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Computer Discovered Mathematics: Inversion of Triangle ABC with respect to the Incircle

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Abstract. We study the inverse image of the sidelines of Triangle ABC with respect to the Incircle of Triangle ABC. The Supplementary material contains more than 1000 theorems discovered by the computer program "Discoverer".

Keywords. inversion, triangle geometry, remarkable point, computer discovered mathematics, Euclidean geometry, "Discoverer".

Mathematics Subject Classification (2010). 51-04, 68T01, 68T99.

1. Introduction

Consider inverse of Triangle ABC wrt the Incircle of Triangle ABC. See Figure 1.

In Figure 1, c is the Incircle ABC, I is the Incenter, Ka, Kb, Kc are the contact points of the Incircle and the sides BC, CA, AB respectively. That is, $KaKbKc$ is the Intouch Triangle of Triangle ABC. recall that the Intouch Triangle is the Cevian Triangle of the Gergonne Point.

Points iA, iB, iC are the inverse images of the vertices A, B, C wrt the Incircle, respectively. We denote by $iAiBiC$ the triangle having as vertices points iA, iB, iC.

The inverse image of the line BC wrt the Incircle is the circle with center J_a. Similarly, the inverse image of the line CA wrt the Incircle is the circle with center J_b and the inverse image of the line AB wrt the Incircle is the circle with center J_c.

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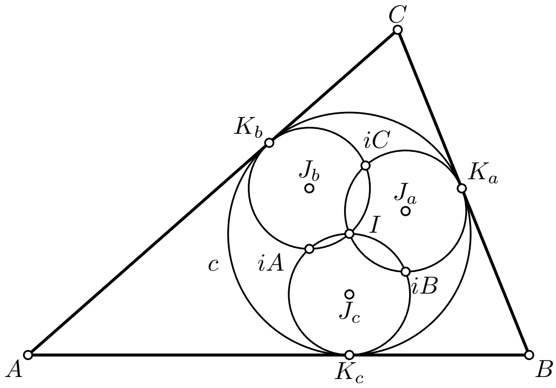

Figure 1.

These three circles are congruent. The radius of the circles is equal to $r/2$, where r is the radius of the Incircle.

The three circles $(J_a),(J_b),(J_c)$ intersect in the Incenter. Also, circles (J_b) and (J_c) intersect in the point iA. Similarly, iB is the point of intersection of circle (J_c) and (J_a), and iC is the point of intersection of circle (J_a) and (J_b).

The triad of circles $(J_a),(J_b),(J_b)$ is a Johnson triad of circles. See [5, Johnson Circles], [3]. In this triad, the triangle $J_aJ_bJ_c$ plays the role of the Johnson triangle, and the circumcircle of triangle $iAiBiC$ plays the role of the Johnson circle.

From the Johnson configuration it follows that triangles $JaJbJc$ and $iAiBiC$ are congruent.

In this paper we denote by $\mathcal{C}$ the triad of circles $(J_a),(J_b),(J_c)$.

Also, below we show that Triangle $iAiBiC$ is the Medial Triangle of the Intouch Triangle $KaKbKc$ of Triangle ABC.

2. Triangle $iAiBiC$

By using the "Discoverer" we will study the triangle $iAiBiC$. See Figure 2.

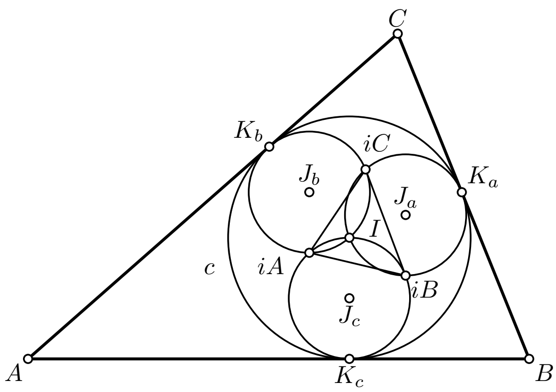

Figure 2.

