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Heronian Triangles of Class J: Congruent Incircles Cevian Perspective

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Abstract. In a previous paper the authors introduced Heronian triangles of class K: a congruent incircles Cevian perspective [2]. In this paper we develop Heronian triangles of class J from the same perspective, where the rational parameter $J := t_b + t_c$ and t_b, t_c are the tangents of its half angles associated with vertices B and C respectively. We show that more relationships exist between the Soddy circles and the incircle of Heronian triangles of class J.

Keywords. Heronian triangle, Congruent incircles Cevian, Soddy circles, Euclidean geometry.

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1. Introduction

In order to introduce the congruent incircles Cevian and the radii of Soddy Circles, we will recall the two propositions used in our previous paper Heronian triangles of class K: a congruent incircles Cevian perspective [2].

Proposition 1.1. *If θ denotes angle ADB for the congruent incircle cevian AD and ρ the radius of the congruent incircles, as shown in Figure 1, then:*

$$\rho = \frac{r}{1+\sqrt{t_b t_c}} = \frac{r}{a}(s - \sqrt{s(s-a)}) \ ,$$

$$\cos\theta = \frac{t_b - t_c}{t_b + t_c} = \frac{b-c}{a} \quad \text{and} \quad \sin\theta = \frac{2\sqrt{t_b t_c}}{t_b + t_c} = \frac{2\sqrt{(s-b)(s-c)}}{a}$$

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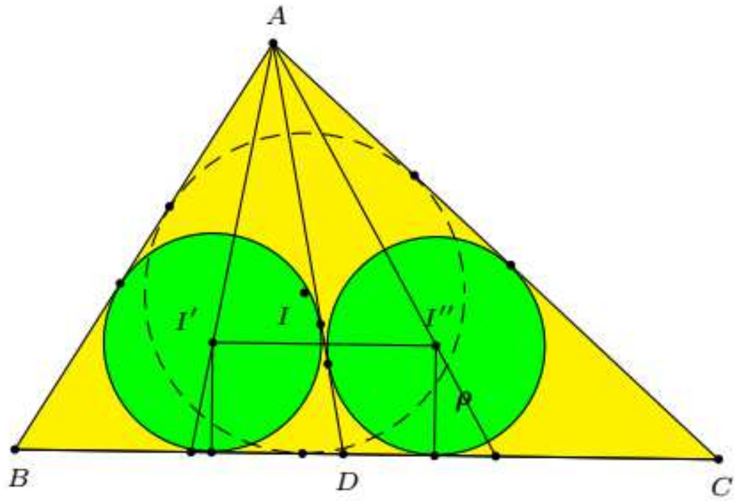

FIGURE 1. Congruent incircles

where $t_a = \tan\frac{A}{2} = \frac{r}{s-a}$, $t_b = \tan\frac{B}{2} = \frac{r}{s-b}$, $t_c = \tan\frac{C}{2} = \frac{r}{s-c}$ are the tangents of the half angles of the ΔABC, s is the semiperimeter and r is the inradius. Also $\frac{r}{r'} = t_b t_c$ where r' is the excircle opposite A and these numbers satisfy the basic relation $t_a t_b + t_b t_c + t_c t_a = 1$.

