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Computer Discovered Mathematics: Harmonic Conjugates

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Abstract. By using the computer program "Discoverer" we find new notable harmonic conjugates in triangle geometry and we find their properties.

Keywords. harmonic conjugates, triangle geometry, remarkable point, computer discovered mathematics, Euclidean geometry, "Discoverer".

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1. Introduction

The computer program "Discoverer", created by the authors, is the first computer program, able easily to discover new theorems in mathematics, and possibly, the first computer program, able easily to discover new knowledge in science. See [2]. In this paper by using the "Discoverer" we find new notable point of the triangle, and we find their properties.

Given four collinear points P, Q, U, V. Recall that the points P and Q are said to divide the other points U and V *harmonically* if

$$\frac{PU}{QU} = -\frac{PV}{QV}$$

They are *harmonic conjugates* of each other with respect to the segment PQ.

Ruler and compass construction of the harmonic conjugates us given in [6, Lesson 6].

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2. Harmonic Conjugates

In projective geometry, the harmonic conjugate point of an ordered triple of points on the real projective line is defined by the following construction [8, Projective harmonic conjugate], [9, Ruler construction of harmonic conjugate, in §1.1.4], [1, Construction 14.4.5] (See Figure 1):

Given three collinear points P, Q, U, let S be a point not lying on their join and let any line through U meet SP, SQ at M, N respectively. If PN and QM meet at K, and SK meets PQ at V, then V is called the harmonic conjugate of U with respect to P, Q.

Recall that the point V does not depend on what point S is taken initially, nor upon what line through U is used to find M and N.

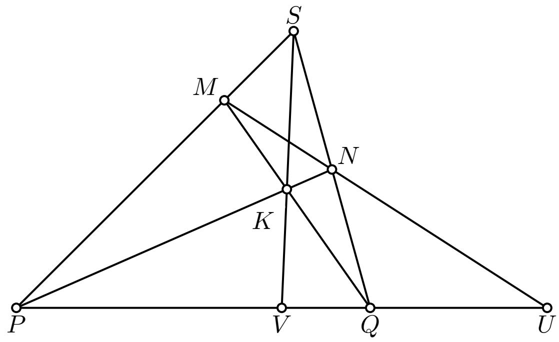

Figure 1.

3. New Notable Points in Triangle Geometry

In the Kinmerling's ETC [4, Tables, Harmonic Conjugates] are listed the HARMONIC CONJUGATES AMONG TRIANGLE CENTERS X(1) TO X(1320).

The "Discoverer" has discovered a number of new notable points which are not available in [4]. See the enclosed Supplementary material. Points D_1 to D_6 below are nit listed in [4]. (Prove it). The "Discoverer" has discovered a number of theorems about the new notable points. We present below a few of these theorems in the form of problems.

The reader may find the definitions in [7], [1] and [3, Definitions].

3.1. **Point D_1.** Denote by D_1 the Harmonic Conjugate of the X(2) Centroid with respect to the X(13) Outer Fermat Point and the X(16) Second Isodynamic Point.

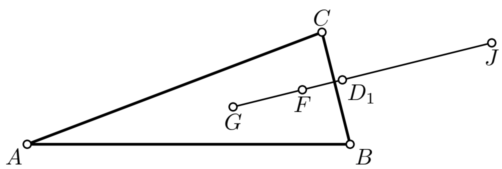

Figure 2.

Figure 2 illustrates the definition of point D_1. In figure 2, G is the Centroid, F is the Outer Fermat Point and J is the Second Isodynamic Point. Then F_1 is the Harmonic Conjugate of G with respect to the F and J.

Problem 3.1. *Prove that point D_1 is the*

(1) *Inner Fermat Point of the First Brocard Triangle of the Antimedial Triangle.*
(2) *Inner Fermat Point of the Outer Fermat Triangle of the First Brocard Triangle.*

Problem 3.2. *Prove that point D_1 lies on the following circles:*

(1) *Lester Circle of the First Brocard Triangle of the Antimedial Triangle.*
(2) *Lester Circle of the Outer Fermat Triangle of the First Brocard Triangle.*

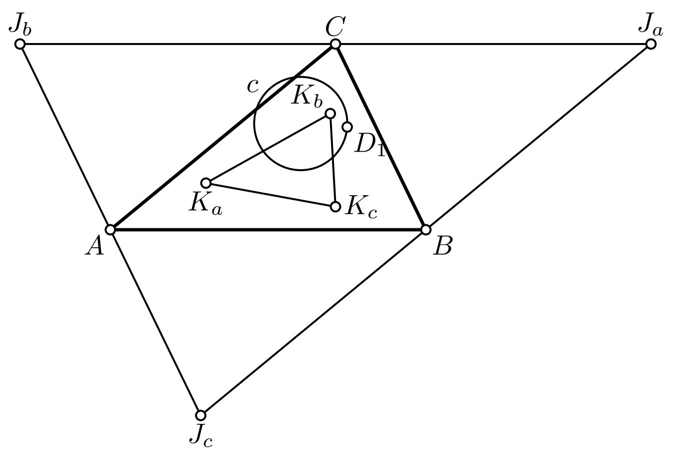

Figure 3.

