c, a, v, w, u, q, r, p$ respectively, and to obtain wR, in vR substitute $a, b, c, u, v, w, p, q, r$ for $b, c, a, v, w, u, q, r, p$ respectively.

Theorem 4.2. *The Feuerbach Point is the Orthopole of the Line through the Circumcenter and the*

(1) *Incenter;*
(2) *X(35) Perspector of the Intangents Triangle and the Kosnita Triangle;*
(3) *X(36) Inverse of the Incenter in the Circumcircle;*
(4) *Circumcenter and X(40) Bevan Point;*
(5) *X(46) Perspector of the Excentral Triangle and the Orthic Triangle;*
(6) *X(55) Internal Center of Similitude of the Incircle and the Circumcircle;*
(7) *X(56) External Center of Similitude of the Incircle and the Circumcircle;*
(8) *X(57) Isogonal Conjugate of the Mittenpunkt;*
(9) *X(65) Orthocenter of the Intouch Triangle;*
(10) *X(354) Weill Point;*
(11) *X(484) Evans Perspector;*
(12) *X(1155) Schroder Point;*
(13) *X(1319) Bevan-Schroder Point;*
(14) *X(3333) Pohoata Point.*

Theorem 4.3. *The X(115) Kiepert Center is the Orthopole of the Line through the Circumcenter and the*

(1) *X(6) Symmedian Point;*
(2) *X(15) First Isodynamic Point;*
(3) *X(16) Second Isodynamic Point;*
(4) *X(32) Third Power Point;*
(5) *X(39) Brocard Midpoint;*
(6) *X(50) Product of the First Isodynamic Point and the Second Isodynamic Point;*
(7) *X(52) Orthocenter of the Orthic Triangle;*
(8) *X(58) Isogonal Conjugate of the Spieker Center;*
(9) *X(61) Isogonal Conjugate of the Outer Napoleon Point;*
(10) *X(62) Isogonal Conjugate of the Inner Napoleon Point;*
(11) *X(182) Center of the Brocard Circl;*

Theorem 4.4. *The X(125) Center of the Jerabek Hyperbola is the Orthopole of the Line through the Circumcenter and the*

(1) *Centroid;*
(2) *Orthocenter;*
(3) *Nine-Point Center;*
(4) *X(20) de Longchamps Point;*
(5) *X(21) Schiffler Point;*
(6) *X(22) Exeter Point;*
(7) *X(23) Far-Out Point;*
(8) *X(24) Perspector of the Kosnita Triangle and the Orthic Triangle;*
(9) *X(25) Product of the Orthocenter and the Symmedian Point;*

(10) *X(26) Circumcenter of the Tangential Triangle;*
(11) *X(27) Quotient of the Orthocenter and the Spieker Center;*
(12) *X(28) Quotient of the Clawson Point and the Spieker Center;*
(13) *X(29) Ceva Product of the Incenter and the Orthocenter;*
(14) *X(381) Center of the Orthocentroidal Circle;*
(15) *X(384) Conway Point;*

Note that the "Discoverer" has discovered many other remarkable points that are orthopoles of lines through the Circumcenter. For example: Points X(122), X(124), X(127), X(130), X(134), X(135), X(136), X(137) and so on.

Theorem 4.5. *The Feuerbach Point is the Orthopole of the Line through the*

(1) *Incenter and the Bevan Point.*
(2) *Incenter and X(55) Internal Center of Similitude of the Incircle and the Circumcircle.*
(3) *Incenter and X(56) External Center of Similitude of the Incircle and the Circumcircle.*
(4) *Bevan Point and X(55) Internal Center of Similitude of the Incircle and the Circumcircle.*
(5) *X(40) Bevan Point and X(56) External Center of Similitude of the Incircle and the Circumcircle.*
(6) *X(55) Internal Center of Similitude of the Incircle and the Circumcircle and X(56) External Center of Similitude of the Incircle and the Circumcircle.*

Theorem 4.6. *The X(115) Kiepert Center is the Orthopole of the Line through the*

(1) *X(15) First Isodynamic Point and X(6) Symmedian Point.*
(2) *X(15) First Isodynamic Point and X(16) Second Isodynamic Point.*
(3) *X(15) First Isodynamic Point and X(39) Brocard Midpoint.*
(4) *X(16) Second Isodynamic Point and X(6) Symmedian Point.*
(5) *X(16) Second Isodynamic Point and X(39) Brocard Midpoint.*
(6) *X(39) Brocard Midpoint and X(6) Symmedian Point.*

Theorem 4.7. *The X(125) Center of the Jerabek Hyperbola is the Orthopole of the Line through the*

(1) *Centroid and the Orthocenter;*
(2) *Centroid and Nine-Point Center;*
(3) *Centroid and X(20) de Longchamps Point*
(4) *Centroid and X(21) Schiffler Point;*
(5) *Centroid and X(22) Exeter Point;*
(6) *X(20) de Longchamps Point and X(4) Orthocenter;*
(7) *X(20) de Longchamps Point and X(21) Schiffler Point.*

Note that the "Discoverer" has discovered many other remarkable points that are orthopoles of lines through remarkable points.

