International Journal of Computer Discovered Mathematics (IJCDM)
ISSN 2367-7775 ©IJCDM
March 2016, Volume 1, No.1, pp.1-10.
Received 15 October 2015. Published on-line 15 November 2015
web: http://www.journal-1.eu/

Computer Discovered Mathematics: Euler Triangles

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Abstract. By using the computer program "Discoverer", we investigate the Euler triangles. We present theorems about the Euler triangle of an arbitrary point P, and we also consider the special cases when P is the Incenter, Centroid, Circumcenter, etc.

Keywords. Euler triangle of a point, triangle geometry, remarkable point, perspector, computer discovered mathematics, Euclidean geometry, "Discoverer".

Mathematics Subject Classification (2010). 51-04, 68T01, 68T99.

1. Introduction

The computer program "Discoverer", created by the authors, is the first computer program, able easily to discover new theorems in mathematics, and possibly, the first computer program, able easily to discover new knowledge in science. See [3].

In this paper, by using the "Discoverer", we investigate the Euler triangles. The paper contains selected theorems about Euler triangles. We expect that the majority of these theorems are new, discovered by a computer. Many of the proofs of theorem are not presented here. We recommend to the reader to find the proofs.

Recall the definition of the Euler triangle of an arbitrary point. See [5]. Let P be a point in the plane of triangle ABC. Denote by Ea the midpoint of points P and A. Similarly, denote by Eb the midpoint of points P and B and by Ec the midpoint of points P and C. Then $EaEbEc$ is the *Euler Triangle of Point* P.

If point P coincides with the Orthocenter of triangle ABC, then the Euler triangle of P coincides with the ordinary Euler triangle. See [15], article "Euler triangle".

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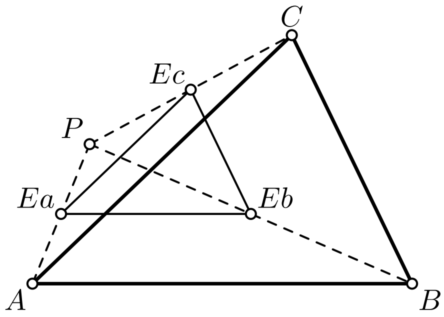

FIGURE 1.

Figure 1 illustrates the definition of an Euler triangle of point P. In Figure 1, P is an arbitrary point and $EaEbEc$ is the Euler triangle of P.

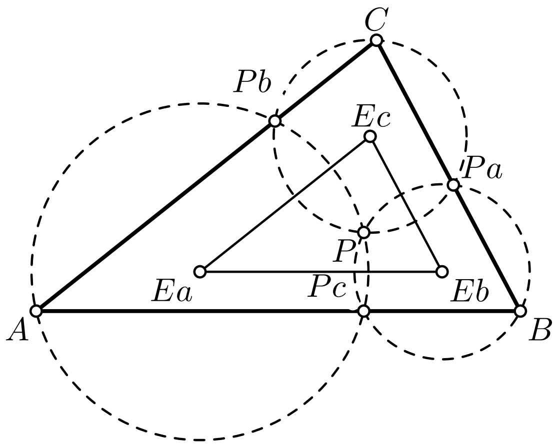

FIGURE 2.

For any point P, denote by Pa, Pb, Pc the projections of P on lines BC, CA, AB respectively. Then the vertices of the Euler triangle $EaEbEc$ of P are the centers of the circumcircles of triangles $APbPc, BPcPa, CPaPb$. These circumcircles intersect in P. (The Miquel theorem). See Figure 2

The following two theorems are published in [5]:

Theorem 1.1. *The barycentric coordinates of the Euler triangle $EaEbEc$ of point $P = (u, v, w)$ are as follows:*

$$Ea = (2u + v + w, v, w), \; Eb = (u, u + 2v + w, w), \; Ec = (u, v, u + v + 2w).$$

Problem 1.1. *For the special case P = Orthocenter, deduce form theorem 1.1 the formulas for the trilinear coordinates of the Euler triangle, given in* [15], *article "Euler triangle".*

