International Journal of Computer Discovered Mathematics (IJCDM)
ISSN 2367-7775 ©IJCDM
June 2016, Volume 1, No.2, pp.25-39.
Received 25 February 2016. Published on-line 5 March 2016
web: http://www.journal-1.eu/


# Generalizations of some triangle geometry results associated with cubics

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**Abstract.** This paper gives generalizations of some theorems and triangle centers in triangle geometry.

**Keywords.** Isoconjugate, barycentric coordinates, collinear, concurrent, pivotal isocubic, circular cubic, coaxial circles, triangle centers.

**Mathematics Subject Classification (2010).** 51M05, 68T01, 68T20.

## 1. Introduction

Beginning with the definitions of isoconjugation, pivotal isocubic [1], [2] and their basic properties in §2, we introduce generalizations of Parry reflection point and Evans perspectors [3]. In §3, we give a simple result on circular cubic and apply it. §3 contains results on circular isogonal cubic and generalizations of some theorems on concurrency of circles. These were inspired from the papers of Bernard Gibert [4] and Paul Yiu [5]. The solutions combine synthetic methods and barycentric coordinates.
This paper uses the following notations:

| | |
|---|---|
| $P^*$ | Isoconjugate of $P$, in §3 $P^*$ is isogonal conjugate of $P$ |
| $W/P$ | Cevian quotient |
| $I$, $I_a$, $I_b$, $I_c$ | Incenter, $A, B, C-$excenters of $\triangle ABC$ |
| $(ABC)$ | Circumcircle of $\triangle ABC$. |

## 2. Pivotal Isocubic

### 2.1. Isoconjugate.

**Definition 2.1.** *Given two points $\Omega = (p, q, r)$ and $P = (x, y, z)$, we call $P^* = (pyz, qzx, rxy)$ is $\Omega -$ isoconjugate of $P$.*

When $\Omega \equiv$ *Lemoine point, centroid*, we obtain isogonal and isotomic conjugation, respectively.

**Proposition 2.1.** *$\Omega-$isoconjugate of the line at infinity is the conic $\mathcal{C}_\infty$ that has equation:*

$$pyz + qzx + rxy = 0$$

**Proposition 2.2.** *$PA$, $P^*A$ intersect $\mathcal{C}_\infty$ at $P_a$, $P'_a$ then $P_aP'_a$ is parallel to $BC$.*

**Proposition 2.3.** *$(PQ \cap P^*Q^*)^* = PQ^* \cap QP^*$ [6]*

**Remark.** *Isoconjugation is the generalization of isogonal conjugation. Projecting $\mathcal{C}_\infty$ to a circle, then isoconjugation becomes isogonal conjugation.*

## 2.2. Basic properties.

**Definition 2.2.** *$W = (\alpha, \beta, \gamma)$. Locus of $P$ such that $W$, $P$, $\Omega-$isoconjugate of $P$ are collinear is a cubic. It has equation:*

$$\alpha x(ry^2 - qz^2) + \beta y(pz^2 - rx^2) + \gamma z(qx^2 - py^2) = 0$$

Pivotal isocubic with pole $\Omega$ and pivot $W$ is denoted by $p\mathcal{K}(\Omega, W)$.

**Proposition 2.4.** *$P$ lies on $p\mathcal{K}(\Omega, W)$.*

1. *$PA$ intersects $p\mathcal{K}(\Omega, W)$ at $D$ then $P^*A$ intersects $p\mathcal{K}(\Omega, W)$ at $D^*$.*
2. *$PA$, $PB$, $PC$ intersect $p\mathcal{K}(\Omega, W)$ at $D$, $E$, $F$. The following triples of points are collinear:*

   $$(A, E, F^*), (A, E^*, F), (B, F, D^*), (B, F^*, D), (C, D, E^*), (C, D^*, E)$$

3. *$W/P$ also lies on $pK(\Omega, W)$ and $P$, $W/P$, $W^*$ are collinear ($W^*$ is called isopivot or secondary pivot). $W^*$ is also tangential point of $W$, $A$, $B$, $C$ on $p\mathcal{K}(\Omega, W)$.*

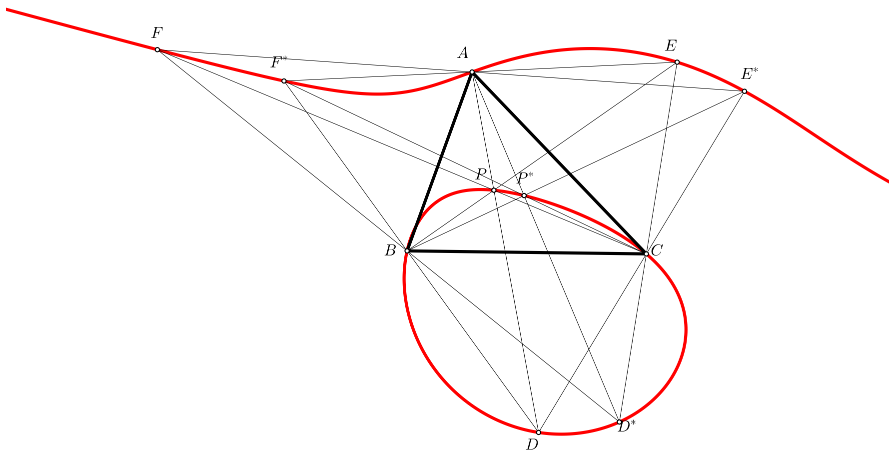


FIGURE 1. Pivotal isocubic

*Proof.* We give proof for 2.
Since $PP^*$, $EE^*$, $FF^*$ are concurrent at $W$, then $\triangle PEF$ and $\triangle P^*E^*F^*$ are perspective. According to Desargues's theorem, $PE \cap P^*E^*$, $PF \cap P^*F^*$, $EF \cap E^*F^*$ are collinear. Therefore, $EF \cap E^*F^*$ lies on $BC$. By 2.3, $(EF \cap E^*F^*)^* = E^*F \cap EF^*$ so $E^*F \cap EF^* = A$. Hence $A$, $E$, $F^*$ are collinear, $A$, $E^*$, $F$ are collinear.

□

2.3. **Generalization of Parry Reflection Point.**

**Proposition 2.5.** *$PA$, $PB$, $PC$ intersect $pK(\Omega, W)$ at $D$, $E$, $F$. $\triangle A_W B_W C_W$ is cevian triangle of $W$, then $DA_W$, $EB_W$, $FC_W$ are concurrent at $W/P^*$.*

This is the generalization of the problem that given by Tran Quang Hung [7].