2.1. **Barycentric Coordinates.**

Theorem 2.1. *The barycentric coordinates of triangle $iAiBiC$ are as follows:*

$$iA = (ab + ac + 2bc - b^2 - c^2, b(b + c - a), c(b + c - a)),$$

$$iB = (a(a+c-b), ab+bc+2ac-a^2-c^2, c(a+c-b)),$$
$$iC = (a(a+b-c), b(a+b-c), ac+bc+2ab-a^2-b^2).$$

Proof. We will find the inverses of vertices A, B, C, respectively, by using [2, §6]. First, we find the squares of the distances from the Incenter I and vertices A, B, C respectively, by using [2, §2]. We obtain:

$$|IA|^2 = \frac{bc(b+c-a)}{a+b+c},$$
$$|IB|^2 = \frac{ac(a+c-b)}{a+b+c},$$
$$|IC|^2 = \frac{ab(a+b-c)}{a+b+c}.$$

The radius of the Incircle is the Inradius. The barycentric coordinates of the inverses of the vertices A, B, C are given in the statement of the theorem. □

2.2. Triangle $iAiBiC$ is the Medial Triangle of the Intouch Triangle.

Theorem 2.2. *Triangle $iAiBiC$ is the Medial Triangle of the Introuch Triangle.*

Proof. By using [2, §4, (14)] we prove that point iA is the midpoint of $KbKc$. Similarly, point iB is the midpoint of $KcKa$ and point iC is the midpoint of $KaKb$. □

2.3. **Area of triangle $iAiBiC$.** From Theorem 2.2 we see that the area of Triangle $iAiBiC$ is the area of the Intouch Triangle divided by 4. See [5, Contact Triangle] for the area of the Intouch Triangle.

Also, we could calculate the area directly, by using [4, §1, (2)]. In both cases we obtain

Theorem 2.3. *The area of triangle $iAiBiC$ is*

$$\frac{(b+c-a)(c+a-b)(a+b-c)}{16abc}\Delta,$$

where Δ is the area of triangle ABC.

2.4. Congruent Triangles.

Problem 2.1. *Prove that the triangle $iAiBiC$ is congruent to the Half-Cevian Triangle of the Nagel Point.*

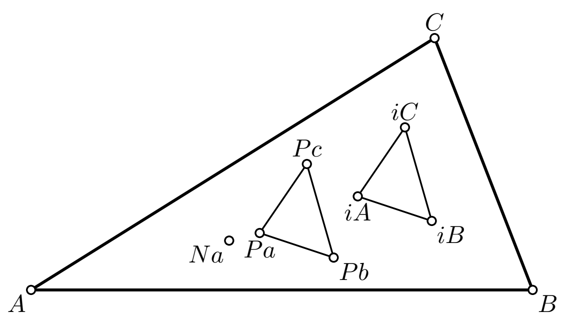

FIGURE 3.

Figure 3 illustrates Theorem 2.1. In Figure 3, Na is the Nagel Point and the $PaPbPc$ is the Half-Cevian Triangle of Na. The triangles $PaPbPc$ and $iAiBiC$ are congruent.

2.5. **Similar Triangles.** Triangle $iAiBiC$ is similar with the Intouch Triangle, since it is the Medial Triangle of the Intouch Triangle. Hence, triangle $iAiBiC$ belongs to the group od triangles similart with the Intouch Triangle. Hence, we obtain the following theorem:

Theorem 2.4. *Triangle $iAiBiC$ is similar to the following triangles:*

(1) *Excentral Triangle.*
(2) *Hexyl Triangle.*
(3) *Fuhrmann Triangle.*
(4) *Yff Central Triangle.*

Theorem 2.5. *The ratio of similarity k of the Excentral Triangle $PaPbPc$ and triangle $iAiBiC$ is*

$$k = \frac{|PbPc|}{|iBiC|} = \frac{|PcPa|}{|iCiA|} = \frac{|PaPb|}{|iAiB|} = \frac{8abc}{(b+c-a)(c+a-b)(a+b-c)}.$$