Theorem 1.2. *For any point P, the Euler Triangle of Point P and the following triangles are perspective:*

(1) *Medial Triangle. The Perspector is the Complement of the Complement of Point P.*
(2) *Half-Medial Triangle. The Perspector is the Complement of Point P.*
(3) *Antimedial Triangle.*
(4) *Inner Grebe Triangle.*
(5) *Outer Grebe Triangle.*

(6) *Johnson Triangle.*
(7) *Inner Yff Triangle*
(8) *Outer Yff Triangle.*

2. Preliminaries

In this section we review some basic facts about barycentric coordinates. We refer the reader to [16],[2],[1],[12],[8],[9],[11],[14]. The labeling of triangle centers follows Kimberling's ETC [10]. For example, X(1) denotes the Incenter, X(2) denotes the Centroid, X(37) is the Grinberg Point, X(75) is the Moses point (Note that in the prototype of the "Discoverer" the Moses point is the X(35)), etc.

The reader may find definitions in [4], Contents, Definitions, and in [15].

The reference triangle ABC has vertices $A = (1, 0, 0), B(0, 1, 0)$ and $C(0, 0, 1)$. The side lengths of $\triangle ABC$ are denoted by $a = BC, b = CA$ and $c = AB$. A point is an element of $\mathbb{R}^3$, defined up to a proportionality factor, that is,

For all $k \in \mathbb{R} - \{0\}$: $P = (u, v, w)$ means that $P = (u, v, w) = (ku, kv, kw)$.

Given a point $P(u, v, w)$. Then P is *finite*, if $u + v + w \neq 0$. A finite point P is *normalized*, if $u + v + w = 1$. A finite point could be put in normalized form as follows: $P = (\frac{u}{s}, \frac{v}{s}, \frac{w}{s})$, where $s = u + v + w$.

The equation of the line joining two points with coordinates u_1, v_1, w_1 and u_2, v_2, w_2 is

$$\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ x & y & z \end{vmatrix} = 0. \tag{2.1}$$

Three lines $p_i x + q_i y + r_i z = 0$, $i = 1, 2, 3$ are concurrent if and only if

$$\begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix} = 0 \tag{2.2}$$

The intersection of two lines $L_1 : p_1 x + q_1 y + r_1 z = 0$ and $L_2 : p_2 x + q_2 y + r_2 z = 0$ is the point

$$(q_1 r_2 - q_2 r_1, r_1 p_2 - r_2 p_1, p_1 q_2 - p_2 q_1) \tag{2.3}$$

Given two points $P = (u_1, v_1, w_1)$ and $Q = (u_2, v_2, w_2)$ in normalized barycentric coordinates, the midpoint M of P and Q is as follows:

$$M = \left(\frac{u_1 + u_2}{2}, \frac{v_1 + v_2}{2}, \frac{w_1 + w_2}{2} \right). \tag{2.4}$$

Given point R with barycentric coordinates p, q, r with respect to triangle DEF, $D = (u_1, v_1, w_1)$, $E = (u_2, v_2, w_2)$, $F = (u_3, v_3, w_3)$. If points D, E, F and R are normalized, the barycentric coordinates u, v, w of R with respect to triangle ABC are as follows:

$$\begin{aligned} u &= u_1 p + u_2 q + u_3 r, \\ v &= v_1 p + v_2 q + v_3 r, \\ w &= w_1 p + w_2 q + q_3 r. \end{aligned} \tag{2.5}$$

Given a point $P(u, v, w)$, the complement of P is the point $(v + w, w + u, u + v)$, the anticomplement of P is the point $(-u + v + w, -v + w + u, -w + u + v)$, the isotomic conjugate of P is the point (vw, wu, uv), and the isogonal conjugate of P is the point (a^2vw, b^2wu, c^2uv).

3. Euler Triangle of Point P

The computer program "Discoverer" has discovered theorems about Euler triangles. Parts of these theorem are given below. The enclosed files are produced by the "Discoverer". They contain lists and tables with theorems.