*Proof.* $P = (x_0, y_0, z_0)$, we suppose that

$$\frac{\overline{WP}}{\overline{WP^*}} = -t$$

Then

$$D = \left(\frac{py_0z_0t}{py_0z_0 + qz_0x_0 + rx_0y_0}, \frac{y_0}{x_0+y_0+z_0}, \frac{z_0}{x_0+y_0+z_0}\right)$$

$$A_W = \left(0, \frac{y_0}{x_0+y_0+z_0} + t\frac{qz_0x_0}{py_0z_0 + qz_0x_0 + rx_0y_0}, \frac{z_0}{x_0+y_0+z_0} + t\frac{rx_0y_0}{py_0z_0 + qz_0x_0 + rx_0y_0}\right)$$

Let $\triangle A'_{P^*}B'_{P^*}C'_{P^*}$ be anticevian triangle of $P^*$ with respect to $\triangle ABC$.

$$A'_{P^*} = (-py_0z_0, qz_0x_0, rx_0y_0)$$

It is easy to verify that $D$, $A_W$, $A'_{P^*}$ are collinear. Symmetrically, $E$, $B_W$, $B'_{P^*}$ are collinear and $F$, $C_W$, $C'_{P^*}$ are collinear. Hence, $DA_W$, $EB_W$, $FC_W$ are concurrent at $W/P^*$. □

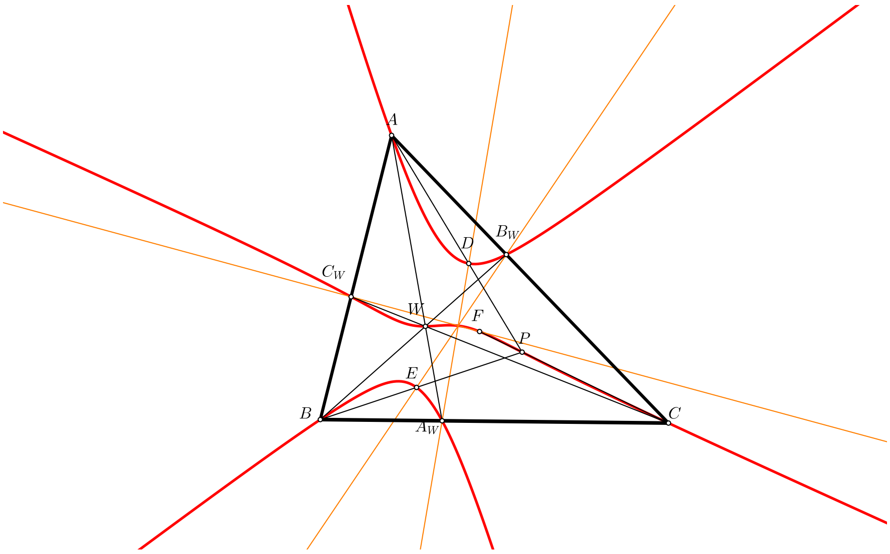


FIGURE 2. Three lines are concurrent at $W/P^*$

**Remark.** *If $W$ lies at infinity, midpoint of $P^*(W/P^*)$ lies on $\mathcal{C}_\infty$.*

2.4. **Generalization of Schiffler point.**

**Proposition 2.6.** *$OI_a$, $OI_b$, $OI_c$ intersect $BC$, $CA$, $AB$ at $D$, $E$, $F$. $AD$, $BE$, $CF$ are concurrent at Schiffler point [8].*

If we consider $I_a$, $I_b$, $I_c$ as the intersections of $IA$, $IB$, $IC$ with Neuberg cubic, then the above property of Schiffler point can be generalized as follow:

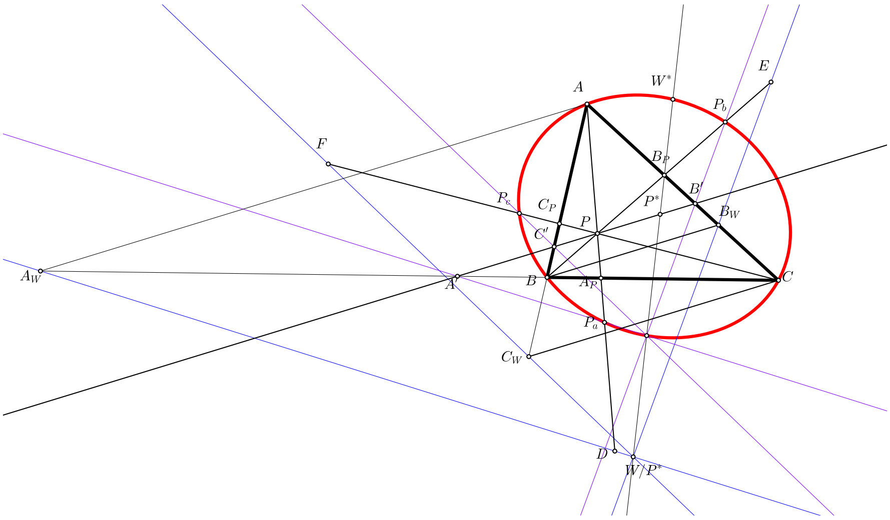


FIGURE 3. $W$ lies at infinity. Midpoint of $P^*(W/P^*)$ lies on $\mathcal{C}_\infty$

**Proposition 2.7.** *$P_1, P_2$ lie on $p\mathcal{K}(\Omega, W)$. $P_iA$, $P_iB$, $P_iC$ intersect $p\mathcal{K}(\Omega, W)$ at $D_i$, $E_i$, $F_i$, where $i = 1$ or $i = 2$. $P_2D_1$, $P_2E_1$, $P_2F_1$ intersect $BC$, $CA$, $AB$ at $X_1$, $Y_1$, $Z_1$.*
*Then $AX_1$, $BY_1$, $CZ_1$, $P_2P_2^*$ are concurrent.*

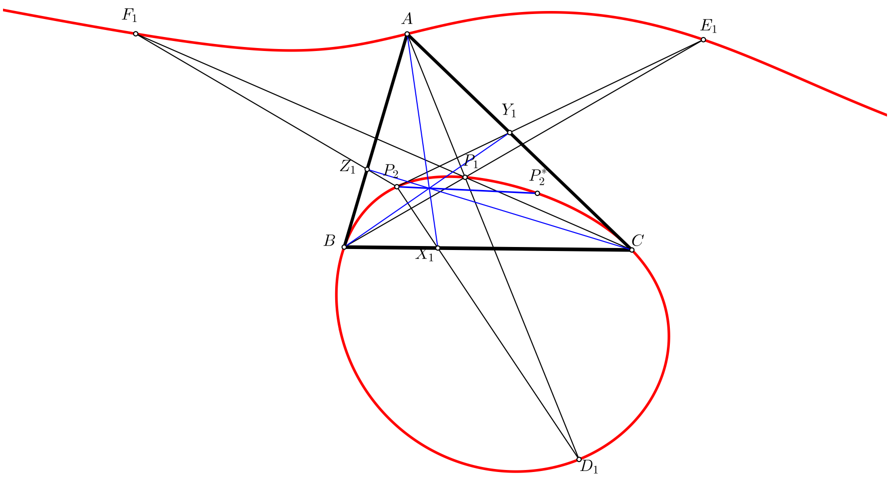


FIGURE 4. Generalization of Schiffler point

## 2.5. Generalization of Evans-Gibert-Neuberg perspectors.

**Proposition 2.8.** *$P_1$, $P_2$ lie on $p\mathcal{K}(\Omega, W)$. $P_iA$, $P_iB$, $P_iC$ intersect $p\mathcal{K}(\Omega, W)$ at $D_i$, $E_i$, $F_i$.*
*Then $D_1D_2$, $E_1E_2$, $F_1F_2$, $P_1^*P_2^*$ are concurrent at a point on $p\mathcal{K}(\Omega, W)$.*

*Proof.* This proof simply follows Cayley - Bacharach theorem.
$P_1P_2$ intersects $p\mathcal{K}(\Omega, W)$ at $P_3$. Consider $p\mathcal{K}(\Omega, W)$ and the degenerated cubic formed by three lines ($\overline{W, P_1, P_1^*}$, $\overline{W, P_2, P_2^*}$, $\overline{W^*, P_3, W/P_3}$) then ($\overline{W, W, W^*}$, $\overline{P_1, P_2, P_3}$, $\overline{P_1^*, P_2^*}$) contains $W/P_3$. Therefore $P_1^*P_2^*$ intersects $p\mathcal{K}(\Omega, W)$ at $W/P_3$.

