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Barycentric Coordinates: Formula Sheet

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Abstract. We present several basic formulas about the barycentric coordinates. The aim of the formula sheet is to serve for references.

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1. Area, Points, Lines

We present several basic formulas about the barycentric coordinates. The aim of the formula sheet is to serve for references.

We refer the reader to [15],[2],[1],[8],[3],[6],[7],[10],[11],[9],[12]. The labeling of triangle centers follows Kimberling's ETC [8].

The reader may find definitions in [13],[14], [5, Contents, Definitions].

The reference triangle ABC has vertices $A = (1, 0, 0)$, $B(0, 1, 0)$ and C(0,0,1). The side lengths of $\triangle ABC$ are denoted by $a = BC, b = CA$ and $c = AB$. A point is an element of $\mathbb{R}^3$, defined up to a proportionality factor, that is,

$$\text{For all } k \in \mathbb{R} - \{0\} : \ P = (u, v, w) \text{ means that } P = (u, v, w) = (ku, kv, kw).$$

Given a point $P(u, v, w)$. Then P is *finite*, if $u + v + w \neq 0$. A finite point P is *normalized*, if $u + v + w = 1$. A finite point could be put in normalized form as follows: $P = (\frac{u}{s}, \frac{v}{s}, \frac{w}{s})$,where $s = u + v + w$.

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We use the Conway's notation:

$$S_A = \frac{b^2+c^2-a^2}{2},\ S_B = \frac{c^2+a^2-b^2}{2},\ S_C = \frac{a^2+b^2-c^2}{2}, \tag{1}$$

If the barycentric coordinates of points $P_i(x_i, y_i, z_i)$, $i = 1, 2, 3$ are normalized, then the area of $\triangle P_1P_2P_3$ is

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \Delta. \tag{2}$$

where Δ is the area of the reference triangle ABC.

The equation of the line joining two points with coordinates u_1, v_1, w_1 and u_2, v_2, w_2 is

$$\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ x & y & z \end{vmatrix} = 0. \tag{3}$$

Three points $P_i(x_i, y_i, z_i)$, $i = 1, 2, 3$ lie on the same line if and only if

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0. \tag{4}$$

The intersection of two lines $L_1 : p_1x + q_1y + r_1z = 0$ and $L_2 : p_2x + q_2y + r_2z = 0$ is the point

$$(q_1r_2 - q_2r_1, r_1p_2 - r_2p_1, p_1q_2 - p_2q_1) \tag{5}$$

Three lines $p_ix + q_iy + r_iz = 0$, $i = 1, 2, 3$ are concurrent if and only if

$$\begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix} = 0 \tag{6}$$

The infinite point of a line $L : px + qy + rz = 0$ is the point (f, g, h), where $f = q - r$, $g = r - p$ and $h = p - q$.

The equation of the line through point $P(u, v, w)$ and parallel to the line $L : px + qy + rz = 0$ is as follows:

$$\begin{vmatrix} f & g & h \\ u & v & w \\ x & y & z \end{vmatrix} = 0 \tag{7}$$

The equation of the line through point $P(u, v, w)$ and perpendicular to the line $L : px + qy + rz = 0$ is as follows (The method discovered by Floor van Lamoen):

$$\begin{vmatrix} F & G & H \\ u & v & w \\ x & y & z \end{vmatrix} = 0 \tag{8}$$

where $F = S_Bg - S_Ch$, $G = S_Ch - S_Af$, and $H = S_Af - S_Bg$.

2. Distance

Given two points $P = (u_1, v_1, w_1)$ and $Q = (u_2, v_2, w_2)$ in normalized barycentric coordinates. Denote $x = u_1 - u_2, y = v_1 - v_2$ and $z = w_1 - w_2$. Then the square of the distance between P and Q is as follows:

$$|PQ|^2 = -a^2yz - b^2zx - c^2xy. \tag{9}$$

3. Change of Coordinates

Given point P with barycentric coordinates p, q, r with respect to triangle DEF, $D = (u_1, v_1, w_1)$, $E = (u_2, v_2, w_2)$, $F = (u_3, v_3, w_3)$. If points D, E, F and P are normalized, the barycentric coordinates u, v, w of P with respect to triangle ABC are as follows:

$$\begin{aligned} u &= u_1p + u_2q + u_3r, \\ v &= v_1p + v_2q + v_3r, \\ w &= w_1p + w_2q + w_3r. \end{aligned} \tag{10}$$