Proof. The side lengths of triangle $iAiBiC$ could be calculated by using [2, §2]. Also, we could use a reference for the side lengths of the Intouch triangle and Theorem 2.2. In both the cases the side lengths of triangle $iAiBiC$ are as follows:

$$|iBiC| = \frac{(b+c-a)\sqrt{(c+a-b)(a+b-c)}}{4\sqrt{bc}},$$

$$|iCiA| = \frac{(c+a-b)\sqrt{(a+b-c)(b+c-a)}}{4\sqrt{ca}},$$

$$|iAiB| = \frac{(a+b-c)\sqrt{(b+c-a)(c+a-b)}}{4\sqrt{ab}}.$$

The side lengths of the Excentral triangle $EaEbEc$, $Ea = (-a, b, c)$, $Eb = (a, -b, c)$, $Ec = (a, b, -c)$ are as follows:

$$|EbEc| = \frac{2a\sqrt{bc}}{\sqrt{(c+a-b)(a+b-c)}},$$

$$|EcEa| = \frac{2b\sqrt{ca}}{\sqrt{(a+b-c)(b+c-a)}},$$

$$|EaEb| = \frac{2c\sqrt{ab}}{\sqrt{(b+c-a)(c+a-b)}}.$$

Now we easily obtain the formula for k. □

Problem 2.2. *Find the ratio of similarity of triangle $iAiBiC$ and triangles given in Theorem 2.4.*

	Point of triangle $iAiBiC$	Point of the Reference triangle
1	X(1) Incenter	X(557)
2	X(2) Centroid	X(354) Weill Point = Centroid of the Intouch Triangle
3	X(4) Orthocenter	X(1) Incenter
4	X(8) Nagel Point	X(177) Incenter of the Intouch Triangle
5	X(20) de Longchamps Point	X(65) Orthocenter of the Intouch Triangle
5	X(69) Retrocenter	X(7) Gergonne Point

TABLE 1.

2.6. **Notable Points.** Table 1 gives several notable points of triangle $iAiBiC$ in terms of notable points of the Reference triangle ABC.

Problem 2.3. *Prove the results given in Table 1.*

Hint about the row 3 of the Table. The Incenter of triangle ABC is the Orthocenter of triangle $iAiBiC$ becuase the triad of circles $(J_a), (J_b), (J_c)$ is a Johnson triad.

Problem 2.4. *By using compass and ruler, construct the Retrocenter of traingle* $iAiBiC$.

Hint. Construct the Geronne Point of Triangle ABC. See Table 1, row 5.

2.7. **New Notable Points.** The Supplementary Material, Folder 1, contains extension of Table 1. It also contains a number of new notable points that are not available in the Kimberling's ETC [4]. The "Discoverer" finds properties of these new notable points. Points below are not available in [4].

2.7.1. *Gergonne Point of triangle iAiBiC.*

Problem 2.5. *The Gergonne Point of triangle iAiBiC is the*

(1) *Mittenpunkt of the Intouch Triangle.*
(2) *Gergonne Point of the Medial Triangle of the Intouch Triangle.*
(3) *Symmedian Point of the Excentral Triangle of the Intouch Triangle.*
(4) *Quotient of the Mittenpunkt of the Intouch Triangle and the Centroid.*
(5) *Pedal Corner Product of the Mittenpunkt of the Intouch Triangle and the Circumcenter.*

Problem 2.6. *The Gergonne Point of triangle iAiBiC lies on the Brocard Circle of the Excentral Triangle of the Intouch Triangle.*

Problem 2.7. *The Point Gergonne Point of triangle iAiBiC lies on the following lines:*

(1) *Line through the First Mid-Arc Point and the Gergonne Point.*
(2) *Line through the Incenter and the Gergonne Point of the Excentral Triangle.*
(3) *Line through the Centroid and the Mittenpunkt of the Intouch Triangle.*

2.8. **Perspectors.** The "Discoverer" has discovered a number of triangls perspective with triangle $iAiBiC$. See the Supplementary Material, Folder 2. Below is a part of these perspectors.