We apply Cayley - Bacharach theorem once again for $p\mathcal{K}(\Omega, W)$, $(\overline{A, P_1, D_1}, \overline{A, P_2, D_2}, \overline{W^*, P_3, W/P_3})$, $(\overline{P_1, P_2, P_3}, \overline{A, A, W^*}, \overline{D_1, D_2})$ then $D_1D_2$ passes through $W/P_3$. □

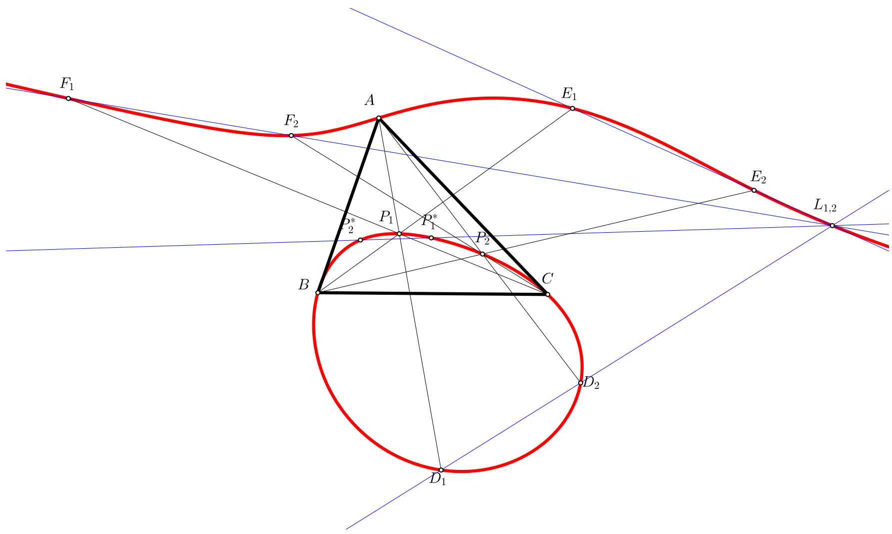


Figure 5.

We denote Evans perspector of $P_1$, $P_2$ by $L_{1,2}$ or $L_{P_1,P_2}$.

**Corollary 2.1.** *$D$, $E$, $F$, $P^*$ share the same tangential point on $p\mathcal{K}(\Omega, W)$.*

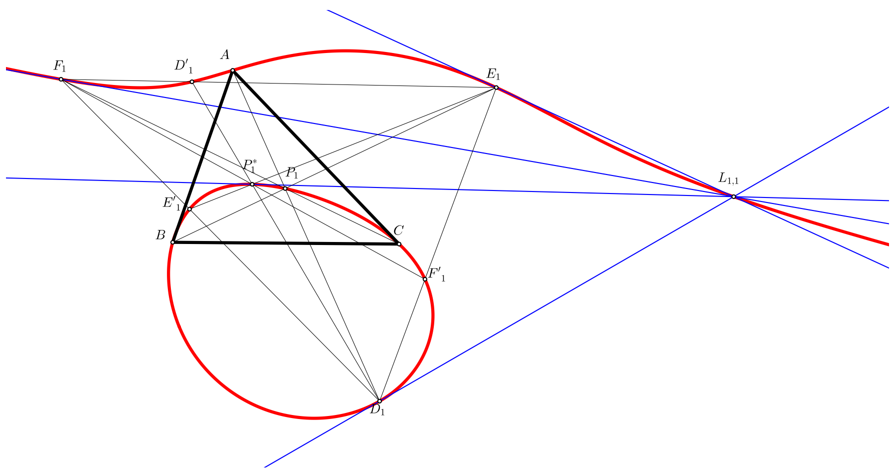


Figure 6. Tangent lines at $D$, $E$, $F$, $P^*$ are concurrent at a point on $p\mathcal{K}(\Omega, W)$

**Corollary 2.2.** *$EF \cap DP^*$, $FD \cap EP^*$, $DE \cap FP^*$ lies on $p\mathcal{K}(\Omega, W)$.*

*Proof.* Apply 2.4 for $D^*, P$: $D^*A$, $D^*B$, $D^*C$ intersect $p\mathcal{K}(\Omega, W)$ at $P^*$, $F$, $E$; $PA$, $PB$, $PC$ intersect $p\mathcal{K}(\Omega, W)$ at $D$, $E$, $F$. Then $EF \cap DP^*$ lies on $p\mathcal{K}(\Omega, W)$. □

**Remark.** *The generalization of Evans-Gibert-Neuberg perspector also contains the generalization of Parry reflection point.*

## 3. Circular cubic

### 3.1. **A simple result.**

**Proposition 3.1.** *$P_1$, $P_2$, $P_3$, $P_4$ lie on a circular cubic. $P_1P_2$, $P_3P_4$ intersect the cubic at $P_{12}$, $P_{34}$. $P_1$, $P_2$, $P_3$, $P_4$ are concyclic if and only if $P_{12}P_{34}$ passes through infinity point of the cubic.*

*Proof.* Circular cubic is a kind of cubic that passes through two circular points at infinity [9] $J_1$, $J_2$. In triangle geometry, $J_1$, $J_2$ are isogonal conjugate, which are intersections of circumcircle and line at infinity. Thus all circles passes through two circular points. A conic contains two circular points if and only if it is a circle. Let $W$ be infinity point(real) of the cubic. $P_1$, $P_2$, $P_3$, $P_4$ are concyclic then $P_1$, $P_2$, $P_3$, $P_4$, $J_1$, $J_2$ lie on a conic. The circular cubic and the degenerated cubic $(\overline{P_1, P_2, P_{12}}, \overline{P_3, P_4, P_{34}}, \overline{J_1, J_2, W})$ have 9 common points $P_1$, $P_2$, $P_3$, $P_4$, $P_{12}$, $P_{34}$, $J_1$, $J_2$, $W$. The denegerated cubic, which is the union of the conic $P_1P_2P_3P_4J_1J_2$ and the line $\overline{P_{12}P_{34}}$ contain 8 of these common points so it passes through $W$, according to Cayley - Bacharach theorem, this implies that $P_{12}P_{34}$ passes through $W$.
Conversely, $P_{12}P_{34}$ passes through $W$, we apply Cayley - Bacharach theorem for the circular cubic, $(\overline{J_1, J_2, W}, \overline{P_1, P_2, P_{12}}, \overline{P_3, P_4, P_{34}})$, $(\overline{W, P_{12}, P_{34}}, P_1P_2P_3J_1J_2)$, then $P_1$, $P_2$, $P_3$, $P_4$, $J_1$, $J_2$ lies on a conic. □

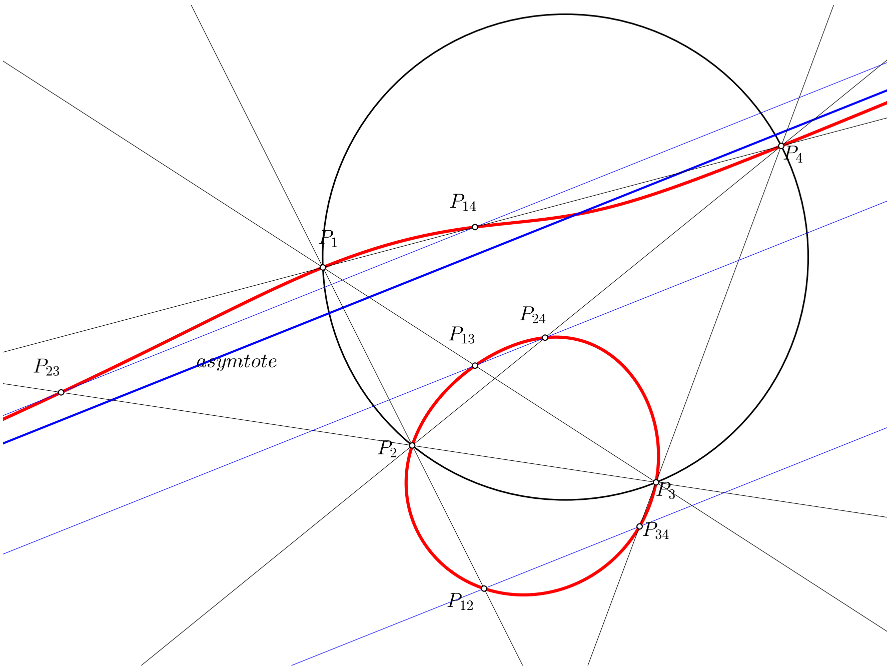


FIGURE 7.

