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Computer Discovered Mathematics: Yff Triangles

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Abstract. By using the computer program "Discoverer" we investigate the Yff Triangles.

Keywords. Inner Yff Triangle, Outer Yff Triangle, triangle geometry, notable point, computer discovered mathematics, Euclidean geometry, "Discoverer".

Mathematics Subject Classification (2010). 51-04, 68T01, 68T99.

1. Introduction

The Yff circles are the two triplets of congruent circle in which each circle is tangent to two sides of a reference triangle. In each case, the triplets intersect pairwise in a single point. Te centers of circles form the *Inner Yff Triangle* and the *Outer Yff Trianglem* respectively.

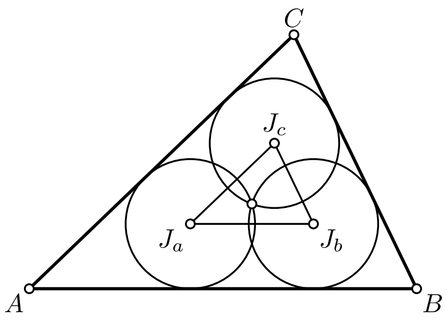

Figure 1.

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Figure 1 illustrates the Inner Yff Circles. In Figure 1, $JaJbJc$ is the Inner Yff Triangle. The point of intersection of the three Inner Yff Circles is the point X(55), the Internal Similitude center of the Incircle of Triangle ABC and the Circumcircle of Triangle ABC.

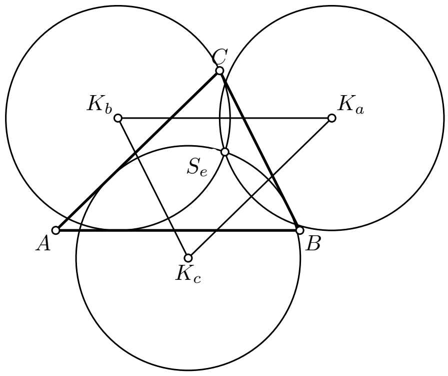

Figure 2.

Figure 2 illustrates the Inner Yff Circles. In Figure 2, $KaKbKc$ is the Outer Yff Triangle. The point of intersection of the three Outer Yff Circles is the point X(56), the External Similitude center of the Incircle of Triangle ABC and the Circumcircle of Triangle ABC.

The reader may find some information about the Yff Circles and Yff Triangles in [11, Yff Circles, Yff Circles Triangles].

In this note we use the computer program " Discoverer" [3] in order to investigate some questions about the Yff Triangles.

We use the barycentric coordinates. For barycentric coordinates we refer the reader to [4], [5], [6], [8]. [9], [12], [2], [1], [10], [11]. Triangle centers are labeled as X(n) in accordance with the Kimberling's Encyclopedia of Triangle Centers ETC [7].

2. Inner Yff Triangle

2.1. **Barycentric Coordinates.** The barycentric coordinates of the Inner Yff Triangle are given in [11, Yff Circles]. Below is an alternative description of the barycentric coordinates:

Theorem 2.1. *The barycentric coordinates of the Inner Yff Triangle $J_aJ_bJ_c$ are as follows:*

$$J_a = (8\Delta^2 + a^2bc, ab^2c, abc^2).$$
$$J_b = (a^2bc, 8\Delta^2 + ab^2c, abc^2).$$
$$J_c = (a^2bc, ab^2c, 8\Delta^2 + abc^2).$$

where Δ is the area of triangle ABC.

2.2. **Homothetic Triangles.** The "Discoverer" has found a number of triangles homothetic to the Inner Yff Triangles. A few of these are as follows:

Theorem 2.2. *The Inner Yff Triangle is homothetic to the following triangles:*

1. *Triangle ABC.*
2. *Medial Triangle.*

(3) *Antimedial Triangle.*
(4) *Euler Triangle.*
(5) *Johnson Triangle.*
(6) *Outer Yff Triangle.*
(7) *Half-Median Triangle.*
(8) *Circumcevian Triangle of the Circumcenter.*
(9) *Triangle of Reflections of the Nine-Point Center in the Sidelines of the Cevian Triangle of the Centroid.*
(10) *Triangle of Reflections of the Incenter in the Sidelines of the Anticevian Triangle of the Incenter.*

Proof. The sides of the Inner Yff Triangle are parallel to the sides of these triangles. □

Theorem 2.3. *The center of homothety of The Inner Yff Triangle and Triangle ABC is the Incenter I of Triangle ABC and the ratio of the homothety is*

$$k = \frac{2\Delta - \rho_1(a+b+c)}{2\Delta},$$

where $\rho_1 = \frac{rR}{r+R}$.