Problem 2.8. *Prove that the Perspector of triangle iAiBiC and the*

(1) *Triangle ABC is the X(1) Incenter.*
(2) *Medial Triangle is the X(142) Mittenpunkt of the Medial Triangle.*
(3) *Intouch Triangle (Homothetic Triangles) is the X(354) Weill Point.*
(4) *Hexyl Triangle (Homothetic Triangles) is the X(3333) Pohoata Point.*
(5) *Yff Central Triangle (Homothetic Triangles) is the X(8083).*
(6) *Circum-Anticevian Triangle of the Incenter (Homothetic Triangles) is the X(57) Isogonal Conjugate of the Mittenpunkt.*
(7) *Pedal Triangle of the Nine-Point Center of the Intouch Triangle is the X(942) Nine-Point Center of the Intouch Triangle.*
(8) *Triangle of Reflections of the Spieker Center in the Sidelines of the Cevian Triangle of the Centroid (Homothetic Triangles) is the X(3874).*
(9) *Triangle of Reflections of the Vertices of the Cevian Triangle of the Gergonne Point in the Incenter (Homothetic Triangles) is the X(65) Orthocenter of the Intouch Triangle.*
(10) *Triangle of Reflections of the Vertices of the Cevian Triangle of the Gergonne Point in the Spieker Center (Homothetic Triangles) is the X(3555).*
(11) *Triangle of Reflections of the Vertices of the Anticevian Triangle of the Incenter in the Spieker Center (Homothetic Triangles) is the X(938).*
(12) *Triangle of Reflections of the Vertices of the Anticevian Triangle of the Incenter in the Bevan Point (Homothetic Triangles) is the X(3339).*
(13) *Triangle of the Orthocenters of the Triangulation Triangles of the Incenter is the X(7) Gergonne Point.*
(14) *Triangle of the Centers of the Taylor Circles of the Triangulation Triangles of the Incenter is the X(3664).*
(15) *Triangle of the de Longchamps Points of the Cevian Corner Triangles of the Nagel Point (Homothetic Triangles) is the X(962).*
(16) *Triangle of the Nine-Point Centers of the Pedal Corner Triangles of the Reflection of the Circumcenter in the Incenter (Homothetic Triangles) is the X(5901).*
(17) *Triangle of the Circumcenters of the Pedal Corner Triangles of the Orthocenter of the Intouch Triangle is the X(5836).*

2.9. **Circumcircle of Triangle $iAiBiC$.** The circumcircle of triangle $iAiBiC$ is the Johnson circle of Triad $\mathcal{C}$. The circle is congruent to circles of the triad. It follows that the radius of the circle is $\frac{r}{2}$, where r is the Inradius, that is, the radius if the Incircle.

Problem 2.9. *The Center of the Circumcircle of triangle iAiBiC is the point X(942) Nine-Point Center of the Intouch Triangle.*

Problem 2.10. *Prove that point X(5083) lies on the Circumcircle of Triangle iAiBiC. This point is the Euler Reflection Point of the Medial Triangle of the Intouch Triangle.*

Problem 2.11. *Prove that Feuerbach Point of the Intouch Triangle lies on the Circumcircle of Triangle iAiBiC. This point is not available in* [4].

Problem 2.12. *Prove that if a point P lies on the Circumcircle of Triangle ABC, then the inverse of P wrt the Incircle lies on the Circumcircle of Triangle $iAiBiC$.*

Problem 2.13. *Prove that the Internal Center of Similitude of the Circumcircle of triangle $iAiBiC$ and the Incircle is the point X(354) Weill Point.*

Problem 2.14. *Prove that the Internal Center of Similitude of the Circumcircle of triangle $iAiBiC$ and the Nine-Point Circle is the pointX(226).*

Problem 2.15. *Prove that the Internal Center of Similitude of the Circumcircle of triangle $iAiBiC$ and the Excentral Circle is the point X(1) Incenter.*

Problem 2.16. *Prove that the Internal Center of Similitude of the Circumcircle of triangle $iAiBiC$ and the Spieker Circle is the point X(3812).*

Problem 2.17. *Prove that the External Center of Similitude of the Circumcircle of triangle $iAiBiC$ and the Incircle is the point X(65) Orthocenter of the Intouch Triangle.*