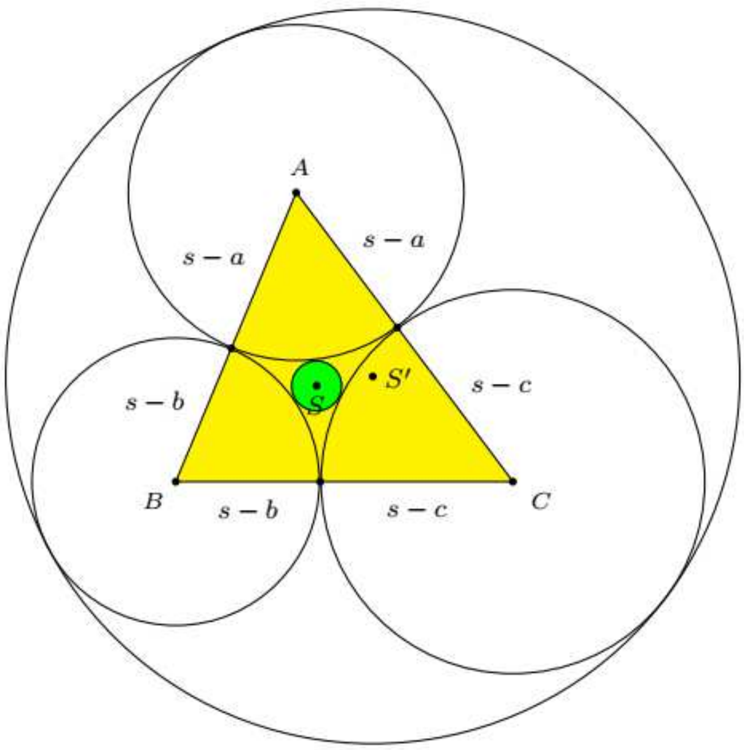

FIGURE 2. Soddy circles

Proposition 1.2. *If $S(r_i)$ and $S(r_o)$ are the inner and outer Soddy circles, as shown in Figure 2, then their radii are given by the formulas:*

$$r_i = \frac{\Delta}{4R + r + 2s} \quad \text{and} \quad r_o = \frac{\Delta}{4R + r - 2s}$$

where Δ is the area of the reference triangle, R its circumradius and r its inradius. Also it is well known that:

$$\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} + \frac{2}{r} = \frac{1}{r_i} \quad \text{and} \quad \frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} - \frac{2}{r} = \frac{1}{r_o}. \quad (1)$$

Multiplying through the above equations by r and noting that $t_a = \tan \frac{A}{2} = \frac{r}{s-a}$ gives:

$$t_a + t_b + t_c + 2 = \frac{r}{r_i} \quad \text{and} \quad t_a + t_b + t_c - 2 = \frac{r}{r_o}. \quad (2)$$

2. Generating Heron triangles

An integer triangle of class J is Heronian if and only if the tangents of its half angles are rational. We set up a J class of Heronian triangles by constraining the sum of two tangents of its half angles to be a constant J, say $J := t_b + t_c$. With reference to Figure 1, let θ be the angle ADB for the congruent incircle cevian AD then by Proposition 1 we have $t_b - t_c = (t_b + t_c)\cos\theta$. Together with $J = t_b + t_c$ and the basic relation (2) and assuming without loss of generality that $b \geq c$ then $0 \leq \cos\theta < 1$ and we have

$$t_a = \frac{4 - J^2(1 - \cos^2\theta)}{4J} \quad (3)$$

$$t_b = \frac{J(1 + \cos\theta)}{2} \quad (4)$$

$$t_c = \frac{J(1 - \cos\theta)}{2}. \quad (5)$$

Clearly, if both J and $\cos\theta$ are rational with $0 < J < \frac{2}{\sin\theta}$ and $0 \leq \cos\theta < 1$, it is possible to create rational triangles with rational areas that are proportional to any Heronian triangle.

Writing $\cos\theta = \frac{m}{n}$ for relatively prime integers $0 \leq m < n$ and using the ratios below to convert these rational triangle to integer triangles

$$s-a \;:\; s-b \;:\; s-c \;=\; \frac{1}{t_a} \;:\; \frac{1}{t_b} \;:\; \frac{1}{t_c}$$

we may take

$$s - a = -2J^2 n(m^2 - n^2)$$
$$s - b = -(m - n)(4n^2 + J^2(m^2 - n^2))$$
$$s - c = (m + n)(4n^2 + J^2(m^2 - n^2)).$$

This gives

$$a = 2n(4n^2 + J^2(m^2 - n^2)) \quad (6)$$
$$b = (m + n)(4n^2 + J^2(m - n)^2) \quad (7)$$
$$c = -(m - n)(4n^2 + J^2(m + n)^2) \quad (8)$$

with $0 < J^2 < \frac{4n^2}{n^2-m^2}$.

Also we have

$$s = 8n^3$$

$$\Delta = -4n^2 J(m^2-n^2)(4n^2+J^2(m^2-n^2))$$

$$r = \frac{-J(m^2-n^2)(4n^2+J^2(m^2-n^2))}{2n}$$

$$R = \frac{(4n^2+J^2(m-n)^2)(4n^2+J^2(m+n)^2)}{8nJ}.$$

Note that the semiperimeter s is not a function of J .

