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Computer Discovered Mathematics: Fuhrmann Triangles

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Abstract. The Fuhrmann Triangle is one of the famous objects in the geometry of the triangle. The Fuhrmann Triangle is defined as follows: Let $PaPbPc$ be the Circumcevian triangle of the Incenter and let $QaQbQc$ be the Cevian triangle of the Centrod. Then the vertices of the Fuhrmann triangle $FaFbFc$ are the reflections of Pa in Qa, Pb in Qb, and Pc in Qc, respectively. We generalize the Fuhrmann triangle to arbitrary points P and Q, and we arrive to the definition of the Fuhrmann triangle of points P and Q. In this paper we study the generalized Fuhrmann triangle and we consider a number of special cases. We use the computer program "Discoverer". The new theorem could be used as problems for high school and university students. Many proofs of theorems are not presented here. We recommend the reader to find the proofs.

Keywords. Fuhrmann Triangle, triangle geometry, computer discovered mathematics, Euclidean geometry, "Discoverer".

Mathematics Subject Classification (2010). 51-04, 68T01, 68T99.

1. Introduction

The computer program "Discoverer", created by the authors, is the first computer program, able easily to discover new theorems in mathematics, and possibly, the first computer program, able easily to discover new knowledge in science. See [4]. In this paper, by using the "Discoverer", we investigate the Fuhrmann triangles.

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The Fuhrmann Triangle is one of the famous objects in the geometry of the triangle. The Fuhrmann Triangle is defined as follows: Let $PaPbPc$ be the Circumcevian triangle of the Incenter and let $QaQbQc$ be the Cevian triangle of the Centrod. Then the vertices of the Fuhrmann triangle $FaFbFc$ are the reflections of Pa in Qa, Pb in Qb, and Pc in Qc, respectively. We generalize the Fuhrmann triangle to arbitrary points P and Q, and we arrive to the definition of the Fuhrmann triangle of points P and Q. In this paper we study the generalized Fuhrmann triangle and we consider a number of special cases.

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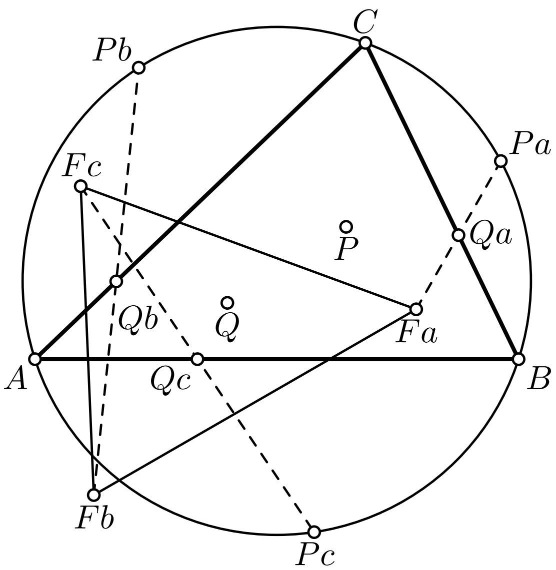

Figure 1.

Figure 1 illustrates the definition. In Figure 1, P and Q are arbitrary points, $PaPbPc$ is th Circumcevian triangle of P, $QaQbQc$ is the Cevian triangle of Q and $FaFbFc$ is the Fuhrmann triangle of P and Q. Then Fa is the reflection of Pa in Qa, Fb is the reflection of Pb in Qb, and Fc is the reflection of Pc in Qc.

2. Preliminaries

In this section we review some basic facts about barycentric coordinates. We refer the reader to [12], [3], [2],[9],[6],[7],[10]. The labeling of triangle centers follows Kimberling's ETC [8]. For example, X(1) denotes the Incenter, X(2) denotes the Centroid, X(37) is the Grinberg Point, X(75) is the Moses point (Note that in the prototype of the "Discoverer" the Moses point is the X(35)), etc.

The reader may find definitions in [5], Contents, Definitions, and in [11].