Problem 2.18. *Prove that the External Center of Similitude of the Circumcircle of triangle $iAiBiC$ and the Nine-Point Circle is the point X(1210).*

Problem 2.19. *Prove that the External Center of Similitude of the Circumcircle of triangle $iAiBiC$ and the Excentral Circle is the point X(3339).*

Problem 2.20. *Prove that the External Center of Similitude of the Circumcircle of triangle $iAiBiC$ and the Inner Johnson-Yff Circle is the point X(1837).*

We could investigate by using the "Discoverer" also other circles related to triangle $iAiBiC$. For exmple:

Problem 2.21. *Prove that the Internal Center of Similitude of the Nine-Point Circle of triangl $iAiBiC$ and the Spieker Circle is the point X(3742).*

3. Central Triangle of the triad $\mathcal{C}$

Triangle $JaJbJc$ is the central triangle of the triad $\mathcal{C}$. From the Johnson configuration it follows that triangle $JaJbJv$ is congruent to triangle $iAiBiC$.

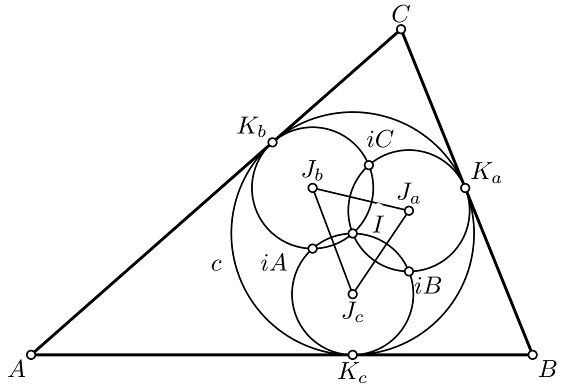

FIGURE 4.

Figure 4 illustrates Triangle $JaJbJc$.

3.1. **Barycentric Coordinates.**

Theorem 3.1. *The barycentric coordinates of triangle $J_aJ_bJ_c$ are as follows:*

$$J_a = (2a^2, 4ab + a^2 + b^2 - c^2, 4ac + a^2 + c^2 - b^2),$$
$$J_b = (4ab + a^2 + b^2 - c^2, 2b^2, 4bc + b^2 + c^2 - a^2),$$
$$J_c = (4ac + a^2 + c^2 - b^2, 4bc + b^2 + c^2 - a^2, 2c^2).$$

Proof. By using [2, §4, (14)] we find point J_a as the midpoint of segment IK_a. Similarly we find J_b and J_c. □

3.2. **Notable Points.** The Supplementary Material, Folder 3 contains description of a number of notable points of triangle $JaJbJc$ in terms of notable points of the Reference Triangle ABC. We recommend the reader to prove these results. For the new points, not available in [4], we recommend the reader to find the barycentric coordinates and properties.

3.3. **Perspectors.**

Problem 3.1. *Traingles ABC and $JaJbJc$ are perspective with Perspector the point X(3296).*

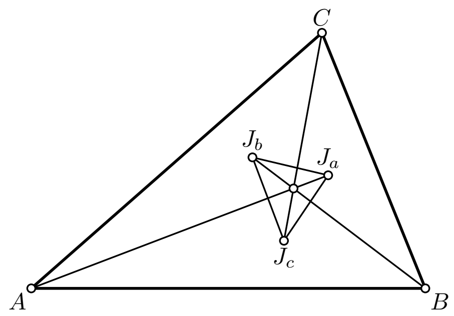

FIGURE 5.

The Figure 5 illustrates Problem 3.1. The lines AJ_a, BJ_b and CJ_c concur in a point. This is point X(3296).

The "Discoverer" has discovered a number of triangles perspective with triangle $JaJbJc$. See the Supplementary Material, Folder 4. Note that List K of Folder 2 contains 23 distinct perspectors and 23 distinct triangles. Hence, if we use Method 1 of [2, §15, Method 1], we can say that the "Discoverer" has discovered 22! = 1124000727777607680000 different methods for construction of triangle $JaJbJc$.

