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Computer Discovered Mathematics: A Note on the Johnson Circles

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Abstract. By using the computer program "Discoverer" we study the Johnson circles.

Keywords. Johnson circles, triangle geometry, remarkable point, computer discovered mathematics, Euclidean geometry, "Discoverer".

Mathematics Subject Classification (2010). 51-04, 68T01, 68T99.

1. Introduction

Johnson's theorem states that if three congruent circles mutually intersect one another in a single point, then the circle passing through their other three pairwise points of intersection is congruent to the original three circles.

The three congruent circles in the Johnson theorem are known as the *Johnson circles*. The circle passing through the other three pairwise points of intersection of the Johnson circles, is known as the *Johnson Circle*. The triangle having as vertices the centers of the Johnson circles is known as the *Johnson triangle.*

Given triangle ABC and its Orthocenter H. It is known that the circles (HBC), (HCA) and (HAB) are Johnson circles.

Figure 1 illustrates the Johnson circles. In Figure 1, H is the Orthocenter, $(J_a), (J_b), (J_c)$ are the Johnson circles, circle (J_a) is the circumcircle of triangle HBC, circle (J_b) is the circumcircle of triangle HCA, circle (J_c) is the circumcircle of triangle HAB. The circles $(J_a), (J_b), (J_c)$ are congruent. In accordance with the Johnson's theorem, the Johnson circles are congruent with the Circumcircle of triangle ABC.

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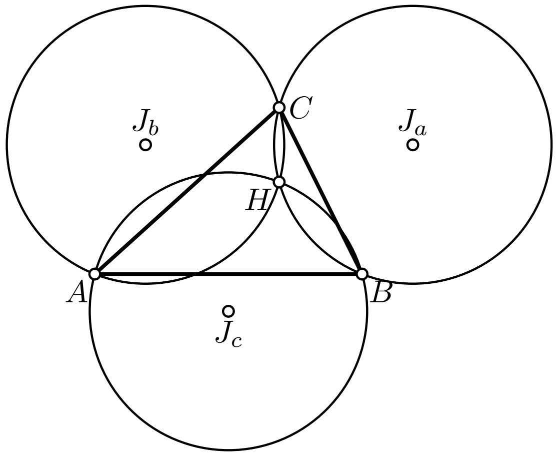

FIGURE 1.

In this paper we denote by $\mathcal{C}$ the Johnson triad of circles $(J_a),(J_b),(J_c)$.

We use barycentric coordinates [5]. See also [1],[2],[6].[7],[9].[10].[11],[14]. For the definitions see [12],[13], [4, Definitions].

2. The Johnson Triangle $J_aJ_bJ_c$

It is known that the Johnson triangle $J_aJ_bJ_c$ and triangle ABC are congruent.

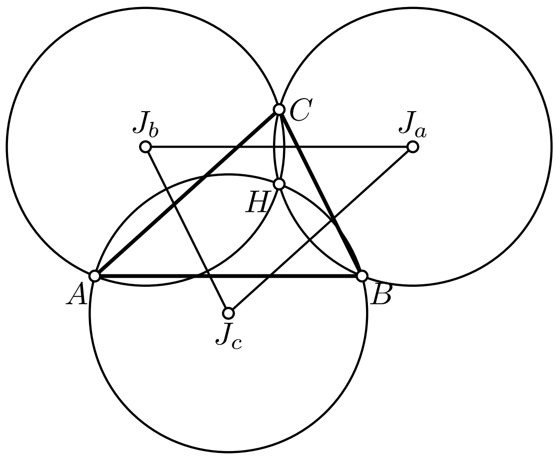

FIGURE 2.

In Figure 2, $JaJbJc$ is the Johnson triangle.

The following theorem is given in [12, Johnson Circles].

Theorem 2.1. *The barycentric coordinates of the Johnson Triangle $J_aJ_bJ_c$ are as follows:*

$$uJ_a = a^2(-a^2+b^2+c^2),$$
$$vJ_a = a^4+c^4-2a^2c^2-a^2b^2-b^2c^2,$$
$$wJ_a = a^4+b^4-2a^2b^2-a^2c^2-b^2c^2.$$
$$uJ_b = b^4+c^4-2b^2c^2-a^2b^2-a^2c^2,$$
$$vJ_b = b^2(-b^2+c^2+a^2),$$
$$wJ_b = a^4+b^4-2a^2b^2-a^2c^2-b^2c^2.$$

$$uJ_c = b^4 + c^4 - 2b^2c^2 - a^2b^2 - a^2c^2,$$
$$vJ_c = a^4 + c^4 - 2a^2c^2 - a^2b^2 - b^2c^2,$$
$$wJ_c = c^2(-c^2 + a^2 + b^2).$$

Table 1 gives the several centers of the Johnson Triangle in terms of the centers of the Reference triangle that are Kimberling centers.