Finally from Proposition 1 we have $\cos\theta = \frac{m}{n} = \frac{b-c}{a}$. By adding and subtracting equations (6) and (7), dividing by $n^3 (= \frac{s}{8})$ and substituting $\frac{m}{n}$ with $\frac{b-c}{a}$ we are able to show the relationship between the sides in terms of J and $(\frac{b-c}{a})^2$ as

$$J = \frac{a(s-a)}{\Delta} \quad \text{and} \quad (\frac{b-c}{a})^2 = \frac{4a+s(J^2-4)}{J^2 s} \quad (9)$$

3. Heronian triangle of class $J = 1$

We have from (3) – (5)

$$t_a = \frac{3+\cos^2\theta}{4} \quad , \quad t_b = \frac{1+\cos\theta}{2} \quad , \quad t_c = \frac{1-\cos\theta}{2} \quad (10)$$

and because $0 \le \cos\theta < 1$, we have $0 < t_c \le t_b < t_a < 1$ and these Heronian triangles will always be acute with a as the longest side. Also because $t_b + t_c = 1$ we have

$$t_a + t_b + t_c < 2 \quad (11)$$

From (2) we have $t_a + t_b + t_c - 2 = \frac{r}{r_o}$ so $r_o < 0$ and the outer Soddy circle will always be internally tangent to the contact circles.

From (6) – (8) we have

$$a = 2n(m^2+3n^2)$$

$$b = (m+n)(m^2-2mn+5n^2)$$

$$c = -(m-n)(m^2+2mn+5n^2).$$

For integers $0 \le m < n \le 6$, we obtain primitive Heronian triangles of class $J = 1$ by dividing a, b, c by their greatest common denominator, as presented in the table below.

m	n	a	b	c	s	Δ	r	R
0	1	6	5	5	8	12	$\frac{3}{2}$	$\frac{25}{8}$
1	2	52	51	25	64	624	$\frac{39}{4}$	$\frac{425}{16}$
1	3	21	20	13	27	126	$\frac{14}{3}$	$\frac{65}{6}$
2	3	186	185	61	216	5580	$\frac{155}{6}$	$\frac{2257}{24}$
1	4	392	365	267	512	47040	$\frac{735}{8}$	$\frac{6497}{32}$
3	4	456	455	113	512	25536	$\frac{399}{8}$	$\frac{7345}{32}$
1	5	95	87	68	125	2850	$\frac{114}{5}$	$\frac{493}{10}$
2	5	790	763	447	1000	165900	$\frac{1659}{10}$	$\frac{16241}{40}$
3	5	105	104	41	125	2100	$\frac{84}{5}$	$\frac{533}{10}$
4	5	910	909	181	1000	81900	$\frac{819}{10}$	$\frac{18281}{40}$
1	6	1308	1183	965	1728	549360	$\frac{3815}{12}$	$\frac{32617}{48}$
5	6	1596	1595	265	1728	210672	$\frac{1463}{12}$	$\frac{38425}{48}$

From these results we can determine a number of characteristics of this class of Heronian triangles.

The relationship between the sides from (9) is $a^2 = \frac{s(s-b)(s-c)}{s-a}$ or alternatively $\Delta = a(s-a)$.

We note that for $m = 0$ (i.e. $\cos\theta = 0$ and $\theta = \frac{\pi}{2}$) generates the only isosceles Heronian triangles of this class and they are all proportional to the Heronian triple (6, 5, 5).

Consider the location of the outer Soddy circle center for this class of Heronian triangle and its radius as in Figure 3.

Theorem 3.1. *The outer Soddy circle center of Heronian triangles of Class $J = 1$ is located on its longest side at the tangent point with the excircle for that longest side. The outer Soddy circle radius is the length of its longest side.*

Proof. Without loss of generality we choose $t_b + t_c = 1$ and from (10) we have $0 < t_c \le t_b < t_a < 1$ and these Heronian triangles will always be acute with a as the longest side. Also we have $t_a = 1 - t_b t_c$.