The reference triangle ABC has vertices $A = (1, 0, 0)$, $B(0, 1, 0)$ and C(0,0,1). The side lengths of $\triangle ABC$ are denoted by $a = BC, b = CA$ and $c = AB$. A point is an element of $\mathbb{R}^3$, defined up to a proportionality factor, that is,

$$\text{For all } k \in \mathbb{R} - \{0\} : \; P = (u, v, w) \text{ means that } P = (u, v, w) = (ku, kv, kw).$$

Given a point $P(u, v, w)$. Then P is *finite*, if $u+v+w \neq 0$. A finite point P is *normalized*, if $u+v+w=1$. A finite point could be put in normalized form as follows: $P = (\frac{u}{s}, \frac{v}{s}, \frac{w}{s})$,where $s = u+v+w$.

Three points $P_i(x_i, y_i, z_i)$, $i = 1, 2, 3$ lie on the same line if and only if

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0. \tag{1}$$

If the barycentric coordinates of points $P_i(x_i, y_i, z_i)$, $i = 1, 2, 3$ are normalized, then the area of $\triangle P_1P_2P_3$ is

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \Delta. \tag{2}$$

where Δ is the area of the reference triangle ABC.

Given two points $P = (u_1, v_1, w_1)$ and $Q = (u_2, v_2, w_2)$ in normalized barycentric coordinates. Denote $x = u_1 - u_2, y = v_1 - v_2$ and $z = w_1 - w_2$. Then the square of the distance between P and Q is as follows (See [9, §15]):

$$d^2 = -a^2yz - b^2zx - c^2xy. \tag{3}$$

Given two points $P = (u_1, v_1, w_1)$ and $Q = (u_2, v_2, w_2)$ in normalized barycentric coordinates, the reflection R of P in Q is as follows:

$$R = (2u_2 - u_1, 2v_2 - v_1, 2w_2 - w_1). \tag{4}$$

3. FUHRMANN TRIANGLES

Theorem 3.1. *Given points in barycentric coordinates $P = (p, q, r)$ and $Q = (u, v, w)$. The barycentric coordinates of the Fuhrmann Triangle $FaFbFc$ of P and Q are as follows:*

$$Fa = (a^2qr(v+w),\ -2va^2qr + vc^2q^2 + vqb^2r + 2vrc^2q + 2vb^2r^2 - c^2q^2w - qb^2rw,$$

$$-2wa^2qr + 2c^2q^2w + 2qb^2rw + wrc^2q + wb^2r^2 - vrc^2q - vb^2r^2),$$

$$Fb = (upa^2r + uc^2p^2 - 2ub^2rp + 2ua^2r^2 + 2urc^2p - pa^2rw - c^2p^2w,\ b^2rp(u+v),$$

$$2pa^2rw + 2c^2p^2w - 2wb^2rp + wa^2r^2 + wrc^2p - ua^2r^2 - urc^2p),$$

$$Fc = (ub^2p^2 + upa^2q + 2uqb^2p + 2ua^2q^2 - 2uc^2pq - b^2p^2v - pa^2qv,$$

$$2b^2p^2v + 2pa^2qv + vqb^2p + va^2q^2 - 2vc^2pq - uqb^2p - ua^2q^2,\ c^2pq(u+v)).$$

Proof. The barycentric coordinates of te Circumcevian triangle $PaPbPc$ of P are known. See e.g. [12, §5.2, Exercise 5, page 64]. The barycentric coordinates of the Cevian triangle $QaQbQc$ of point Q are well known. By using (4) we find the reflection Fa of Pa in Qa, Similarly, we find the reflection Fb of Pb in Qb, and reflection Fc of Pc in Qc. The barycentric coordinates of points Fa, Fb and Fc are given in the statement of the theorem. □

4. Special Case: Fuhrmann Triangle

The Fuhrmann Triangle of the Incenter and Centroid is the known Fuhrmann Triangle. See [11, Fuhrmann Triangle], [2, §17.2.10 Fuhrmann triangle], [1, Fuhrmann triangle].