With this simple result, we can prove a large amounts of concyclic points on circular cubics: *Neuberg cubic, Orthopivotal cubic* [10],...

- The Lester circle contains $X_3$, $X_5$, $X_{13}$, $X_{14}$, $X_{5671}$.
- The first Evans circle contains $X_1$, $X_{484}$, $X_{1276}$, $X_{1277}$.
- The second Evans circle contains $X_{74}$, $X_{101}$, $X_{399}$, $X_{1276}$, $X_{1277}$.

According to the above proof, we call $P_{12}$, $P_{34}$ is a concyclic pair of the circle $(P_1P_2P_3P_4)$. Therefore, there must be hundreds of groups of concyclic triangle centers. For short, we lists the concyclic pairs and collinear triangle centers $X_n$ on Neuberg cubic, where $n < 8000$.

| $P$ | $P \in X_iX_j$ | $P^*$ | $P^* \in X_iX_j$ |
|---|---|---|---|
| 1 | (3,484),(4,3465),(13,5672)<br>(14,5673),(15,1276),(16,1277)<br>(74,3464),(399,3065),(1138,5677)<br>(1157,3483),(1263,5685),(2132,7164)<br>(3484,7165),(5623,7326),(5624,7325)<br>(5668,7060),(5669,7059),(5670,7327)<br>(5671,7329) | 1 | (3,484),(4,3465),(13,5672)<br>(14,5673),(15,1276),(16,1277)<br>(74,3464),(399,3065),(1138,5677)<br>(1157,3483),(1263,5685),(2132,7164)<br>(3484,7165),(5623,7326),(5624,7325)<br>(5668,7060),(5669,7059),(5670,7327)<br>(5671,7329) |
| 3 | (1,484),(3,1157),(15,16)<br>(74,399),(1138,5671),(1263,5684)<br>(2133,5670),(3065,5685),(3440,5674)<br>(3441,5675),(3464,3466),(5672,7059)<br>(5673,7060),(5677,7164) | 4 | (1,3465),(4,3484),(13,5669)<br>(14,5668),(74,5667),(484,3483)<br>(616,1337),(617,1338),(1138,2132)<br>(1157,3482),(1276,1277),(3440,5682)<br>(3441,5681),(5677,7329),(5680,7164) |
| 13 | (1,5672),(4,5669),(13,5674)<br>(14,399),(16,616),(74,5623)<br>(484,1277),(3441,5624),(3479,5668)<br>(5677,7325) | 15 | (1,1276),(3,16),(14,617)<br>(15,1337),(74,5668),(1138,5624)<br>(3065,5673),(3464,7326),(3465,7059) |
| 14 | (1,5673),(4,5668),(13,399)<br>(14,5675),(15,617),(74,5624)<br>(484,1276),(3440,5623),(3480,5669)<br>(5677,7326) | 16 | (1,1277),(3,15),(13,616)<br>(16,1338),(74,5669),(1138,5623)<br>(3065,5672),(3464,7325),(3465,7060) |
| 30 | (3,4),(13,15),(14,16),(399,1138)<br>(484,3065),(616,3440),(617,3441)<br>(1157,1263),(1276,7060),(1277,7059)<br>(1337,3479),(1338,3480),(2132,2133)<br>(3464,7164)(3465,3466),(3481,3482)<br>(3483,7165),(5672,7326),(5673,7325)<br>(5677,7327),(5680,7328),(5685,7329) | 74 | (1,3464),(3,399),(4,5667)<br>(13,5623),(14,5624),(15,5668)<br>(16,5669),(74,2132),(484,3465)<br>(1138,5670),(1157,3484),(1263,5671)<br>(1276,5672),(1277,5673),(1337,5674)<br>(1338,5675),(2133,5676),(3065,5677)<br>(3440,5678),(3441,5679),(3466,5680)<br>(3479,5681),(3480,5682),(3481,5683)<br>(3482,5684),(3483,5685) |
| 399 | (1,3065),(3,74),(13,14)<br>(484,7329),(1276,7325),(1277,7326)<br>(1337,3441),(1338,3440),(2133,5667)<br>(3464,7327),(3465,7164),(3466,3483) | 1138 | (1,5677),(4,2132),(15,5624)<br>(16,5623),(74,5670),(484,3464)<br>(616,5675),(617,5674),(1157,5667)<br>(3465,5685),(3479,5679),(3480,5678)<br>(3484,5684),(5672,5673),(5680,7165) |
| 484 | (1,3),(4,3483),(13,1277)<br>(14,1276),(74,3465),(399,7329)<br>(484,1263),(1138,3464),(2132,7327)<br>(2133,5680),(3466,3484),(3482,7165)<br>(5667,7164),(5668,7325),(5669,7326) | 3065 | (1,399),(74,5677),(1157,3465)<br>(2132,3466),(3065,5671),(3483,5684)<br>(5667,7165),(5670,7164) |

Notice that, $X_3$, $X_3$, $X_{1157}$ are collinear means $X_{1157}$ is tangential point of $X_3$ on Neuberg cubic.
But the circular isogonal cubic seems to be the most special circular cubic - pivotal

isogonal cubic whose pivot at infinity. From now, we denote pivot at infinity of circular isogonal cubic by $W$.

**Proposition 3.2.** *$P$ lies on circular isogonal cubic $p\mathcal{K}(X_6, W)$ of $\triangle ABC$. $PA$, $PB$, $PC$ intersect $p\mathcal{K}(X_6, W)$ at $D$, $E$, $F$.*

(1) *$p\mathcal{K}(X_6, W)$ contains $I$, $I_a$, $I_b$, $I_c$.*
(2) *$D$, $E$, $F$ lies on $(PBC)$, $(PCA)$, $(PAB)$, respectively.*
(3) *The following quadruples of points are concyclic:*

$$(B, C, E^*, F), (C, A, F^*, D), (A, B, D^*, E)$$

$$(B, C, E, F^*), (C, A, F, D^*), (A, B, D, E^*)$$

(4) *Denote $\Psi_A$ is the composition of the inversion $\mathbf{I}_A^{AB.AC}$ and the reflection in bisector of $\angle(AB, AC)$. $M$ lies on $p\mathcal{K}(X_6, W)$ if and only if $\Psi_A(M)$ lies on $p\mathcal{K}(X_6, W)$.*

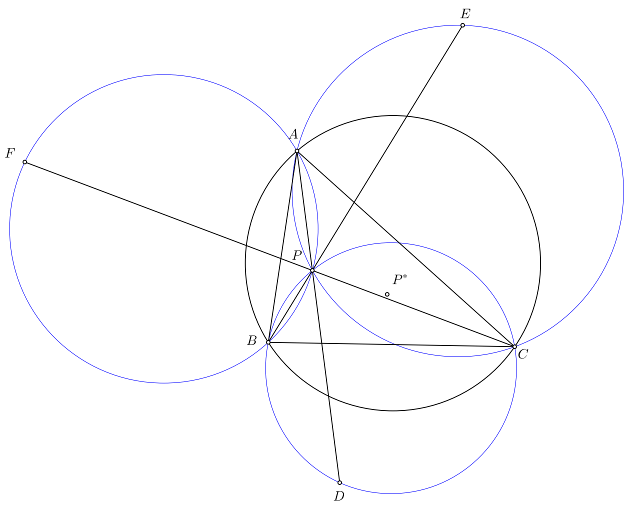


FIGURE 8.