4. Division of a Segment

Given points P, Q, R which lie on the same line. Then there exists a unique real number λ such that

$$\overrightarrow{PR} = \lambda\overrightarrow{QR}. \tag{11}$$

We say that point R divides segment $\overrightarrow{PQ}$ in ratio λ. From 11 we obtain

$$R - P = \lambda(R - Q),$$

so that

$$R = \frac{P - \lambda Q}{1 - \lambda}.$$

If λ is inside the segment PQ we have $\lambda < 0$ and we say that point R divides the segment $\overrightarrow{PQ}$ internally. In this case, if $\lambda = -\frac{p}{q}, p > 0, q > 0$ we obtain

$$R = \frac{qP + pQ}{q + p}.$$

If λ is outside the segment PQ we have $\lambda > 0$ and we say that point R divides the segment $\overrightarrow{PQ}$ externally. In this case, if $\lambda = \frac{p}{q}, p > 0, q > 0$ we obtain

$$R = \frac{qP - pQ}{q - p}.$$

If points P and Q are in normalized barycentric coordinates, $P = (u_1, v_1, w_1)$ and $Q = (u_2, v_2, w_2)$ we obtain for the formula for the internal division:

$$R = \left(\frac{qu_1 + pu_2}{q + p}, \frac{qv_1 + pv_2}{q + p}, \frac{qw_1 + pw_2}{q + p}.\right). \tag{12}$$

and the formula for the external division:

$$R = \left(\frac{qu_1 - pu_2}{q - p}, \frac{qv_1 - pv_2}{q - p}, \frac{qw_1 - pw_2}{q - p}.\right). \tag{13}$$

If $p = q = 1$, we obtain the formula for the midpoint R of the segment PQ:

$$R = \left(\frac{u_1 + u_2}{2}, \frac{v_1 + v_2}{2}, \frac{w_1 + w_2}{2}\right). \tag{14}$$

Also, we obtain the formula for reflection R of P in Q:

$$R = (2u_2 - u_1, 2v_2 - v_1, 2w_2 - w_1). \tag{15}$$

5. Homothety

Given a point O and a real number k. Point X is the *homothetic image* of point $P \neq O$ wrt the homothety with center O and factor k if

$$\overrightarrow{OX} = k\overrightarrow{OP}. \tag{16}$$

From 16 we obtain

$$X - O = k(P - O).$$

so that

$$X = O + k(P - O).$$

If points O and P are in normalized barycentric coordinates, $O = (uO, vO, wO)$, $P = (uP, vO, wP)$, we obtain

$$\begin{aligned} uX &= uO + k(uP - uO), \\ vX &= vO + k(vP - vO), \\ wX &= wO + k(wP - wO). \end{aligned} \tag{17}$$

6. Inversion

Given circle $c = (O, R)$. Point X is the inverse of point $P \neq O$ with respect to the circle c if points X and P lie on the ray with endpoint O and

$$OX.OP = R^2. \tag{18}$$

Hence

$$\frac{OX}{OP} = \frac{R^2}{|OP|^2}.$$

The vectors $\overrightarrow{OX}$ and $\overrightarrow{OP}$ are collinear, so that there exists a number λ such that

$$\overrightarrow{OX} = \lambda\, \overrightarrow{OP}.$$

We obtain

$$\lambda = \frac{\overline{OX}}{\overline{OP}} = \frac{OX}{OP} = \frac{R^2}{|OP|^2} \tag{19}$$

By using 19 we obtain

$$\overrightarrow{OX} = \lambda\, \overrightarrow{OP} = \frac{R^2}{|OP|^2}\, \overrightarrow{OP}.$$

Hence

$$X - O = \frac{R^2}{|OP|^2}\, (P - O),$$

so that

$$X = O + \frac{R^2}{|OP|^2}\, (P - O).$$

If we use normalized barycentric coordinates, $O = (uO, vO, wO)$, $P = (uP, vP, wP)$, $X = (uX, vX, wX)$, we obtain

$$\begin{aligned} uX &= uO + \frac{R^2}{|OP|^2}\,(uP - uO), \\ vX &= vO + \frac{R^2}{|OP|^2}\,(vP - vO), \\ wX &= wO + \frac{R^2}{|OP|^2}\,(wP - wO). \end{aligned} \tag{20}$$

7. Popular Notable Points

The barycentric coordinates of the form $(f(a, b, c), f(b, c, a), f(c, a, b))$ are shortened to $[f(a, b, c)]$.