From Theorem 3.1 we obtain as corollary the following theorem:

Theorem 4.1. *The barycentric coordinates of the Fuhrmann Triangle $FaFbFc$ are as follows:*

$$Fa = (a^2, -a^2 + c^2 + bc, -a^2 + b^2 + bc),$$
$$Fb = (-b^2 + c^2 + ac, b^2, -b^2 + a^2 + ac),$$
$$Fc = (-c^2 + b^2 + ab, -c^2 + a^2 + ab, c^2).$$

The "Discoverer" has discovered more than 300 different triangles similar to the Fuhrmann triangle. A part of these triangles is presented in Theorem 4.2.

Theorem 4.2. *The following theorems are similar to the Fuhrmann triangle:*

(1) *Intouch Triangle.*
(2) *Excentral Triangle.*
(3) *Hexyl Triangle.*
(4) *Yff Central Triangle.*
(5) *Circum-Incentral Triangle.*
(6) *Circum-Anticevian Triangle of the Incenter.*
(7) *Pedal Triangle of the Inverse of the Incenter in the Circumcircle.*
(8) *Circumcevian Triangle of the Inverse of the Incenter in the Circumcircle.*
(9) *Triangle of Reflections of the Incenter in the Sidelines of Triangle ABC.*
(10) *Triangle of the Nine-Point Centers of the Triangulation Triangles of the Incenter.*
(11) *Triangle of the Feuerbach Points of the Cevian Corner Triangles of the Gergonne Point.*
(12) *Triangle of the de Longchamps Points of the Cevian Corner Triangles of the Nagel Point.*
(13) *Triangle of the Nine-Point Centers of the Pedal Corner Triangles of the Reflection of the Circumcenter in the Incenter.*
(14) *Half-Cevian Triangle of the Nagel Point.*
(15) *Half-Circumcevian Triangle of the Incenter.*

Proof. (1) The Fuhrmann Triangle and the Intouch Triangl are similar.

The barycentric coordinates of the Fuhrmann triangle are given in Theorem 4.1.

The Intouch triangle of a triangle ABC, is the triangle $GaGbGc$ formed by the points of tangency of the Incircle of ABC with ABC. Also, the Intouch triangle is Cevian triangle of the Gergonne point Ge. Hence, the barycentric coordinates of the Intouch Triangle $GaGbGc$ are as follows:

$$Ga = (0, (a + b - c)(b + c - a), (b + c - a)(c + a - b)),$$
$$Gb = ((c + a - b)(a + b - c), 0, (b + c - a)(c + a - b)),$$
$$Gc = ((c + a - b)(a + b - c), (a + b - c)(b + c - a), 0).$$

Denote

$$D = \sqrt{a^3 + b^3 + c^3 + 3abc - a^2b - a^2c - b^2a - b^2c - c^2a - c^2b}$$

By using (3) we find the side lengths a_1, b_1 and c_1 of the Fuhrmann triangle as follows:

$$a_1 = \frac{D\sqrt{a}}{\sqrt{(c+a-b)(a+b-c)}}, \quad b_1 = \frac{D\sqrt{b}}{\sqrt{(a+b-c)(b+c-a)}},$$

$$c_1 = \frac{D\sqrt{c}}{\sqrt{(b+c-a)(c+a-b)}}.$$

Similarly, we find the side lengths $a2, b2$ and $c2$ of the Intouch triangle as follows:

$$a_2 = \frac{(b+c-a)\sqrt{(c+a-b)(a+b-c)}}{2\sqrt{bc}}, \quad b_2 = \frac{(c+a-b)\sqrt{(a+b-c)(b+c-a)}}{2\sqrt{ca}},$$

$$c_2 = \frac{(a+b-c)\sqrt{(b+c-a)(c+a-b)}}{2\sqrt{ab}}.$$

Denote

$$k_a = \frac{a_1}{a_2}, \quad k_b = \frac{b_1}{b_2}, \quad k_c = \frac{c_1}{c_2},$$

Then we have

$$k = k_a = k_b = k_c = \frac{2D\sqrt{abc}}{(b+c-a)(c+a-b)(a+b-c)},$$

where k is the ratio of similitude of the Fuhrmann triangle and the Intouch triangle.