*Proof.* We introduce a proof for 4.
$(DA, DB) = (DP, DB) = (CP, CB) = (CA, CP^*)$, $(AB, AD) = (AP^*, AC)$, hence $\triangle ABD$ and $\triangle AP^*C$ are similar. Therefore, $AP^*.AD = AB.AC$. Since $AP$ is reflection of $AD^*$ in bisector of $\angle(AB, AC)$, then $\Psi_A(P) = D^*$. □

## 3.2. **Pencils of Circles.**

**Proposition 3.3.** *$PA$, $PB$, $PC$ intersect $(PBC)$, $(PCA)$, $(PAB)$ at $D$, $E$, $F$. $(TAD)$, $(TBE)$, $(TCF)$ are coaxial if and only if $T$ lies on circular isogonal cubic which contains $P$. Furthermore, when $(TAD)$, $(TBE)$, $(TCF)$ are coaxial, both common points lie on circular isogonal cubic which contain $P$ and the radical axis passes through $P^*$.*

This is the generalization of Musselman theorem [11]. When $P$, $T$ coincide with orthocenter and circumcenter, we obtain Musselman theorem.

*Proof.* Let $P = (u, v, w)$, $T = (x_0, y_0, z_0)$. $(TAD)$, $(TBE)$, $(TCF)$ have equations:

$$L_a(x, y, z)(x + y + z) - (a^2yz + b^2zx + c^2xy) = 0$$

$$L_b(x,y,z)(x+y+z)-(a^2yz+b^2zx+c^2xy)=0$$
$$L_c(x,y,z)(x+y+z)-(a^2yz+b^2zx+c^2xy)=0$$
$$L_a(x,y,z)=\left(\Big(\frac{a^2vw}{u}z_0+Tw\Big)y-\Big(\frac{a^2vw}{u}y_0+Tv\Big)z\right)/(wy_0-vz_0)$$
$$L_b(x,y,z)=\left(\Big(\frac{b^2wu}{v}x_0+Tu\Big)z-\Big(\frac{b^2wu}{v}z_0+Tw\Big)x\right)/(uz_0-wx_0)$$
$$L_c(x,y,z)=\left(\Big(\frac{c^2uv}{w}y_0+Tv\Big)x-\Big(\frac{c^2uv}{w}+Tu\Big)y\right)/(vx_0-uy_0)$$
$$\text{where } T=\frac{a^2y_0z_0+b^2z_0x_0+c^2x_0y_0}{x_0+y_0+z_0}$$

$L_a(x,y,z)$, $L_b(x,y,z)$, $L_c(x,y,z)$ are radical axis of $(TAD)$, $(TBE)$, $(TCF)$ and $(ABC)$, respectively. $(TAD)$, $(TBE)$, $(TCF)$ are coaxial if and only if these radical axis are concurrent. This means:

$$\Big(\frac{a^2vw}{u}z_0+Tw\Big)\Big(\frac{b^2wu}{v}x_0+Tu\Big)\Big(\frac{c^2uv}{w}y_0+Tv\Big)=\Big(\frac{a^2vw}{u}y_0+Tv\Big)\Big(\frac{b^2wu}{v}z_0+Tw\Big)\Big(\frac{c^2uv}{w}x_0+Tu\Big)$$
$$\Leftrightarrow \Big(\frac{a^2v}{u}z_0+T\Big)\Big(\frac{b^2w}{v}x_0+T\Big)\Big(\frac{c^2u}{w}y_0+T\Big)=\Big(\frac{a^2w}{u}y_0+T\Big)\Big(\frac{b^2u}{v}z_0+T\Big)\Big(\frac{c^2v}{w}x_0+T\Big)$$
$$\Leftrightarrow T\Big(\frac{a^2}{u}(y_0w-z_0v)+\frac{b^2}{v}(z_0u-x_0w)+\frac{c^2}{w}(x_0v-y_0u)\Big)$$
$$=\frac{b^2c^2}{vw}ux_0(y_0w-z_0v)+\frac{c^2a^2}{wu}vy_0(z_0u-x_0w)+\frac{a^2b^2}{uv}wz_0(x_0v-y_0u)$$
$$\Leftrightarrow (a^2y_0z_0+b^2z_0x_0+c^2x_0y_0)\Big(a^2vw(y_0w-z_0v)+b^2wu(z_0u-x_0w)+c^2uv(x_0v-y_0u)\Big)$$
$$=(x_0+y_0+z_0)\Big(b^2c^2u^2x_0(y_0w-z_0v)+c^2a^2v^2y_0(z_0u-x_0w)+a^2b^2w^2z_0(x_0v-y_0u)\Big)$$
$$\Leftrightarrow \sum_{cyclic}\Big(a^2vw(u+v+w)-u(a^2vw+b^2wu+c^2uv)\Big)x_0(c^2y_0^2-b^2z_0^2)=0$$

Infinity point of $PP^*$ is $W=\Big(a^2vw(u+v+w)-u(a^2vw+b^2wu+c^2uv),\dots,\dots\Big)$ so the last equation means $TT^*$ passes through $W$. Hence $(TAD)$, $(TBE)$, $(TCF)$ are coaxial if and only if $T$ lies on circular isogonal cubic $p\mathcal{K}(X_6,W)$. Let two common points be $T$ and $T'$. Since $(T'AD)$, $(T'BE)$, $(T'CF)$ are coaxial, then $T'$ lies on $p\mathcal{K}(X_6,W)$. According to 3.1, $T$, $T'$, $P^*$ are collinear. □

**Proposition 3.4.** *$P$ lies on $p\mathcal{K}(X_6,W)$. $PA$, $PB$, $PC$ intersect $(PBC)$, $(PCA)$, $(PAB)$ at $D$, $E$, $F$. Then $(P^*AD)$, $(P^*BE)$, $(P^*CF)$ are coaxial. These circles pass through $P^*$, $Q$ and $QP^*$, $QD$, $QE$, $QF$ are tangent to $p\mathcal{K}(X_6,W)$ at $P^*$, $D$, $E$, $F$.*

**Proposition 3.5.** *The perspector $L_{1,2}$ lies on:*

$$(P_2^*AD_1),(P_2^*BE_1),(P_2^*CF_1),(P_1^*AD_2),(P_1^*BE_2),(P_1^*CF_2)$$
$$(AE_1F_2),(AE_2F_1),(BF_1D_2),(BD_2F_1),(CD_1E_2),(CD_2E_1)$$

The following property gives us a construction for circular isogonal cubic with a given point on it.