From (2) $\frac{r}{r_o} = t_a + t_b + t_c - 2 = t_a - 1 = -t_b t_c$, hence $a = s-b+s-c = \frac{r}{t_b} + \frac{r}{t_c} = \frac{r}{t_b t_c}$ and we have $r_o = -a$.

Therefore the outer Soddy circle is internally tangent to its contact circles with a radius equal to the length of its longest side.

In a standard configuration of Soddy circles where the outer Soddy circle is internally tangent to the contact circles (see Figure 2), the distance of the Soddy center S' from the B and C vertices is given as $BS = |r_o| - (s-b)$ and $S'C = |r_o| - (s-c)$.

Consider the triangle $\Delta BS'C$, if it exists, then by the triangle inequality theorem $BS' + S'C < BC$. However for Class $J = 1$, $|r_o| = a$ and we have $BS' + S'C = a$ therefore B, S', C are co-linear and S' lies on BC.

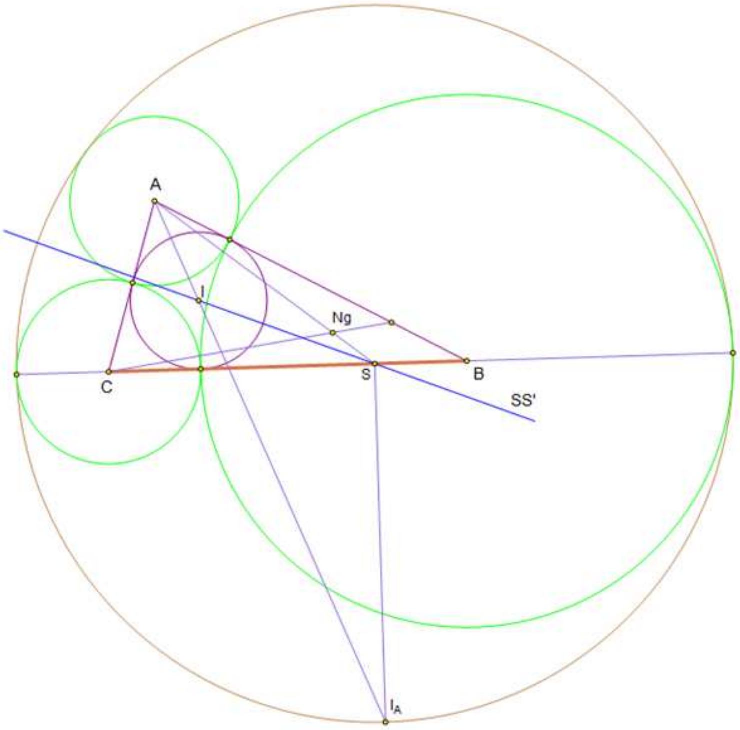

FIGURE 3. Heronian triangle of Class $J = 1$

Finally, from (11) we have $t_a + t_b + t_c < 2$, hence S' is also the isoperimetric point - see (X175) Kimberling at [3] and Hajja, Yff at [1].

Therefore we have $AB + BS' = AC + S'C$ and $BS' + S'C = BC$ this gives $BS' = s - c$ and S' is located at the tangent point of the A excircle with the line BC . For this Heronian class, S' is also located at the midpoint of the perimeter of the triangle from vertex A . □

4. HERONIAN TRIANGLE OF CLASS $J = 2$

We have from (3) – (5)

$$t_a = \frac{\cos^2\theta}{2}, \qquad t_b = 1 + \cos\theta, \qquad t_c = 1 - \cos\theta$$

and because these tangents of the half angles of the triangle must be > 0 and $0 < \cos\theta < 1$ then $t_b > 1$ and these Heronian triangles will always be obtuse with b as the longest side . We note that

$$t_b > \frac{1}{2} > t_a > 0 \quad \text{and} \quad t_b > 1 > t_c > 0$$

and because $t_b + t_c = 2$ we have $2 < t_a + t_b + t_c < \frac{5}{2}$.

Also from (2) we have $t_a + t_b + t_c - 2 = \frac{r}{r_o}$ so $r_o = s - a > 0$ and the outer Soddy Circle will always be externally tangent to the contact circles.