(2) The Fuhrmann Triangle and the Excentral Triangle are similar.

Intouch Triangle and the Excentral Triangle are similar. See [2, §17.2.5]. □

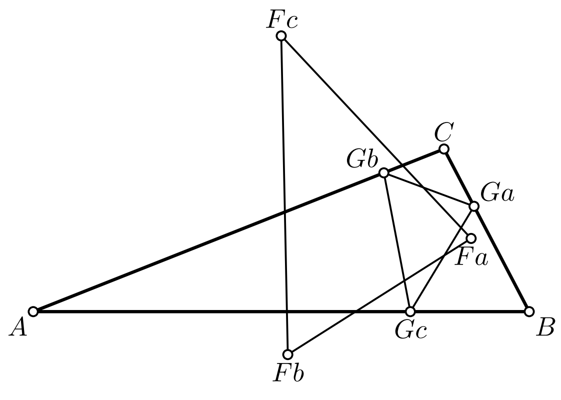

FIGURE 2.

Figure 2 illustrates Theorem 4.2, (1). In Figure 2, $FaFbFc$ is the Fuhrmann Triangle, and $GaGbGc$ is the Intouch Triangle.

Problem 4.1. *Find the ratios of similitude in Theorem 4.2, (2) to (15).*

The computer program "Discoverer" has discovered a number of theorems about triangle perspective with the Fuhrmann triangle. A few of these theorems are presented in the Table 1.

	The Fuhrmann triangle is perspective with the	**Perspector**
1	Medial Triangle	X(3)
2	Excentral Triangle	X(191)
3	Antimedial Triangle	X(8)
4	Tangential Triangle	X(3)
5	Neuberg Triangle	X(3)
6	Reflected Neuberg Triangle	X(3)
7	Johnson Triangle	X(3)
8	First Brocard Triangle	X(3)
9	Outer Kiepert-Gallatly Triangle	X(3)
10	Kosnita Triangle	X(3)
11	Honsberger Triangle	X(3)
12	Pedal Triangle of the Inverse of the Incenter in the Circumcircle	X(56)
13	Circum-Incentral Triangle	X(3)
14	Circum-Anticevian Triangle of the Incenter	X(3)
15	Cevian Triangle of the Spieker Center	X(191)
16	Cevian Triangle of the Isotomic Conjugate of the Incenter	X(8)
17	Triangle of Reflections of the Inverse of the Incenter in the Circumcircle in the Sidelines of Triangle ABC	X(1)
18	Half-Circumcevian Triangle of the Incenter	X(2)
19	Triangle of the Centers of the Orthocentroidal Circles of the Anticevian Corner Triangles of the Incenter	X(5220)
20	Triangle of Reflections of the Vertices of the Pedal Triangle of the Inverse of the Incenter in the Circumcircle in the Feuerbach Point	X(5592)
21	Triangle of Reflections of the Vertices of the Pedal Triangle of the Inverse of the Incenter in the Circumcircle in the Inverse of the Incenter in the Circumcircle (Homothetic Triangles)	X(5204)
22	Triangle of Reflections of the Vertices of the Pedal Triangle of the Inverse of the Incenter in the Circumcircle Bevan-Schroder Point (Homothetic Triangles)	X(1388)

Table 1

The first row in Table 1 can be rewritten as follows:

Theorem 4.3. *The Fuhrmann Triangle and the Medial Triangle are perspective with the Circumcenter as perspector.*