Theorem 4.8. *The following points are not available in the Kimberling's Encyclopedia ETC* [7]*: The Orthopole of the Line through the*

(1) *Incenter and Centroid.*

(2) *Incenter and the Orthocenter.*
(3) *Incenter and X(5) Nine-Point Center.*
(4) *Incenter and X(6) Symmedian Point.*
(5) *Incenter and X(8) Nagel Point.*
(6) *Incenter and X(9) Mittenpunkt.*
(7) *Incenter and X(10) Spieker Center.*
(8) *Centroid and X(6) Symmedian Point.*
(9) *Centroid and X(7) Gergonne Point.*
(10) *Centroid and X(8) Nagel Point.*
(11) *Centroid and X(9) Mittenpunkt.*
(12) *Centroid and X(37) Grinberg Point.*
(13) *Circumcenter and X(7) Gergonne Point.*
(14) *Circumcenter and X(11) Feuerbach Point.*
(15) *Circumcenter and X(12) Feuerbach Perspector.*

Note that the above results could be easily extended by the "Discoverer".

5. Proofs

Proof of theorem 4.1.

We use barycentric coordinates. By using (3.1) we find the equation of the line $\mathcal{L}$ through points $P = (u, v, w)$ and $Q = (p, q, r)$ as follows:

$$(vr - qw)x + (wp - ru)y + (uq - pv)z = 0.$$

Then by using (3.2) we find the equation of the Line L_1 through point A and perpendicular to line $\mathcal{L}$. Similarly, we find the equation of the Line L_2 through point B and perpendicular to line $\mathcal{L}$, and the equation of the Line L_3 through point C and perpendicular to line $\mathcal{L}$.

By using (3.3) we find the intersection point Qa of lines L_1 and BC, Similarly, we find the intersection point Qb of lines L_2 and CA, and the intersection point Qc of lines L_3 and BC,

Then by using (3.2) we find the equation of the Line L_4 through point Qa and perpendicular to line BC. Similarly, we find the equation of the Line L_5 through point Qb and perpendicular to line CA, and the equation of the Line L_6 through point Qc and perpendicular to line AB.

By using (3.4) we see that the line L_4, L_5 and L_6 concur in a point. By using (3.3) we find the orthopole R as the point of intersection of Lines L_1 and L_2. The barycentric coordinates of the orthopole are given in the statement of the theorem. □

Proof of theorem 4.2.1.

The Incenter has barycentric coordinates $I = (a, b, c)$ and the Circumcenter has barycentric coordinates $O = (a^2(b^2 + c^2 - a^2), b^2(c^2 + a^2 - b^2), c^2(a^2 + b^2 - c^2))$. In the statement of theorem 4.1 we substitute P for the Incenter and Q for the Circumcenter. Then the barycentric coordinates of the orthopole are as follow:

$$((-a + b + c)(b - c)^2, (-b + c + a)(c - a)^2, (-c + a + b)(a - b)^2.).$$

This is the Feuerbach point. □

REFERENCES

[1] P. Douillet, *Translation of the Kimberling's Glossary into barycentrics*, 2012, http://www.ddekov.eu/e2/htm/links/glossary.pdf
[2] S. Grozdev and D. Dekov, Computer-Generated Encyclopeda of Euclidean Geometry, http://www.ddekov.eu/e2/index.htm
[3] S. Grozdev and D. Dekov, *A Survey of Mathematics Discovered by Computers*, International Journal of Computer Discovered Mathematics, 2015, vol.0, no.0, 3-20. http://www.journal-1.eu/2015/01/Grozdev-Dekov-A-Survey-pp.3-20.pdf.
[4] S. Grozdev and V. Nenkov, *Three Remarkable Points on the Medians of a Triangle* (Bulgarian), Sofia, Archimedes, 2012.
[5] S. Grozdev and V. Nenkov, *On the Orthocenter in the Plane and in the Space* (Bulgarian), Sofia, Archimedes, 2012.
[6] Hristo Hitov, *The Geometry of the Triangle*, Sofia, Narodna Prosveta, 1990.
[7] C. Kimberling, *Encyclopedia of Triangle Centers - ETC*, http://faculty.evansville.edu/ck6/encyclopedia/ETC.html
[8] Gerry Leversha, *The Geometry of the Triangle*, The United Kingdom Mathematical Trust, The Pathways Series no.2, 2013.
[9] G. Paskalev and I. Tchobanov, *Remarkable Points in the Triangle* (in Bulgarian), Sofa, Narodna Prosveta, 1985.
[10] E. Weisstein, *MathWorld - A Wolfram Web Resource*, http://mathworld.wolfram.com/
[11] *Wikipedia*, https://en.wikipedia.org/wiki/
[12] P. Yiu, *Introduction to the Geometry of the Triangle*, 2001, new version of 2013, http://math.fau.edu/Yiu/YIUIntroductionToTriangleGeometry130411.pdf