	Center of the Johnson Triangle	**Center of the Reference triangle**
1	X(1) Incenter	X(355) Center of the Fuhrmann Circle
2	X(2) Centroid	X(381) Center of the Orthocentroidal Circle
3	X(3) Circumcenter	X(4) Orthocenter
4	X(4) Orthocenter	X(3) Circumcenter
5	X(5) Nine-Point Center	X(5) Nine-Point Center

TABLE 1.

The Supplementary material contains an extension of the Table 1. We have investigated 201 triangle centers of the Johnson triangle. Between them 51 are Kimberling centers and the rest of 150 centers are new notable points of the Triangle.

Problem 2.1. *Prove the theorems given in Table 1.*

Hint. Triangles $JaJbJc$ and ABC are homothetic with Nine-Point Center as center of homothety and factor -1. Use formula [5, (17)].

Problem 2.2. *Prove that the Perspector of the Johnson Triangle and the*

(1) *Medial Triangle is the X(3) Circumcenter.*
(2) *Orthic Triangle is the X(155) Orthocenter of the Tangential Triangle.*
(3) *Antimedial Triangle is the X(4) Orthocenter.*
(4) *Tangential Triangle is the X(3) Circumcenter.*
(5) *Euler Triangle is the X(381) Center of the Orthocentroidal Circle.*
(6) *Inner Grebe Triangle is the X(6215).*
(7) *Outer Grebe Triangle is the X(6214).*
(8) *Fuhrmann Triangle is the X(3) Circumcenter.*
(9) *Inner Yff Triangle is the X(12) Feuerbach Perspector.*
(10) *Outer Yff Triangle is the X(11) Feuerbach Point.*
(11) *Half-Median Triangle is the X(1656).*

Proof. (1). Let $MaMbMc$ be the Medial Triangle and O be the Circumcenter. Then Points $MaJaO$ lie on the same line. (1) We use [5, §1, (4)]. We obtain

$$\begin{bmatrix} 0 & 1 & 1 \\ a^2(b^2+c^2-a^2) & a^4+c^4-2a^2c^2-a^2b^2-b^2c^2 & a^4+b^4-2a^2b^2-a^2c^2-b^2c^2 \\ a^2(b^2+c^2-a^2) & b^2(c^2+a^2-b^2) & c^2(a^2+b^2-c^2) \end{bmatrix} = 0.$$

Similarly, points $MbJbO$ lie on the same line, and points $McJcO$ lie on the same line. Hence, the Circumcenter O is the point of intersection of lines $MaJa, MbJb$ and $McJc$. □

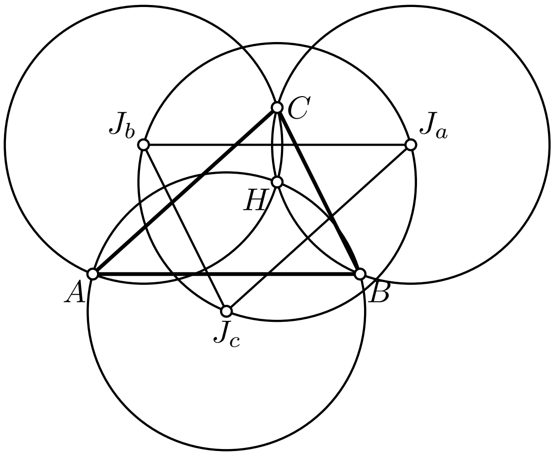

FIGURE 3.

The following problems include the Circumcircle of triangle $JaJbJc$. See Figure 3,

Problem 2.3. *Prove that the Internal Center of Similitude of the Circumcircle of the Johnson Triangle and the Circumcircle is the X(5) Nine-Point Center.*

Problem 2.4. *Prove that the External Center of Similitude of the Circumcircle of the Johnson Triangle and the Incircle is the X(1479) Center of the Outer Johnson-Yff Circle.*

The Supplementary material contains a number of similitude centers of the Circumcircle of triangle $JaJbJc$.

3. MONGE TRIANGLE OF THE JOHNSON CIRCLES

The Monge Triangle of the Johnson Circles is the Euler Triangle of Triangle ABC. Hence, in fact we study here the Euler triangle of Triangle ABC.

Let Ma be the internal center of similitude of circles (Jb) and (Jc). Similarly, define Mb as the internal center of similitude of circles (Jc) and (Ja), and define Mc as the internal center of similitude of circles (Ja) and (Jb). Then $MaMbMc$ is the *Monge triangle of the Johnson circles*. In Figure 4, the triangle inside triangle ABC is the Monge triangle.

It is obvious that the Monge Triangle of the triad $(\mathcal{C})$ is the Medial Triangle of the Johnson triangle $JabJc$. Hence, we can easily find the barycentric coordinates of the vertices of the Monge Triangle of the triad $(\mathcal{C})$.