Popular triangle points in triangle ABC:

– Incenter = X(1) = $I = [a]$.
– Centroid = X(2) = $G = [1]$.
– Circumcenter = X(3) = $O = [(a^2(b^2 + c^2 - a^2)]$.
– Orthocenter = X(4) = $H = [(c^2 + a^2 - b^2)(a^2 + b^2 - c^2)]$.
– Nine-Point Center = X(5) = $N = [a^2(b^2 + c^2) - (b^2 - c^2)^2]$
– Symmedian Point = X(6) = $K = [a^2]$.
– Gergonne Point = X(7) = $Ge = [(c + a - b)(a + b - c)]$.
– Nagel Point = X(8) = $Na = [(b + c - a)]$.
– Mittenpunky = X(9) = $[a(b + c - a)]$.
– Spieker Center = X(10) = $Sp = [b + c]$.
– Feuerbach Point = X(11) = $[(b + c - a)(b - c)^2]$.
– Grinberg Point = X(37) = $[a(b + c)]$. (In honor of Darij Grinberg).
– Moses Point = X(75) = $[bc]$. (In honor of Peter Moses).
– tK = Third Brocard Point = X(76) = $[b^2c^2]$.
– Brisse Point (In honor of Edward Brisse) (suggest point).
– Stothers Point (In honor of Wilson Stothers) (suggest point).
– Paskalev Point (In honor of Georgi Paskalev) (suggest point).

8. Popular Notable Triangles

Given a point $P = (u, v, w)$.

The Cevian Triangle $P_aP_bP_c$ of P has barycentric coordinates as follows:

$$P_a = (0, v, w),\ P_b = (u, 0, w),\ P_c = (u, v, 0). \tag{21}$$

The Anticevian Triangle $P^aP^bP^c$ of P has barycentric coordinates as follows:

$$P^a = (-u, v, w),\ P^b = (u, -v, w),\ P^c = (u, v, -w). \tag{22}$$

The Pedal Triangle $P_{[a]}P_{[b]}P_{[c]}$ of P has barycentric coordinates as follows:

$$\begin{aligned} P_{[a]} &= (0, S_Cu + a^2v, S_Bu + a^2w), \\ P_{[b]} &= (S_Cv + b^2u, 0, S_Av + b^2w), \\ P_{[c]} &= (S_Bw + c^2u, S_Aw + c^2v, 0). \end{aligned} \tag{23}$$

The Euler Triangle $E_aE_bE_c$ of P has barycentric coordinates as follows:

$$E_a = (2u + v + w, v, w),\\ E_b = (u, u + 2v + w, w),\\ E_c = (u, v, u + v + 2w). \tag{24}$$

The Half-Cevian Triangle $HC_aHC_bHC_c$ of P has barycentric coordinates as follows:

$$HC_a = (v + w, v, w),\ HC_b = (u, u + w, w),\ HC_c = (u, v, u + v). \tag{25}$$

9. Popular Notable Lines

The equation of the Euler Line of triangle ABC is as follows:

$$(b-c)(b+c)(b^2+c^2-a^2)x+(c-a)(c+a)(c^2+a^2-b^2)y+(a-b)(a+b)(a^2+b^2-c^2)z = 0.$$

10. Popular Notable Circles

The equation of the Circumcircle of triangle ABC is as follows:

$$a^2yz + b^2zx + c^2xy = 0.$$

11. Transformations of Points

Given a point $P(u, v, w)$,

– the complement of P is the point $(v + w, w + u, u + v)$,
– the anticomplement of P is the point $(-u + v + w, -v + w + u, -w + u + v)$,
– the isotomic conjugate of P is the point (vw, wu, uv),
– and the isogonal conjugate of P is the point (a^2vw, b^2wu, c^2uv).

We will use the following notations:

– cP = complement of P.
– aP = anticomplement of P.
– gP = isogonal conjugate of P.
– tP = isotomic conjugate of of P.
– iP = inverse of P wrt given circle.

They easily combine between themselves and/or with other notations as in

– ctP = complement of isotomic conjugate of P.
– giP = isogonal conjugate of the inverse of P.