Theorem 3.1. *Let P and Q be named points labeled $\mathcal{P}$ and $\mathcal{Q}$, respectively. Then $\mathcal{Q}$ of the Euler triangle of the $\mathcal{P}$ is the midpoint of the $\mathcal{P}$ and the $\mathcal{Q}$.*

Proof. We use barycentric coordinates. Let $P = (u, v, w)$ and $Q = (p, q, r)$ be points. By using (2.4) we see that the midpoint M of P and Q is the point

$$M = (2up+uq+ur+pv+pw, pv+2vq+vr+uq+qw, pw+qw+2wr+ur+vr).$$

Let $EaEbEc$ be the Euler triangle of point P. Suppose that point R has barycentric coordinates p, q, r with respect to the Euler triangle $EaEbEc$. Suppose that points Ea, Eb, Ec and R have normalized coordinates. Then by using (2.5) we obtain the barycentric coordinates of R with respect to triangle ABC as follows:

$$R = (2up+uq+ur+pv+pw, pv+2vq+vr+uq+qw, pw+qw+2wr+ur+vr).$$

It is easy to see that points M and R coincide. □ □

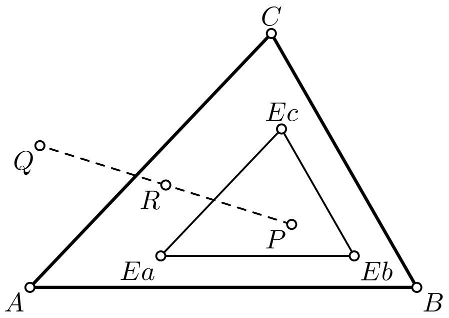

FIGURE 3.

Figure 3 illustrates theorem 3.1. In Figure 3, P and Q are arbitrary points, $EaEbEc$ is the Euler triangle of point P, R is the midpoint of points P and Q. If P and Q are named points labeled by $\mathcal{P}$ and $\mathcal{Q}$ respectively, then $\mathcal{Q}$ of the Euler triangle of the $\mathcal{P}$ is the midpoint of the $\mathcal{P}$ and the $\mathcal{Q}$.

Example 3.1. *If $\mathcal{P}$ is the Incenter and $\mathcal{Q}$ is the Centroid, we obtain the following statement: The Centroid of the Euler triangle of the Incenter is the midpoint of the Incenter and the Centroid.*

Example 3.2. *If $\mathcal{P} = \mathcal{Q} =$ Centroid, we obtain the following statement: The Centroid of the Euler triangle of the Centroid is the Centroid.*

Parts of the theorems below give more details. The reader may find additional details in the enclosed files. The enclosed files are created by the computer program "Discoverer".

About investigation of midpoints and reflections, see [6],[7].

4. Special case: P = Incenter

The barycentric coordinates of the Euler triangle of the Incenter are as follows (See Theorem 1.1):

$$Ea = (2a + b + c, b, c),\ Eb = (a, a + 2b + c, c),\ Ec = (a, b, a + b + 2c).$$

Theorem 4.1. *The following table gives the centers of the Euler triangle of the Incenter (EulerT - left) in terms of the centers of the reference triangle (RefT - right):*

X(n) wrt EulerT	Center of Euler triangle of the Incenter	X(n) wrt RefT
X(2)	Centroid	X(551)
X(3)	Circumcenter	X(1385)
X(4)	Orthocenter	X(946)
X(5)	Nine-Point Center	X(5901)
X(6)	Symmedian Point	X(1386)
X(7)	Gergonne Point	X(5542)
X(8)	Nagel Point	X(10)
X(9)	Mittenpunkt	X(1001)
X(10)	Spieker Center	X(1125)
X(11)	Feuerbach Point	X(1387)
X(20)	de Longchamps Point	X(4297)
X(35)	Perspector of the Intangents Triangle and the Kosnita Triangle	X(2646)
X(36)	Inverse of the Incenter in the Circumcircle	X(1319)
X(40)	Bevan Point	X(3)
X(43)	Perspector of the Excentral Triangle and the Symmedial Triangle	X(995)
X(44)	Harmonic Conjugate of the Grinberg Point with respect to the Mittenpunkt and the Symmedian Point	X(3246)

Table 1

In the enclosed files the reader may find an extension of the above table.