Proof. The barycentric coordinates of the Fuhrmann triangle $FaFbFc$ are given in Theorem 4.1. Th barycentric coordinates of the Medial triangle $MaMbMc$ are well known: $Ma = (0, 1, 1), Mb = (1, 0, 1), Mc = (1, 1, 0)$. The barycentric coordinates of the Circumcenter are also well known:

$$O = (a^2(b^2 + c^2 - a^2), b^2(c^2 + a^2 - b^2), c^2(a^2 + b^2 - c^2)).$$

By using (1) we prove that points Ma, O and Fa lie on the same line:

$$\begin{vmatrix} 0 & 1 & 1 \\ a^2(b^2+c^2-a^2) & b^2(c^2+a^2-b^2) & c^2(a^2+b^2-c^2) \\ a^2 & c^2-a^2+bc & b^2-a^2+bc \end{vmatrix} = 0.$$

Similarly, we prove that points Mb, O and Fb lie on the same line, and points Mc, O and Fc lie on the same line. Hence, point O lies on the lines $MaFa, MbFb$ and $McFc$. □

Problem 4.2. *Prove the corresponding theorem in Table 1, rows 2-22.*

Problem 4.3. *Prove that the Fuhrmann Triangle and any Kiepert triangle are perspective with the Circumcenter as perspector.*

Corollary to Problem 4.3. *The Fuhrmann Triangle is perspective with the following triangles: the Outer Fermat Triangle, The Inner Fermat triangle, the Outer Vecten Triangle and the Inner Vecten Triangle, and the perspector is the Circumcenter.*

5. SPECIAL CASE: FUHRMANN TRIANGLE OF THE SYMMEDIAN POINT AND THE CENTROID

Theorem 5.1. *The barycentric coordinates of the Fuhrmann Triangle $FaFbFc$ of the Symmedian Point and the Centroid are as follows:*

$$Fa = (a^2, 2c^2-a^2, 2b^2-a^2),$$
$$Fb = (2c^2-b^2, b^2, 2a^2-b^2),$$
$$Fc = (2b^2-c^2, 2a^2-c^2, c^2).$$

Theorem 5.2. *The area of the Fuhrmann Triangle of the Symmedian Point and the Centroid is*

$$\left|\frac{4(b^2+a^2+c^2)(a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2)}{(2b^2+2c^2-a^2)(2a^2+2c^2-b^2)(2a^2+2b^2-c^2)}\right|\Delta$$

where Δ is the area of triangle ABC.

Proof. We use (2). □

Table 2 gives a few centers of the Fuhrmann Triangle of the Symmedian Point and the Centroid in terms of the centers of the Reference triangle ABC.

	Center of the Fuhrmann Triangle of the Symmedian Point and the Centroid	**Center of the Reference Triangle**
1	X(3) Circumcenter	X(1352)
2	X(15) First Isodynamic Point	X(621)
3	X(16) Second Isodynamic Point	X(622)
4	X(187) Schoute Center	X(316)

Table 2

We select a few of the triangles similar to the Fuhrmann Triangle of the Symmedian Point and the Centroid:

Theorem 5.3. *The Fuhrmann Triangle of the Symmedian Point and the Centroid is similar to the following triangles:*

(1) *Second Brocard Triangle.*
(2) *Circum-Medial Triangle.*
(3) *Pedal Triangle of the Centroid.*
(4) *Antipedal Triangle of the Symmedian Point.*
(5) *Pedal Triangle of the Far-Out Point.*
(6) *Triangle of the Euler Reflection Points of the Cevian Corner Triangles of the Steiner Point.*
(7) *Triangle of the Centers of the Orthocentroidal Circles of the Pedal Corner Triangles of the Nine-Point Center.*

Proof. (1) The Fuhrmann Triangle of the Symmedian Point and the Centroid and the Second Brocard Triangl are similar.

The barycentric coordinates of the Fuhrmann triangle $FaFbFc$ of the Symmedian Point and the Centroid are given in Theorem .