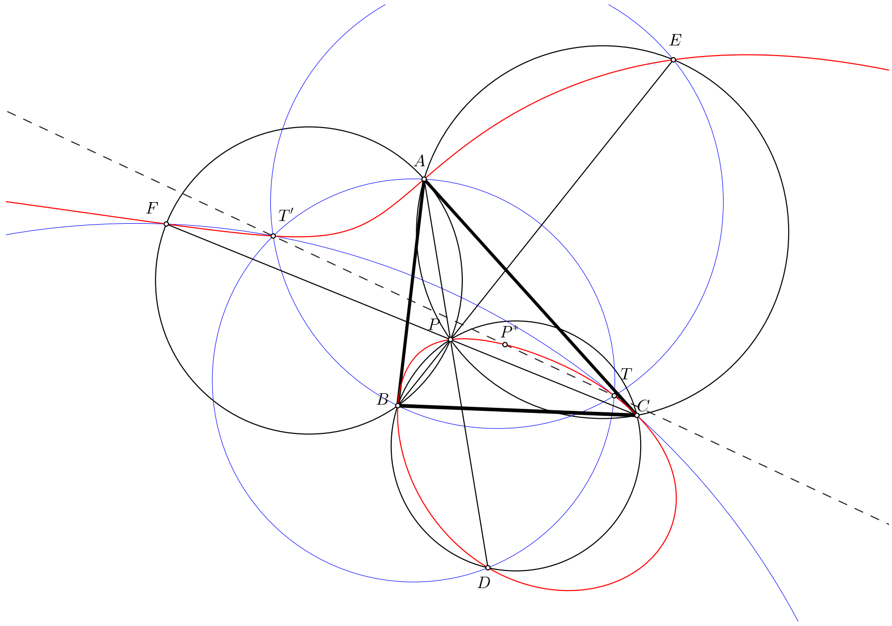


FIGURE 9. Generalization of Musselman theorem

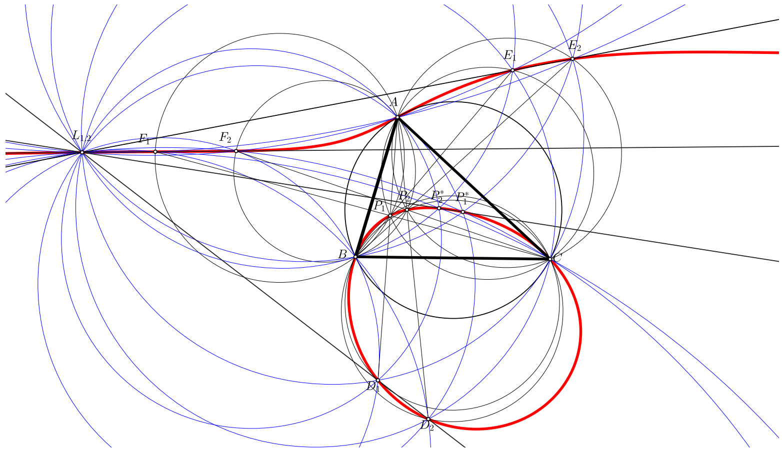


FIGURE 10. 12 circle pass through "Evans perspector"

**Proposition 3.6.** *$S$ varies on $PP^*$, $S^*A$, $S^*B$, $S^*C$ intersect $(ABC)$ at $A'$, $B'$, $C'$. $PA$, $PB$, $PC$ intersect $(PBC)$, $(PCA)$, $(PAB)$ at $D$, $E$, $F$. $(ADA')$, $(BEB')$, $(CFC')$ are concurrent at $T^+$ and $T_-$. $T^+$, $T_-$ lie on the circular isogonal cubic which contains $P$.*

Let $P = (u, v, w)$ and $P^* = (a^2vw, b^2wu, c^2uv)$

$$\frac{\overline{SP}}{\overline{SP^*}} = -t$$

$$S = \Big(\frac{u}{u+v+w} + t\frac{a^2vw}{a^2vw + b^2wu + c^2uv}, \dots, \dots\Big)$$

$$S^* = \left(a^2/\Big(\frac{u}{u+v+w} + t\frac{a^2vw}{a^2vw + b^2wu + c^2uv}\Big), \dots, \dots\right) = (x_{S^*}, y_{S^*}, z_{S^*})$$

$$(ADA') : \frac{a^2vw}{u}\Big(\frac{y}{y_{S^*}} - \frac{z}{z_{S^*}}\Big)(x+y+z) + \Big(\frac{v}{y_{S^*}} - \frac{w}{z_{S^*}}\Big)(a^2yz + b^2zx + c^2xy) = 0$$

$$(BEB'): \frac{b^2wu}{v}\Big(\frac{z}{z_{S^*}}-\frac{x}{x_{S^*}}\Big)(x+y+z)+\Big(\frac{w}{z_{S^*}}-\frac{u}{x_{S^*}}\Big)(a^2yz+b^2zx+c^2xy)=0$$

$$(CFC'): \frac{c^2uv}{w}\Big(\frac{x}{x_{S^*}}-\frac{y}{y_{S^*}}\Big)(x+y+z)+\Big(\frac{u}{x_{S^*}}-\frac{v}{y_{S^*}}\Big)(a^2yz+b^2zx+c^2xy)=0$$

$S^*P^*$ is radical axes of these circles. First coordinate of a point $M$ on $S^*P^*$, include $T^{\pm}$ is:

$$\frac{a^2vw}{a^2vw+b^2wu+c^2uv}+k\frac{x_{S^*}}{x_{S^*}+y_{S^*}+z_{S^*}}$$

Since $T^{\pm}$ lies on $(ADA')$, let $M$ lies on $(ADA')$, we obtain a quadratic equation of $k$. After solved it, we obtain two values of $k$, and then the coordinates of two common points $T^{\pm}$.

**Proposition 3.7.** *$P_1$, $P_2$, $T$ lies on $p\mathcal{K}(X_6, W)$. $P_iA$, $P_iB$, $P_iC$ intersect $(P_iBC)$, $(P_iCA)$, $(P_iAB)$ at $D_i$, $E_i$, $F_i$.*
*Then $(TD_1D_2)$, $(TE_1E_2)$, $(TF_1F_2)$, $(TP_1^*P_2^*)$ are coaxial.*

**Proposition 3.8.** *$\triangle DEF$ is cevian triangle of $P$ with respect to $\triangle ABC$. Locus of $T$ such that $(TAD)$, $(TBE)$, $(TCF)$ is a pivotal circular cubic.*

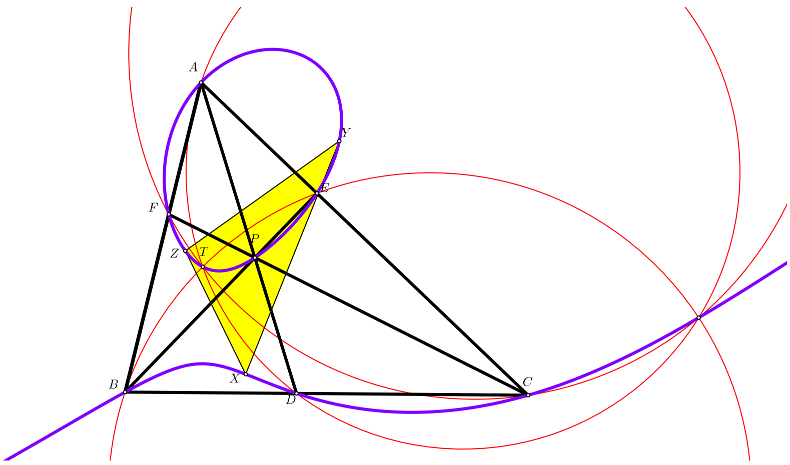


FIGURE 11. QA-Cu1

Let $P=(u,v,w)$. Locus of $T$ is a cubic has equation:

$$a^2\Big(a^2(u+v)(u+w)-b^2u(u+v)-c^2u(u+w)\Big)yz(wy-vz)$$
$$+b^2\Big(b^2(v+w)(v+u)-c^2v(v+w)-a^2v(v+u)\Big)zx(uz-wx)$$
$$+c^2\Big(c^2(w+u)(w+v)-a^2w(w+u)-b^2w(w+v)\Big)xy(vx-uy)=0$$

This is known as QA-DT-P4 cubic of the quadrangle $A$, $B$, $C$, $P$, or QA-Cu1 in Chris van Tienhoven's website [12]. Let $X$, $Y$, $Z$ be Miquel points of quadrilaterals $(AB, AC, PB, PC)$, $(BC, BA, PC, PA)$, $(CA, CB, PA, PB)$ then $X$, $Y$, $Z$ lies on QA-Cu1 and QA-Cu1 is circular isogonal cubic of $\triangle XYZ$. Under the inversion $I_P^k$, $I_P^k(D)$, $I_P^k(E)$, $I_P^k(F)$ are intersections other than $P$ of $(PI_P^k(B)I_P^k(C))$, $(PI_P^k(C)I_P^k(A))$, $(PI_P^k(A)I_P^k(B))$, the image of QA-Cu1 is the cubic in 3.3 of $\triangle I_P^k(A)I_P^k(B)I_P^k(C)$. In general, reflection of a circular cubic in a circle centered at a point on it, is also a circular cubic.