Figure 3 illustrates problem 3.2. In Figure 3, $JaJbJc$ is the Antimedial triangle, $KaKbKc$ is the First Brocard Triangle and c is the Lester Circle of the First Brocard Triangle of the Antimedial Triangle. Then point D_1 lies on circle c.

Problem 3.3. *Prove that point D_1 lies on the following lines:*

(1) *Line through the Inner Fermat Point and the Orthocenter.*
(2) *Line through the First Isodynamic Point and the Reflection of the Centroid in the Circumcenter.*
(3) *Line through the Outer Napoleon Point and the Harmonic Conjugate of the Orthocenter with respect to the Centroid and the Circumcenter (the Orthocenter is not between the Centroid and Circumcenter).*
(4) *Line through the Inner Napoleon Point and the Harmonic Conjugate of the Centroid with respect to the Nine-Point Center and the Orthocenter (the Centroid is not between the Nine-Point Center and Orthocenter).*
(5) *Line through the Center of the Radical Circle of the Neuberg Circles and the Perspector of the Antimedial Triangle and the Outer Fermat Triangle.*
(6) *Image of the Euler Line under the Homothety with Center the Center of the Brocard Circle and Ratio -1.*

3.2. **Point D_2.** Denote by D_2 the Harmonic Conjugate of the X(2) Centroid with respect to the X(14) Inner Fermat Point and the X(15) First Isodynamic Point. The Centroid is not between the Inner Fermat Point and First Isodynamic Point.

Problem 3.4. *Prove that point D_2 lies on the*

(1) *Lester Circle of the First Brocard Triangle of the Antimedial Triangle.*

(2) *Lester Circle of the Inner Fermat Triangle of the First Brocard Triangle.*

Problem 3.5. *Prove that point D_2 lies on the following lines:*

(1) *Line through the Orthocenter and the Outer Fermat Point.*
(2) *Line through the Second Isodynamic Point and the Reflection of the Centroid in the Circumcenter.*
(3) *Line through the Outer Napoleon Point and the Harmonic Conjugate of the Centroid with respect to the Nine-Point Center and the Orthocenter (the Centroid is not between the Nine-Point Center and Orthocenter).*
(4) *Line through the Inner Napoleon Point and the Harmonic Conjugate of the Orthocenter with respect to the Centroid and the Circumcenter (the Orthocenter is not between the Centroid and Circumcenter).*
(5) *Line through the Center of the Radical Circle of the Neuberg Circles and the Perspector of the Antimedial Triangle and the Inner Fermat Triangle.*
(6) *Image of the Euler Line under the Homothety with Center the Center of the Brocard Circle and Ratio -1.*

3.3. **Point D_3.** Denote by X_1 the Harmonic Conjugate of the X(11) Feuerbach Point with respect to the X(1) Incenter and the X(12) Feuerbach Perspector. The Feuerbach Point is not between the Incenter and Feuerbach Perspector.

Problem 3.6. *Prove that point D_3 is the*

(1) *Harmonic Conjugate of the Feuerbach Perspector with respect to the Incenter and the Johnson Midpoint. The Feuerbach Perspector is outside the segment from the Incenter and Johnson Midpoint.*
(2) *Internal Center of Similitude of the Incircle of Triangle ABC and the Nine-Point Circle of the Inner Yff Triangle.*

Problem 3.7. *Prove that point D_3 lies on the following lines:*

(1) *Line through the Incenter and the Nine-Point Center (The Feuerbach Line).*
(2) *Line through the Spieker Center and the Weill Point.*
(3) *Line through the Internal Center of Similitude of the Incircle and the Circumcircle and the de Longchamps Point.*
(4) *Line through the External Center of Similitude of the Incircle and the Circumcircle and the Harmonic Conjugate of the Orthocenter with respect to the Centroid and the Circumcenter.*
(5) *Line through the Center of the Inner Johnson-Yff Circle and the Reflection of the Circumcenter in the Orthocenter.*

3.4. **Other Points.** Denote by D_4 the Harmonic Conjugate of the X(4) Orthocenter with respect to the X(9) Mittenpunkt and the X(10) Spieker Center.

Problem 3.8. *Prove that point D_4 is the intersection of the Line through the Orthocenter and the Spieker Center and the Line through the Centroid of Triangle ABC and the Nine-Point Center of the Intouch Triangle.*

Denote by D_5 the Harmonic Conjugate of the X(4) Orthocenter with respect to the X(10) Spieker Center and the X(19) Clawson Point.

Problem 3.9. *Prove that point D_5 is the intersection of the Line through the Orthocenter and the Spieker Center and the Line through the Perspector and Homothetic Center of the Kosnita Triangle and the Orthic Triangle and the Perspector of the Extouch Triangle and the Tangential Triangle.*

Denote by D_6 the Harmonic Conjugate of the X(09) Mittenpunkt with respect to the X(04) Orthocenter and the X(10) Spieker Center.

Problem 3.10. *Prove that point D_6 is the intersection of the Line through the Orthocenter and the Spieker Center and the and the Line through the Nagel Point and the Perspector of the Half-Bisector Triangle and the Intouch Triangl.*

Supplementary Material

The enclosed supplementary material contains theorems related to the topic.

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