Theorem 3.1. *The barycentric coordinates of the Monge troangle $MaMbMc$ of the Johnson circles are as follows:*

$$\begin{aligned}
uMa &= 2a^2b^2 + 2a^2c^2 + 4b^2c^2 - 2b^4 - 2c^4,\\
vMa &= (-a^2 + b^2 + c^2)(-c^2 + a^2 + b^2),\\
wMa &= (-a^2 + b^2 + c^2)(-b^2 + c^2 + a^2).\\
uMb &= (-b^2 + c^2 + a^2)(-c^2 + a^2 + b^2),\\
vMb &= 4a^2c^2 - 2c^4 - 2a^4 + 2a^2b^2 + 2b^2c^2,\\
wMb &= (-a^2 + b^2 + c^2)(-b^2 + c^2 + a^2).
\end{aligned}$$

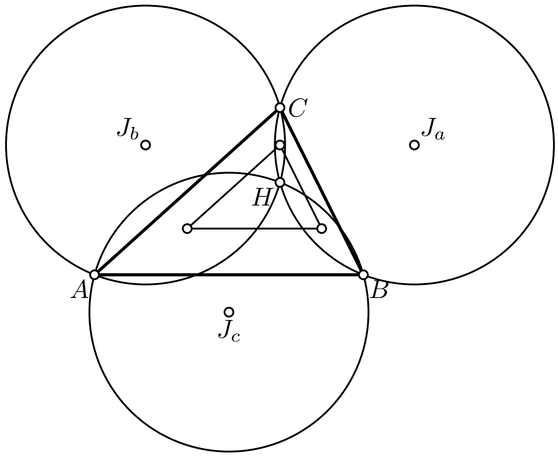

FIGURE 4.

$$uMc = (-b^2 + c^2 + a^2)(-c^2 + a^2 + b^2),$$
$$vMc = (-a^2 + b^2 + c^2)(-c^2 + a^2 + b^2),$$
$$wMc = 4a^2b^2 - 2b^4 - 2a^4 + 2a^2c^2 + 2b^2c^2.$$

4. INNER MOSES TRIANGLE OF THE JOHNSON CIRCLES

Let Na be the internal center of similitude of circles (Ja) and $(JaJbJc)$. Similarly, define Nb as the internal center of similitude of circles (Jb) and $(JaJbJc)$, and define Nc as the internal center of similitude of circles (Jc) and $(JaJbJc)$. Then $NaNbNc$ is the *Inner Moses triangle of the Johnson triad of circles.* In Figure 5, triangle $NaNbNc$ is the Inner Moses triangle.

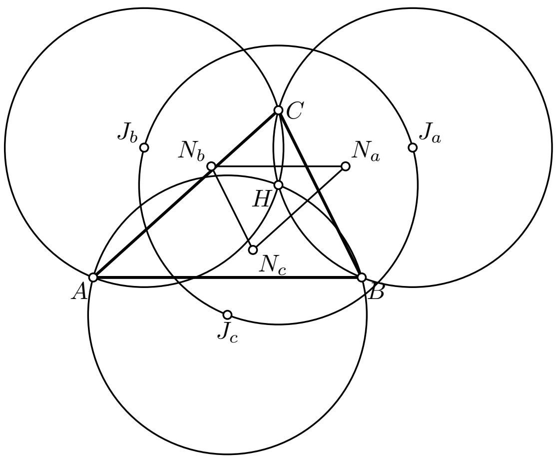

FIGURE 5.

Theorem 4.1. *The barycentric coordinates of the Inner Moses Triangle $NaNbNc$ of the Johnson circles are as follows:*

$$uNa = 2a^4 - a^2b^2 - a^2c^2 + 2b^2c^2 - b^4 - c^4,$$
$$vNa = a^2b^2 + 4a^2c^2 + b^2c^2 - 2c^4 - 2a^4 + b^4,$$
$$wNa = 4a^2b^2 + a^2c^2 + b^2c^2 - 2b^4 - 2a^4 + c^4.$$
$$uNb = a^2b^2 + a^2c^2 + 4b^2c^2 - 2b^4 - 2c^4 + a^4,$$

$$vNb = 2b^4 - b^2c^2 - a^2b^2 + 2a^2c^2 - c^4 - a^4.$$
$$wNb = 4a^2b^2 + a^2c^2 + b^2c^2 - 2b^4 - 2a^4 + c^4.$$
$$uNc = a^2b^2 + a^2c^2 + 4b^2c^2 - 2b^4 - 2c^4 + a^4,$$
$$vNc = a^2b^2 + 4a^2c^2 + b^2c^2 - 2c^4 - 2a^4 + b^4,$$
$$wNc = 2c^4 - a^2c^2 - b^2c^2 + 2a^2b^2 - a^4 - b^4$$

Supplementary Material

The enclosed supplementary material contains theorems related to the topic.

Acknowledgement

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