4. MONGE TRIANGLE OF THE TRIAD $\mathcal{C}$

In the Johnson triad $\mathcal{C}$, denote by M_a the Internal Similitude Center of Circles (Jb) and (Jc), denote by M_b the Internal Similitude Center of Circles (Jc) and (Ja) and denote by M_c the Internal Similitude Center of Circles (Ja) and (Jb). Then $M_aM_bM_c$ is the *Monge Triangle of the Triad* $\mathcal{C}$. (In honor of Gaspard Monge). See Figure 6. For the general definition of the Monge triangle (the old name is the Inner Johnson Triangle) see `http://www.ddekov.eu/j/2007/JCGEG200731.pdf`.

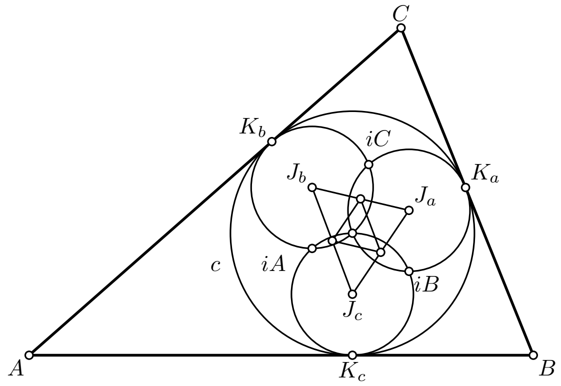

FIGURE 6.

Theorem 4.1. *The barycentric coordinates of the Monge triangle $MaMbMc$ are as follows:*

$$Ma = (ca^2+cb^2+ba^2+bc^2+8abc-b^3-c^3, b(6bc+b^2+c^2-a^2), c(6bc+b^2+c^2-a^2)),$$
$$Mb = (a(6ac+a^2+c^2-b^2), ab^2+ac^2+ca^2+cb^2+8abc-a^3-c^3, c(6ac+a^2+c^2-b^2)),$$
$$Mc = (a(6ab+a^2+b^2-c^2), b(6ab+a^2+b^2-c^2), ba^2+bc^2+ab^2+ac^2+8abc-a^3-b^3).$$

We recommend the reader to investigate the Monge Triangle.

5. INNER MOSES TRIANGLE OF THE TRIAD $\mathcal{C}$

In the Johnson triad $\mathcal{C}$, denote by N_a the Internal Similitude Center of Circles (Jb) and the circumcircle of Triangle $iAiBiC$, denote by N_b the Internal Similitude Center of Circles (Jc) and the circumcircle of Triangle $iAiBiC$, and denote by N_c the Internal Similitude Center of Circles (Ja) and the circumcircle of Triangle $iAiBiC$. Then $N_aN_bN_c$ is the *Inner Moses Triangle of the Triad* $\mathcal{C}$ (in honor of Peter Moses). See Figure 7. For the general definition of the Inner Moses triangle see `http://www.ddekov.eu/j/2007/JCGEG200733.pdf`.

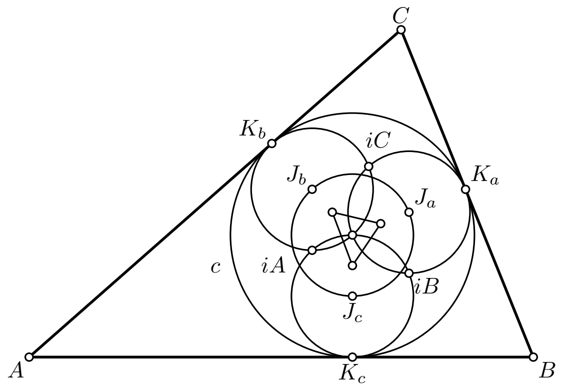

FIGURE 7.

Theorem 5.1. *The barycentric coordinates of the Inner Moses triangle $NaNbNc$ of the triad $\mathcal{C}$ and the circumcircle of triangle $JaJbJc$ are as follows:*

$$Na = (6a^2, 8ab + a^2 + b^2 - c^2, 8ac + a^2 + c^2 - b^2),$$
$$Nb = (8ab + a^2 + b^2 - c^2, 6b^2, 8bc + b^2 + c^2 - a^2),$$
$$Nc = (8ac + a^2 + c^2 - b^2, 8bc + b^2 + c^2 - a^2, 6c^2),$$

We recommend the reader to investigate the Inner Moses Triangle. Below we give several results.