From (6) – (8) and dividing by the common factor of 4, we have

$$a = 2m^2n$$

$$b = (m+n)(m^2 - 2mn + 2n^2)$$
$$c = -(m-n)(m^2 + 2mn + 2n^2).$$

For integers $1 \leq m < n \leq 6$, we obtain primitive Heronian triangles of class $J = 2$ by dividing a, b, c by their greatest common denominator, as presented in the table below.

m	n	a	b	c	s	Δ	r	R
1	2	4	15	13	16	24	$\frac{3}{2}$	$\frac{65}{8}$
1	3	3	26	25	27	36	$\frac{4}{3}$	$\frac{325}{24}$
2	3	12	25	17	27	90	$\frac{10}{3}$	$\frac{85}{6}$
1	4	8	125	123	128	480	$\frac{15}{4}$	$\frac{1025}{16}$
3	4	72	119	65	128	2016	$\frac{63}{4}$	$\frac{1105}{16}$
1	5	5	123	122	125	300	$\frac{12}{5}$	$\frac{2501}{40}$
2	5	20	119	111	125	1050	$\frac{42}{5}$	$\frac{629}{10}$
3	5	45	116	89	125	1800	$\frac{72}{5}$	$\frac{2581}{40}$
4	5	80	117	53	125	1800	$\frac{72}{5}$	$\frac{689}{10}$
1	6	12	427	425	432	2520	$\frac{35}{6}$	$\frac{5185}{24}$
5	6	300	407	157	432	19800	$\frac{275}{6}$	$\frac{5809}{24}$

We note that because $0 < \cos\theta < 1$ it is not possible to have isosceles Heronian triangles of this class.

The relationship between the sides from (9) is

$$a^3 = s(b-c)^2 \quad \text{or alternatively} \quad \Delta = \frac{a(s-a)}{2}.$$

Consider the location of the outer Soddy circle center for this class of Heronian triangle and its radius as in Figure 4.

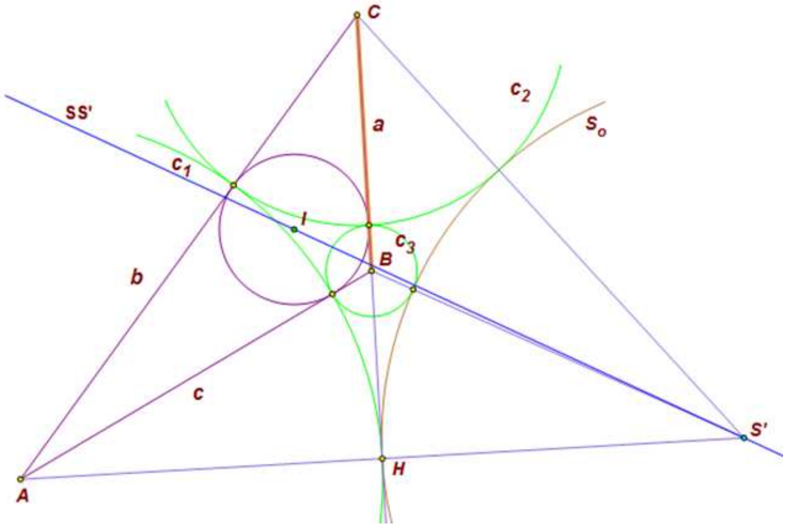

FIGURE 4. Heronian triangle of Class $J = 2$

Theorem 4.1. *The outer Soddy circle center of Heronian triangles of Class $J = 2$ is located at the reflection of one of its vertices about its associated side where the vertex is the one whose tangent of its half angle does not contribute to J . The* *outer Soddy circle radius is equal to the radius of the contact circle associated with the same vertex.*

Proof. Without loss of generality we choose $t_b + t_c = 2$. We know that from (2) $t_a + t_b + t_c - 2 = \frac{r}{r_o}$, so we get the radius of the outer Soddy circle $r_o = s - a > 0$ as the contact circle associated with t_a that does not contribute to J . With $r_o > 0$ the outer Soddy Circle will always be externally tangent to the contact circles, hence $BS' = s - b + r_o = s - b + s - a = c$ and $\Delta ABS'$ is isosceles.

Consequently S' is the reflection of A about BC . □

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