If we apply theorem 3.1 to the first row of Table 1, we obtain: The Centroid of the Euler triangle of the Incenter is the midpoint of the Incenter and the Centroid. This is point X(551).

Additional identification of points of the Euler triangle of the Incenter the reader may find in the enclosed files.

Theorem 4.2. *The Perspector of the Euler Triangle of the Incenter and the*

(1) *Medial Triangle is the X(1125) Complement of the Spieker center.*
(2) *Intouch Triangle is the X(3649).*
(3) *Antimedial Triangle is the X(3616).*
(4) *Johnson Triangle is the X(5886).*
(5) *Half-Median Triangle is the X(10) Spieker Center.*

Proof of Theorem 4.2.1.

(1) We apply theorem 1.2.1. The Complement of the Incenter is the Spieker Center. The Complement of the Spieker Center is the point X(1125). □

(2) The barycentric coordinates of the Euler triangle $EaEbEc$ of the Incenter are given in this section. The barycentric coordinates of the Intouch triangle $TaTbTc$ are as follows:

$$Ta = (0, (b-c+a)(b+c-a), (c-a+b)(c+a-b)),$$

$$Tb = ((a-b+c)(a+b-c), 0, (c-a+b)(c+a-b)),$$

$$Tc = ((a-b+c)(a+b-c), (b-c+a)(b+c-a), 0).$$

By using (2.1) we find the equations of the lines $L_1 = EaTa$, $L_2 = EbTb$, and $L_3 = EcTc$ as follows:

$$L_1 : (b-c)(b+c-a)x + (2a+b+c)(c+a-b)y - (2a+b+c)(a+b-c)z = 0.$$

$$L_2 : (a+2b+c)(b+c-a)x + (a-c)(c+a-b)y - (a+2b+c)(a+b-c)z = 0$$

$$L_3 : -(b+c-a)(a+b+2c)x + (c+a-b)(a+b+2c)y - (a-b)(a+b-c)z = 0.$$

By using (2.2) we conclude that the lines L_1, L_2 and L_3 concur in a point. Then by using (2.3) we find the intersection point R of lines L_1 and L_2 as follows:

$$R = ((b+c)(2a+b+c)(a-b+c)(a+b-c),$$

$$(c+a)(2b+c+a)(b-c+a)(b+c-a),$$

$$(a+b)(2c+a+b)(c-a+b)(c+a-b)).$$

This is point X(3649). □

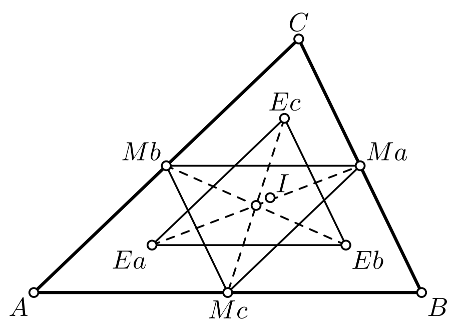

FIGURE 4.

Figure 4 illustrates theorem 4.2.1. In Figure 4, I is the Incenter, $EaEbEc$ is the Euler triangle of the Incenter, and $MaMbMc$ is the Medial triangle. Then lines $EaMa$, $EbMb$ and $EcMc$ concur in a point. This is the point X(1125), the Complement of the Spieker Center.

5. Special case: P = Centroid

The barycentric coordinates of the Euler triangle of the Centroid are as follows (See Theorem 1.1):

$$Ea = (4, 1, 1),\ Eb = (1, 4, 1),\ Ec = (1, 1, 4).$$

Note that the enclosed files contain results about identification of remarkable points of the Euler triangle of the Centroid.