The barycentric coordinates of the Second Brocard Triangle $BaBbBc$ are as follows:

$$Ba = (b^2 + c^2 - a^2, b^2, c^2), \quad Bb = (a^2, c^2 + a^2 - b^2, c^2),$$
$$Bc = (a^2, b^2, a^2 + b^2 - c^2).$$

Denote

$$D = \sqrt{a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2}$$

By using (3) we find the side lengths a_1, b_1 and c_1 of the Fuhrmann triangle of the Symmedian Point and the Centroid as follows:

$$a_1 = \frac{2aD}{\sqrt{(2c^2 + 2a^2 - b^2)(2a^2 + 2b^2 - c^2)}}, \quad b_1 = \frac{2bD}{\sqrt{(2a^2 + 2b^2 - c^2)(2b^2 + 2c^2 - a^2)}},$$

$$c_1 = \frac{2cD}{\sqrt{(2b^2 + 2c^2 - a^2)(2c^2 + 2a^2 - b^2)}}..$$

Similarly, we find the side lengths $a2, b2$ and $c2$ of the Second Brocard triangle (See also [11, Second Brocard Triangle]) as follows:

$$a_2 = \frac{aD}{\sqrt{(c^2 + 2a^2 - b^2)(2a^2 + 2b^2 - c^2)}}, \quad b_2 = \frac{bD}{\sqrt{(a^2 + 2b^2 - c^2)(2b^2 + 2c^2 - a^2)}},$$

$$c_2 = \frac{cD}{\sqrt{(b^2 + 2c^2 - a^2)(2c^2 + 2a^2 - b^2)}}..$$

Denote

$$k_a = \frac{a_1}{a_2}, \quad k_b = \frac{b_1}{b_2}, \quad k_c = \frac{c_1}{c_2},$$

Then we have

$$k = k_a = k_b = k_c = 2,$$

where k is the ratio of similitude of the Fuhrmann triangle of the Symmedian Point and the Centroid and the Second Brocard triangle. □

Figure 3 illustrates Theorem 5.3. In Figure3, $FaFbFc$ is the Fuhrmann Triangle of the Symmedian Point and the Centroid and $BaBbBc$ is the Second Brocard Triangle.

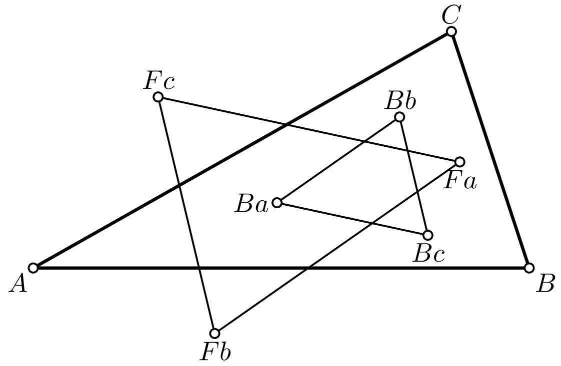

FIGURE 3.

Theorem 5.4. *Prove that the Fuhrmann Triangle of the Symmedian Point and the Centroid and the First Brocard Triangle are perspective and the perspector is the Centroid.*

Proof. The barycentric coordinates of the Fuhrmann Triangle $FaFbFc$ of the Symmedian Point and the Centroid ar given in Theorem 5. The Centroid has barycentric coordinates $G = (1, 1, 1)$. Recall that the First Brocard Triangle $BaBcBc$ has barycentric coordinates

$$Ba = (a^2, c^2, b^2), \quad Bb = (c^2, b^2, a^2), \quad Bc = (b^2, a^2, c^2).$$

By using (1) we prove that the points G, Ba and Fa lie on the same line:

$$\begin{vmatrix} 1 & 1 & 1 \\ a^2 & c^2 & b^2 \\ a^2 & 2c^2 - a^2 & 2b^2 - a^2 \end{vmatrix} = 0.$$

Similarly, points G, Bb and Fb lie on the same line, and points G, Bc and Fc lie on the same line. Hencem the Centroid G lies on lines $BaFa, BbFb$ and $BcFc$. $\square$

Problem 5.1. *Prove that the Fuhrmann Triangle of the Symmedian Point and the Centroid and the Second Brocard Triangle are perspective and the perspector is the Centroid.*