Note that $0 < k < 1$.

Proof. It is obvious that the Incenter is the center of the homothety. By using [4, §2] we find the lengths of segments IA and IJa, and then we find $k = \frac{IJa}{IA}$. □

Problem 2.1. *Find the centers and ratios of homotheties given in Theorem 2.2.*

2.3. **Notable Points of the Inner Yff Triangle.** By using the homothety, given in Theorem 2.3 and the formula given in [4, §5], we can find easily the barycentric coordinates of the notable points of the Inner Yff triangle, provided that the barycentric coordinates of the corresponding notable points of triangle ABC are given.

Theorem 2.4. *The Circumcenter of the Inner Yff Triangle is the point X(55), the Internal Similitude Center of the Incircle and Circumcircle of triangle ABC.*

Theorem 2.5. *The Orthocenter of the Inner Yff Triangle is the point X(1478).*

The reader may find a few additional examples in the Supplementary material. Note that the Folder 1, list K in the Supplementary material extends the table, given in [11, First Yff Circles Triangle].

The "Discoverer" has investigated 201 selected notable points of the Inner Yff Triangle. Of these 10 are available in the Kimberling's ETC [7] and the rest of 191 points are not available in the ETC. These are new notable points. We recommend the reader to investigate the new notable points.

2.4. **Similitude Centers of Circles of the Inner Yff Triangle.**

Theorem 2.6. *The Internal Center of Similitude of the Circumcircle of Triangle ABC and the Circumcircle of the Inner Yff Triangle is the point X(35).*

We can use the following description of point X(35) [7, X(35)]: Let A' be the inverse-in-circumcircle of the A-excenter, and define B' and C' cyclically. Then the lines AA', BB', CC' concur in X(35).

The "Discoverer" has investigated 841 Internal Similitude centers of selected notable circles of the Inner Yff Triangle. Of these 32 are included in [7] and the rest of 809 points are new notable points. See Folder 2 in the Supplementary material.

Theorem 2.7. *The External Center of Similitude of the Circumcircle of Triangle ABC and the Circumcircle of the Inner Yff Triangle is the Incenter of Triangle ABC.*

The "Discoverer" has investigated 830 External Similitude centers of selected notable circles of the Inner Yff Triangle. Of these 46 are included in [7] and the rest of 784 points are new notable points. See Folder 3 in the Supplementary material.

The teachers may use these results in order to create problems for high-schools (and olympiads, in the case or more complicated results).

2.5. **Monge Triangle of the Inner Yff Circles.** Denote by $JaJbJc$ the Inner Yff Triangle. Denote by Ma, Mb, Mc the internal similitude centers of the Inner Yff circles (Jb) and (Jc), (Jc) and (Ja), and (Ja) and (Jb) respectively. Then triangle $MaMbMb$ is the *Monge triangle of the Inner Yff Circles.* It easy to see that the Monge triangle of the Inner Yff Circles is the Medial Triangle of the Inner Yff Triangle.

In Figure 3 the Monge triangle of the Inner Yff Circles is the triangle inscribed in triangle $JaJbJc$.

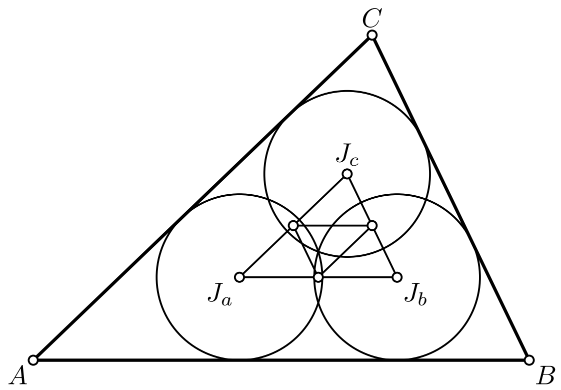

FIGURE 3.

By using the barycentric coordinates of the Inner Yff Triangle we can easily find the barycentric coordinates of the Monge triangle of the Inner Yff Circles.