### 3.3. Cyclologic triangles.

**Proposition 3.9.** *$A$, $B$, $C$, $P$, $Q$ lies on a circular cubic. $(PBC)$, $(PCA)$, $(PAB)$, $(QBC)$, $(QCA)$, $(QAB)$ intersect the cubic at $D$, $E$, $F$, $X$, $Y$, $Z$.*

(1) *$(AEF)$, $(BFD)$, $(CDE)$ are concurrent at a point on the cubic.*
(2) *$BF \cap CE = D'$, $CD \cap AF = E'$, $AE \cap BD = F'$. $D'$, $E'$, $F'$ and $(D'BC)$, $(E'CA)$, $(F'AB)$ are concurrent at a point on the cubic.*
(3) *$\triangle DEF$ and $\triangle XYZ$ are cyclologic, two cyclology centers lie on the cubic.*

In case of circular isogonal cubic, we obtain more interesting properties, which generalize Parry reflection point.

**Proposition 3.10.** *$P_1$, $P_2$ lies on $p\mathcal{K}(X_6, W)$ of $\triangle ABC$. $P_iA$, $P_iB$, $P_iC$ intersect $(P_iBC)$, $(P_iCA)$, $(P_iAB)$ at $D_i$, $E_i$, $F_i$.*
*Then $(P_1^*D_1D_2)$, $(P_1^*E_1E_2)$, $(P_1^*F_1F_2)$, $(D_2E_1F_1)$, $(E_2F_1D_1)$, $(F_2D_1E_1)$ are concurrent at $R_1$ on $p\mathcal{K}(X_6, W)$.*

*Proof.* $(AE_1F_1), (BF_1D_1), (CD_1E_1), (P_1^*AD_1), (P_1^*BE_1), (P_1^*CF_1)$ are concurrent at $Q_1$. $\Psi_A : (AE_1F_1), (AP_1^*D_1) \to E_1F_1, P_1^*D_1$ so $\Psi_A : Q_1 \mapsto D_1P_1^* \cap E_1F_1$. $Q_1A$, $Q_1B$, $Q_1C$ intersect $p\mathcal{K}(X_6, W)$ at $X_1$, $Y_1$, $Z_1$. $X_1^* = E_1F_1 \cap D_1P_1^*$ according to 3.2. From 2.8, $D_2X_1$, $E_2Y_1$, $F_2Z_1$, $P_2^*Q_1^*$ are concurrent at $R_1$ on $p\mathcal{K}(X_6, W)$. Furthermore, according to 3.1, we obtain that $(D_2E_1F_1)$ passes through $R_1$. $(E_2F_1D_1)$, $(F_2D_1E_1)$ pass through $R_1$, likewise. From 3.7, $(P_1^*D_1D_2)$, $(P_1^*E_1E_2)$, $(P_1^*F_1F_2)$ are concurrent at $P_1^*$ and $R_1'$ where $R_1'$ is the third common point of $P_1^*L_{1,2}^*$ and $p\mathcal{K}(X_6, W)$. All we need to do now is show that $R_1 \equiv R_1'$, we rewrite this as follow: *Given cubic $p\mathcal{K}(X_6, W)$ and three points $W$, $P_1$, $P_2$ on it. $WP_1$, $WP_2$ intersect the cubic at $P_1^*$, $P_2^*$; $P_1^*P_2^*$ intersect the cubic at $L_{1,2}$, tangent line at $P_1^*$ intersect the cubic at $Q_1$; $WL_{1,2}$, $WQ_1$ intersect the cubic at $L_{1,2}^*$, $Q_1^*$. Then $Q_1^*P_2^*$ intersects $L_{1,2}^*P_1^*$ at a point on the cubic.* Its proof simply follows Cayley - Bacharach theorem: $Q_1^*P_2^*$ intersect $p\mathcal{K}(X_6, W)$ at $R_1$. $p\mathcal{K}(X_6, W)$ and the degenerated cubic $(\overline{W, L_{1,2}, L_{1,2}^*}, \overline{P_1^*, P_1^*, Q_1}, \overline{Q_1^*, P_2^*, R_1})$ pass through 9 points: $W$, $L_{1,2}$, $L_{1,2}^*$, $P_1^*$(double point), $Q_1$, $Q_1^*$, $P_2^*$, $R_1$. Then the degenerated cubic $(\overline{W, Q_1, Q_1^*}, \overline{P_1^*, P_2^*, L_{1,2}}, \overline{L_{1,2}^*, P_1^*})$ passes through $R_1$, this means $L_{1,2}^*$, $P_1^*$, $R_1$ are collinear. □

We call $R_1$ is a cyclology center of $(P_1, P_2)$. Notice that cyclology centers of $(P_1, P_2)$ and $(P_2, P_1)$ are different.

**Proposition 3.11.** *$P$ lies on circular isogonal $p\mathcal{K}(X_6, W)$. $PA$, $PB$, $PC$ intersect $(PBC)$, $(PCA)$, $(PAB)$ at $D$, $E$, $F$. $(DEF)$ intersects $p\mathcal{K}(X_6, W)$ at the fourth point $P'$.*
*$(P^*P'D)$, $(P^*P'E)$, $(P^*P'F)$ are tangent to $p\mathcal{K}(X_6, W)$ at $D$, $E$, $F$.*

**Proposition 3.12.** *These consequences don't contain cubic*
*$PA$, $PB$, $PC$ intersect $(PBC)$, $(PCA)$, $(PAB)$ at $D$, $E$, $F$. $W$ is the infinity point on $PP^*$.*

(1) *$W/I$ lies on Bevan circle. Particularly, $X_{3464}$ is the fourth common point of Bevan circle and Neuberg cubic.*
(2) *$(P^*DI_a)$, $(P^*EI_b)$, $(P^*FI_c)$, $(I_aEF)$, $(I_bFD)$, $(I_cDE)$ are concurrent.*
(3) *$\triangle A_WB_WC_W$ is cevian triangle of $W$ with respect to $\triangle ABC$.*
*$(PDD^*)$, $(PEE^*)$, $(PFF^*)$, $(DE^*F^*)$, $(EF^*D^*)$, $(FD^*E^*)$, $(II_aD^*)$, $(II_bE^*)$, $(II_cF^*)$, $(D^*I_bI_c)$, $(E^*I_cI_a)$, $(F^*I_aI_b)$, $(APA_W)$, $(BPB_W)$, $(CPC_W)$ are concurrent at $W/P$ [13] [14].*