Theorem 5.5. *The Fuhrmann Triangle of the Symmedian Point and the Centroid and the Antimedial Triangle are perspective and the perspector is the X(69) Retrocenter.*

Problem 5.2. *Prove that the Fuhrmann Triangle of the Symmedian Point and the Orthocenter and the Triangle of Reflections are perspective. The perspector is not available in Kimberling's ETC* [8]*. Find the perspector.*

6. SPECIAL CASE: FUHRMANN TRIANGLE OF THE ORTHOCENTER AND THE CENTROID

Theorem 6.1. *The barycentric coordinates of the Fuhrmann Triangle of the Orthocenter and the Centroid are as follows:*

$$Fa = (a^2(a^2 + b^2 - c^2)(c^2 + a^2 - b^2), -(c^2 + a^2 - b^2)(b^4 - 2c^2b^2 + a^4 + c^4 - 2c^2a^2),$$
$$-(a^2 + b^2 - c^2)(b^4 - 2c^2b^2 - 2b^2a^2 + c^4 + a^4)),$$
$$Fb = ((b^2 + c^2 - a^2)(b^4 - 2c^2b^2 + a^4 + c^4 - 2c^2a^2), -b^2(b^2 + c^2 - a^2)(a^2 + b^2 - c^2),$$

$$(a^2+b^2-c^2)(b^4-2b^2a^2+a^4+c^4-2c^2a^2)),$$
$$Fc=((b^2+c^2-a^2)(b^4-2c^2b^2-2b^2a^2+c^4+a^4),$$
$$(c^2+a^2-b^2)(b^4-2b^2a^2+a^4+c^4-2c^2a^2),-c^2(c^2+a^2-b^2)(b^2+c^2-a^2)).$$

Theorem 6.2. *The area of the Fuhrmann Triangle of the Orthocenter and the Centroid is*

$$\left|\frac{a^6-a^4b^2-a^4c^2+3a^2c^2b^2-a^2c^4-a^2b^4-b^4c^2+b^6-b^2c^4+c^6}{a^2b^2c^2}\right|\Delta$$

where Δ is the area of triangle ABC.

Proof. We use (2). □

Theorem 6.3. *The Fuhrmann Triangle of the Orthocenter and the Centroid and the Antimedial Triangle are perspective and the perspector is the X(20) de Longchamps Point.*

Problem 6.1. *Prove that the Fuhrmann Triangle of the Orthocenter and the Centroid and the Tangential Triangle are perspective and the perspector is the point X(2917).*

Table 3 gives a few centers of the Fuhrman Triangle of the Orthocenter and the Centroid in terms of the centers of the Reference triangle ABC.

	Center of the Fuhrman Triangle of the Orthocenter and the Centroid	**Center of the Reference Triangle**
1	X(1) Incenter	X(5693)
2	X(3) Circumcenter	X(3) Circumcenter
3	X(5) Nine-Point Center	X(5876)
4	X(20) de Longchamps Point	X(6241)
5	X(36) Inverse of the Incenter in the Circumcircle	X(6326)
6	X(54) Kosnita Point	X(2888)
7	X(64) Isogonal Conjugate of the de Longchamps Point	X(5925)
8	X(74) Ceva Product of the First Isodynamic Point and the Second Isodynamic Poin	X(20) de Longchamps Point
9	X(110) Euler Reflection Point	X(4) Orthocenter
10	X(125) Center of the Jerabek Hyperbola	X(5562)
11	X(399) Parry Reflection Point	X(382)

Table 3

7. Special Case: Fuhrmann Triangle of the Nine-Point Center and the Centroid

Theorem 7.1. *The Fuhrmann Triangle of the Nine-Point Center and the Centroid and the Antimedial Triangle are perspective and the perspector is the Circumcenter.*

8. Special Case: Fuhrmann Triangle of the Circumcenter and the Orthocenter

Theorem 8.1. *The Fuhrmann Triangle of the Circumcenter and the Orthocenter and the Triangle of Reflections are perspective and the perspector is the point X(6243).*

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