Table 2 gives the centers of the Inner Moses Triangle of the triad $\mathcal{C}$ in terms of the centers of the Reference triangle ABC that are Kimberling centers $X(n)$. Denote by T the Inner Moses Triangle of the triad $\mathcal{C}$.

	Center of Triangle T	Center of the Reference triangle
1	X(3) Circumcenter	X(1) Incenter
2	X(4) Orthocenter	X(5045)
3	X(64) Isogonal Conjugate of the de Longchamps Point	X(3635)
4	X(381) Center of the Orthocentroidal Circle	X(5049)
5	X(399) Parry Reflection Point	X(1387)

TABLE 2.

Problem 5.1. *Prove the theorems given in Table 2.*

6. JOHNSON MIDPOINT OF THE TRIAD $\mathcal{C}$

In the Johnson four-circle-configuration of the Reference triangle, the Johnson midpoint is the midpoint of each of segments joining the vertices of a reference triangle with the vertices of the Central Triangle of the triad. This is point X(495) in [4]. See [5, Johnson Midpoint].

Hence, in the Johnson four-circle-configuration of the triad $\mathcal{C}$, the Johnson Point is the midpoint of the segments $iAJa, iBJb$ and $iCJc$. See Figure 8.

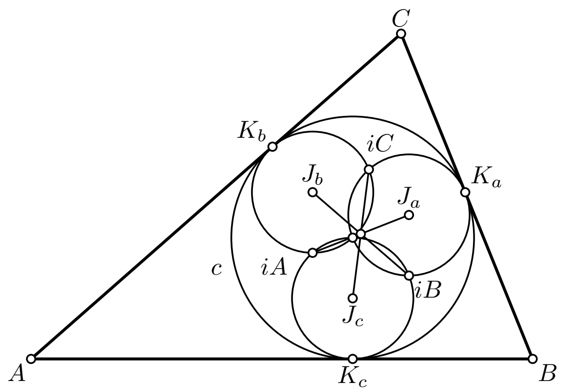

FIGURE 8.

Theorem 6.1. *The Johnson Midpoint J of the triad $\mathcal{C}$ has barycentric coordinates:*

$$uJ = a(ba^2 + ca^2 + cb^2 + bc^2 + 6abc - b^3 - c^3),$$
$$vJ = b(ab^2 + ac^2 + ca^2 + cb^2 + 6abc - c^3 - a^3),$$
$$wJ = c(ab^2 + ac^2 + ba^2 + bc^2 + 6abc - a^3 - b^3).$$

Proof. By using [4, §4, (14)], we find the midpoint of segment $iAJa$. □

Corollary. The Johnson Midpoint of the triad $\mathcal{C}$ is the point X(5045) in [4].

Problem 6.1. *The Johnson Midpoint of the Triad $\mathcal{C}$ is the Midpoint of the Incenter and the Nine-Point Center of the Intouch Triangle.*

Problem 6.2. *The Johnson Midpoint of the Triad $\mathcal{C}$ is the Midpoint of the Incenter and the Nine-Point Center of the Intouch Triangle.*

Problem 6.3. *The Johnson Midpoint of the Triad $\mathcal{C}$ is the Circumcenter of the Half-Median Triangle of the Intouch Triangle.*

Problem 6.4. *The Johnson Midpoint of the Triad $\mathcal{C}$ lies on the*

(1) *Lester Circle of the Medial Triangle of the Intouch Triangle.*
(2) *Lester Circle of the Half-Median Triangle of the Intouch Triangle.*
(3) *Brocard Circle of the Half-Median Triangle of the Intouch Triangle.*
(4) *Circle passing through the Bevan Point, Feuerbach Point and Nine-Point Center.*

SUPPLEMENTARY MATERIAL

The enclosed supplementary material contains theorems related to the topic.

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