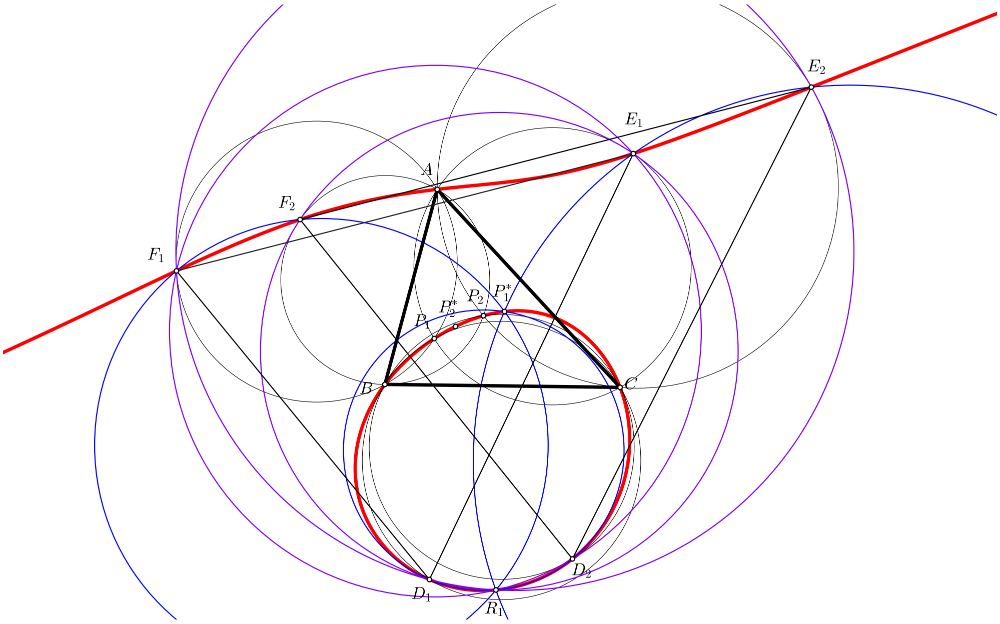


FIGURE 12. Cyclologic triangles inscribed in the circular isogonal $p\mathcal{K}(X_6, W)$

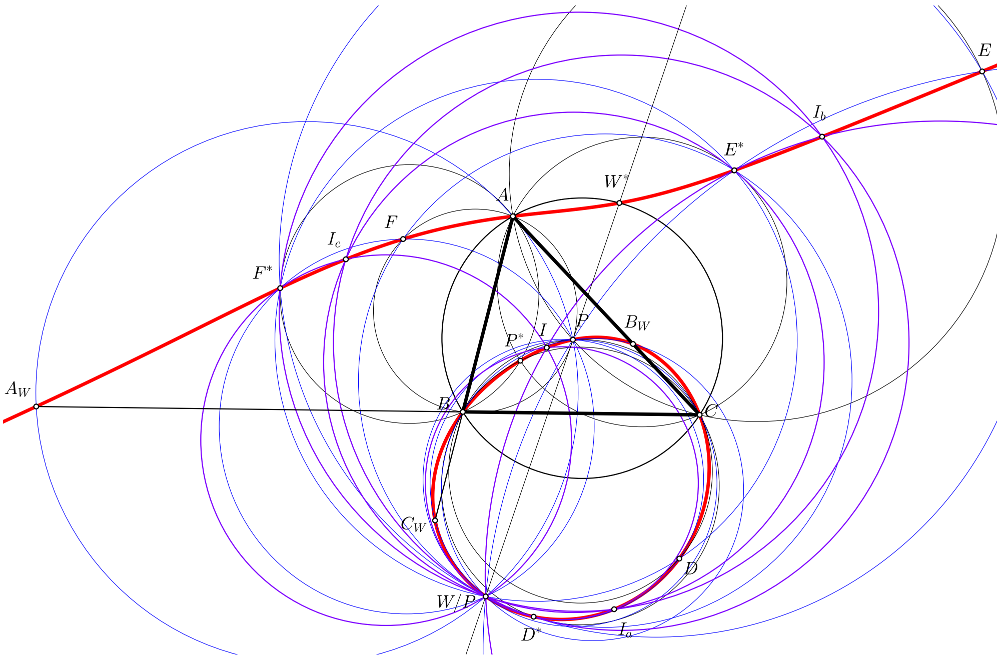


FIGURE 13.

We list here some pairs of points on Neuberg cubic **K001** and their *cyclology centers*.

| Triangle centers | Cyclology centers |
|---|---|
| $(X_1, X_1)$ | $X_{3464}$ |
| $(X_1, X_3)$, $(X_3, X_1)$ | $X_{5667}$, $X_{8485}$ |
| $(X_1, X_4)$, $(X_4, X_1)$ | $X_{399}$, $X_1X_{1263} \cap X_3X_{3065}$ |
| $(X_1, X_{13})$, $(X_{13}, X_1)$ | $X_{5668}$, $X_{8435}$ |
| $(X_1, X_{14})$, $(X_{14}, X_1)$ | $X_{5669}$, $X_{8436}$ |
| $(X_1, X_{15})$, $(X_{15}, X_1)$ | $X_{5623}$, $X_{8523}$ |
| $(X_1, X_{16})$, $(X_{16}, X_1)$ | $X_{5624}$, $X_{8524}$ |
| $(X_3, X_3)$ | $X_4X_{8439} \cap$ **K001** |
| $(X_3, X_4)$, $(X_4, X_3)$ | $X_{5667}$, $X_{399}$ |
| $(X_3, X_{13})$, $(X_{13}, X_3)$ | $X_{15}X_{8439} \cap X_4X_{8471}$, $X_{8441}$ |
| $(X_3, X_{14})$, $(X_{14}, X_3)$ | $X_{16}X_{8439} \cap X_4X_{8479}$, $X_{8442}$ |
| $(X_3, X_{15})$, $(X_{15}, X_3)$ | $X_{8463}$, $X_{5678}$ |
| $(X_3, X_{16})$, $(X_{16}, X_3)$ | $X_{8453}$, $X_{5679}$ |
| $(X_4, X_4)$ | $X_3X_{1263} \cap$ **K001** |
| $(X_4, X_{13})$, $(X_{13}, X_4)$ | $X_{8173}$, $X_{617}$ |
| $(X_4, X_{14})$, $(X_{14}, X_4)$ | $X_{8172}$, $X_{616}$ |
| $(X_4, X_{15})$, $(X_{15}, X_4)$ | $X_{8467}$, $X_{8519}$ |
| $(X_4, X_{16})$, $(X_{16}, X_4)$ | $X_{8475}$, $X_{8520}$ |
| $(X_{13}, X_{13})$ | $X_{8451}$ |
| $(X_{13}, X_{14})$, $(X_{14}, X_{13})$ | $X_{8174}$, $X_{8175}$ |
| $(X_{13}, X_{15})$, $(X_{15}, X_{13})$ | $X_{5668}$, $X_{5623}$ |
| $(X_{13}, X_{16})$, $(X_{16}, X_{13})$ | $X_{14}X_{3479} \cap X_{15}X_{3441}$, $X_{14}X_{3441} \cap X_{15}X_{8492}$ |
| $(X_{14}, X_{14})$ | $X_{8461}$ |
| $(X_{14}, X_{15})$, $(X_{15}, X_{14})$ | $X_{16}X_{3440} \cap X_{13}X_{3480}$, $X_{13}X_{3440} \cap X_{16}X_{8491}$ |
| $(X_{14}, X_{16})$, $(X_{16}, X_{14})$ | $X_{5669}$, $X_{5624}$ |
| $(X_{15}, X_{15})$ | $X_{13}X_{8491} \cap$ **K001** |
| $(X_{15}, X_{16})$, $(X_{16}, X_{15})$ | $X_{8466}$, $X_{8474}$ |
| $(X_{16}, X_{16})$ | $X_{14}X_{8492} \cap$ **K001** |

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