Theorem 2.8. *The Nagel Point of the Monge Triangle of the Inner Yff Circles is the Incenter of Triangle ABC.*

The "Discoverer" has investigated 201 selected notable points of the Monge Triangle of the Inner Yff Circles. Of these 6 are available in [7] and the rest of 195 points are new notable points. See Folder 4 in the Supplementary material.

The teacher may use these results in order to create problems for high-schools (and olympiads, in the case or more complicated results).

2.6. **Inner Moses Triangle of the Inner Yff Circles.** Denote by Qa, Qb, Qc the internal similitude centers of the Circumcircle of the Inner Yff Triangle and Yff circles $(Ja), (Jb), (Jc)$ respectively. Then triangle $QaQbQc$ is the *Inner Moses Triangle of the Inner Yff Circles.*

In Figure 4 the Moses triangle of the Inner Yff Circles is the triangle inside the Inner Yff Triangle $JaJbJc$.

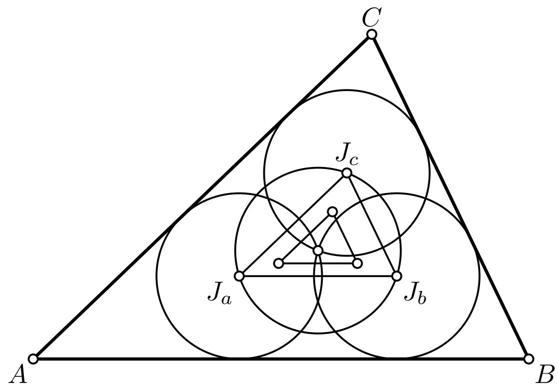

FIGURE 4.

We recommend the reader to investigate this triangle.

3. OUTER YFF TRIANGLE

3.1. **Barycentric Coordinates.** The barycentric coordinates of the Outer Yff Triangle are given in [11, Yff Circles]. Below is an alternative description of the barycentric coordinates:

Theorem 3.1. *The barycentric coordinates of the Outer Yff Triangle $K_aK_bK_c$ are as follows:*

$$K_a = (a^2bc - 8\Delta^2, ab^2c, abc^2).$$
$$K_b = (a^2bc, ab^2c - 8\Delta^2, abc^2).$$
$$K_c = (a^2bc, ab^2c, abc^2 - 8\Delta^2).$$

where Δ is the area of triangle ABC.

3.2. **Homothetic Triangles.** The "Discoverer" has found a number of triangles homothetic to the Outer Yff Triangles. Note that the triangles homothetic to the Outer Yff Triangle are the same as triangles homothetic to the Inner Yff Triangles (and clearly, the same as triangle homothetic to Triangle ABC).

Theorem 3.2. *Triangle ABC and the Inner Yff Triangle are homothetic. The center of homothety is the Incenter I of triangle ABC and the ratio of the homothety is*

$$k = \frac{2\Delta - \rho_2(a+b+c)}{2\Delta},$$

where $\rho_1 = \frac{rR}{R-r}$.

Note that $k < 0$.

Proof. It is obvious that the Incenter is the center of homothety. By using [4, §2] we find the lengths of segments IA and KJa, and then we find $k = \frac{IKa}{IA}$. ☐

3.3. **Notable Points of the Outer Yff Triangle.** By using the homothety, given in Theorem 3.2 and the formula given in [4, §5], we can find easily the barycentric coordinates of the notable points of the Outer Yff triangle, provided that the barycentric coordinates of the corresponding notable points of triangle ABC are given.

Theorem 3.3. *The Circumcenter of the Outer Yff Triangle is the point X(56), the External Similitude Center of the Incircle and Circumcircle of triangle ABC.*

Theorem 3.4. *The Orthocenter of the Outer Yff Triangle is the point X(1479).*

The reader may find a few additional examples in the Supplementary material. Note that the Folder 5, list K in the Supplementary material extends the table, given in [11, Second Yff Circles Triangle].

The "Discoverer" has investigated 201 selected notable points of the Outer Yff Triangle. Of these 12 are available in the Kimberling's ETC [7] and the rest of 189 points are not available in the ETC. These are new notable points. We recommend the reader to investigate the new notable points.