12. Products of Points

Given points $P_1 = (u_1, v_1, w_1)$ and $P_2 = (u_2, v_2, w_2)$,

– the product of P_1 and P_2 is the point (u_1u_2, v_1v_2, w_1w_2),
– the quotient of P_1 and P_2, $u_2v_2w_2 \neq 0$, is the point $(\frac{u_1}{u_2}, \frac{v_1}{v_2}, \frac{w_1}{w_2})$.

13. Similitude Centers of two Circles

Given circles $c_1 = (O_1, r_1)$ and $c_2 = (O_2, r_2)$.
The internal similitude center of circles is as follows:

$$Si = \frac{r_2 O_1 + r_1 O_2}{r_2 + r_1}.$$

That is, point Si divides internally the segment $\overrightarrow{O_1 O_2}$ in ratio $\frac{r_1}{r_2}$.
The external similitude center of circles is as follows:

$$Se = \frac{r_2 O_1 - r_1 O_2}{r_2 - r_1}.$$

That is, point Se divides externally the segment $\overrightarrow{O_1 O_2}$ in ratio $\frac{r_1}{r_2}$.

14. The Dergiades Method

Given three points which are not on the same line. There are a few methods for constructing the equation of the circle through these three points. The methods are almost equivalent. Paul Yiu [16, §15] presents a method due to the Greek mathematician Nikolaos Dergiades. The method is as follows:

The equation of the circle passing through three given points $P_1 = (u_1, v_1, w_1)$, $P_2 = (u_2, v_2, w_2)$ and $P_3 = (u_3, v_3, w_3)$ is as follows:

$$a^2yz + b^2zx + c^2xy - (x + y + z)(px + qy + rz) = 0$$

where

$$p = \frac{D_1}{s_1 s_2 s_3 D},\ q = \frac{D_2}{s_1 s_2 s_3 D},\ r = \frac{D_3}{s_1 s_2 s_3 D},$$

with

$$s_1 = u_1 + v_1 + w_1,\ s_2 = u_2 + v_2 + w_2,\ s_3 = u_3 + v_3 + w_3,\ D = \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix},$$

$$D_1 = \begin{vmatrix} a^2v_1w_1 + b^2w_1u_1 + c^2u_1v_1 & s_1v_1 & s_1w_1 \\ a^2v_2w_2 + b^2w_2u_2 + c^2u_2v_2 & s_2v_2 & s_2w_2 \\ a^2v_3w_3 + b^2w_3u_3 + c^2u_3v_3 & s_3v_3 & s_3w_3 \end{vmatrix},$$

$$D_2 = \begin{vmatrix} s_1u_1 & a^2v_1w_1 + b^2w_1u_1 + c^2u_1v_1 & s_1w_1 \\ s_2u_2 & a^2v_2w_2 + b^2w_2u_2 + c^2u_2v_2 & s_2w_2 \\ s_3u_3 & a^2v_3w_3 + b^2w_3u_3 + c^2u_3v_3 & s_3w_3 \end{vmatrix},$$

$$D_3 = \begin{vmatrix} s_1u_1 & s_1v_1 & a^2v_1w_1 + b^2w_1u_1 + c^2u_1v_1 \\ s_2u_2 & s_2v_2 & a^2v_2w_2 + b^2w_2u_2 + c^2u_2v_2 \\ s_3u_3 & s_3v_3 & a^2v_3w_3 + b^2w_3u_3 + c^2u_3v_3 \end{vmatrix}.$$

15. Geometric Constructions Methods

Method 1. Perspector-Perspector method.
We want to construct by using compass and ruler triangle $T = TaTbTc$. Given triangle $Tp = PaPbPc$ perspective with triangle T with perspector P, and triangle $Tq = QaQbQc$ perspective with triangle T with perspector Q. We construct triangle T as follows: Ta is the intersection of lines PPa and QQa, Tb is the intersection of lines PPb and QQb and Tc is the intersection of lines PPc and QQc.

Method 2. Perspector-Circumcircle method.
We want to construct by using compass and ruler triangle $T = TaTbTc$. Given triangle $Tp = PaPbPc$ perspective with triangle T with perspector P, and the circumcircle c of triangle T. We construct triangle T as follows: Ta is the intersection of line PPa and circle c, Tb is the intersection of line PPb and circle c, and Tc is the intersection of line PPc and circle c.

The above methods are especially effective if we use the computer program "Discoverer" which easily discovers perspective triangles and their perspectors.

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