3.4. **Similitude Centers of Circles of the Outer Yff Triangle.**

Theorem 3.5. *The Internal Center of Similitude of the Circumcircle of Triangle ABC and the Nine-Point Circle of the Outer Yff Triangle is the point X(497).*

The "Discoverer" has investigated 841 Internal Similitude centers of selected notable circles of the Outer Yff Triangle. Of these 54 are included in [7] and the rest of 787 points are new notable points. See Folder 6 in the Supplementary material.

Theorem 3.6. *The External Center of Similitude of the Incircle of Triangle ABC and the Nine-Point Circle of the Outer Yff Triangle is the Feuerbach point of triangle ABC.*

The "Discoverer" has investigated 830 External Similitude centers of selected notable circles of the Outer Yff Triangle. Of these 32 are included in [7] and the rest of 798 points are new notable points. See Folder 7 in the Supplementary material.

The teachers may use these results in order to create problems for high-schools (and olympiads, in the case or more complicated results).

3.5. **Monge Triangle of the Outer Yff Circles.** Denote by $KaKbKC$ the Outer Yff Triangle. Denote by Ma, Mb, Mc the internal similitude centers of the Outer Yff circles (Kb) and (Kc), (Kc) and (Ka), and (Ka) and (Kb), respectively. Then triangle $MaMbMb$ is the *Monge triangle of the Outer Yff Circles.* It easy to see that the Monge triangle of the Outer Yff Circles is the Medial Triangle of the Outer Yff Triangle.

In Figure 5 the Monge triangle of the Outer Yff Circles is the triangle inscribed in triangle $JaJbJc$.

By using the barycentric coordinates of the Outer Yff Triangle we can easily find the barycentric coordinates of the Monge triangle of the Outer Yff Circles.

Theorem 3.7. *The Orthocenter of the Monge Triangle of the Outer Yff Circles is the point X(56), the External similitude center of the Incenter of triangle ABC and the Circumcircle of triangle ABC.*

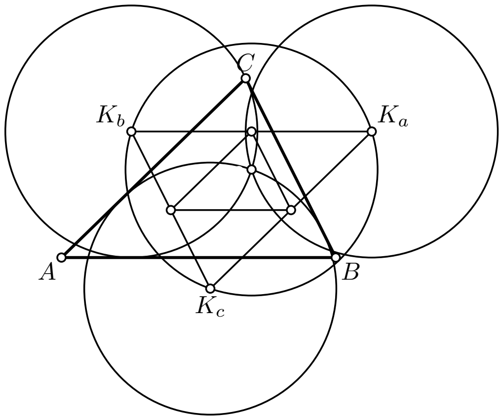

FIGURE 5.

The "Discoverer" has investigated 201 selected notable points of the Monge Triangle of the Outer Yff Circles. Of these 7 are available in [7] and the rest of 194 points are new notable points. See Folder 8 in the Supplementary material.

The teacher may use these results in order to create problems for high-schools (and olympiads, in the case or more complicated results).

3.6. **Outer Moses Triangle of the Outer Yff Circles.** Denote by $KaKbKc$ the Outer Yff Triangle. Denote by Qa, Qb, Qc the internal similitude centers of the Circumcircle of the Outer Yff Triangle and Yff circles $(Ka), (Kb), (Kc)$ respectively. Then triangle $QaQbQc$ is the *Outer Moses Triangle of the Outer Yff Circles.*

In Figure 6 the Moses triangle of the Outer Yff Circles is the triangle inside the Outer Yff Triangle $KaKbKc$.

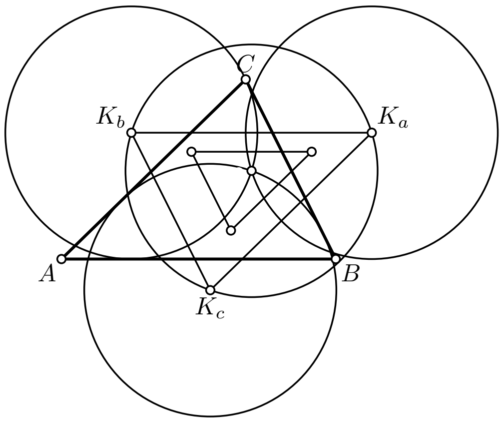

FIGURE 6.

We recommend the reader to investigate this triangle.

SUPPLEMENTARY MATERIAL

The enclosed supplementary material contains theorems related to the topic.

ACKNOWLEDGEMENT

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