Theorem 5.1. *The Perspector of the Euler Triangle of the Centroid and the Johnson Triangle is the X(5055).*

6. Special case: P = Circumcenter

The barycentric coordinates of the Euler triangle of the Circumcenter are as follows (See Theorem 1.1):

$$Ea = (3a^2b^2 + 3a^2c^2 + 2b^2c^2 - 2a^4 - b^4 - c^4, b^2(c^2 + a^2 - b^2), c^2(a^2 + b^2 - c^2)),$$
$$Eb = (a^2(b^2 + c^2 - a^2), 3a^2b^2 + 2a^2c^2 + 3b^2c^2 - a^4 - 2b^4 - c^4, c^2(a^2 + b^2 - c^2)),$$
$$Ec = (a^2(b^2 + c^2 - a^2), b^2(c^2 + a^2 - b^2), 2a^2b^2 + 3a^2c^2 + 3b^2c^2 - a^4 - b^4 - 2c^4).$$

Note that the enclosed files contain results about identification of remarkable points of the Euler triangle of the Circumcenter.

Theorem 6.1. *The Perspector of the Euler Triangle of the Circumcenter and the*

1. *Medial Triangle is the X(140) Nine-Point Center of the Medial Triangle.*
2. *Antimedial Triangle is the X(631).*
3. *Lucas Central Triangle is the X(3) Circumcenter.*
4. *Johnson Triangle is the X(2) Centroid.*
5. *Inner Yff Triangle is the X(56) External Center of Similitude of Circumcircle and Incircle.*
6. *Outer Yff Triangle is the X(55) Internal Center of Similitude of Circumcircle and Incircle.*
7. *Kosnita Triangle is the X(182) Center of the Brocard Circle.*
8. *Half-Median Triangle is the X(5) Nine-Point Center.*

7. Special case: P = Orthocenter. The ordinary Euler Triangle

If point P coincides with the Orthocenter, then the Euler triangle of P coincides with the ordinary Euler triangle.

The barycentric coordinates of the Euler triangle of the Orthocenter are as follows (See Theorem 1.1):

$$Ea = (4c^2b^2 - 2c^4 - 2b^4 + 2c^2a^2 + 2a^2b^2, (a^2 + b^2 - c^2)(b^2 + c^2 - a^2),$$
$$(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)).$$
$$Eb = ((c^2 + a^2 - b^2)(a^2 + b^2 - c^2), 2c^2b^2 - 2c^4 - 2a^4 + 4c^2a^2 + 2a^2b^2,$$
$$(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)).$$
$$Ec = ((c^2 + a^2 - b^2)(a^2 + b^2 - c^2), (a^2 + b^2 - c^2)(b^2 + c^2 - a^2),$$
$$2c^2b^2 - 2a^4 - 2b^4 + 2c^2a^2 + 4a^2b^2).$$

Theorem 7.1. *Table 2 below completes the table given in* [15], *article "Euler Triangle".*

X(n) wrt EulerT	**Center of Euler triangle of the Orthocenter**	**X(n) wrt RefT**
X(6)	Symmedian Point	X(5480)
X(7)	Gergonne Point	X(5805)
X(13)	Outer Fermat Point	X(5478)
X(14)	Inner Fermat Point	X(5479)
X(15)	First Isodynamic Point	X(7684)
X(16)	Second Isodynamic Point	X(7685)
X(21)	Schiffler Point	X(6841)
X(52)	Orthocenter of the Orthic Triangle	X(5446)
X(54)	Kosnita Point	X(3574)
X(55)	Internal Center of Similitude of the Incircle and the Circumcircle	X(7680)
X(56)	External Center of Similitude of the Incircle and the Circumcircle	X(7681)
X(57)	Isogonal Conjugate of the Mittenpunkt	X(7682)
X(58)	Isogonal Conjugate of the Spieker Center	X(7683)
X(64)	Isogonal Conjugate of the de Longchamps Point	X(6247)
X(65)	Orthocenter of the Intouch Triangle	X(7686)
X(72)	Quotient of the Grinberg Point and the Orthocenter	X(5777)
X(76)	Third Brocard Point	X(6248)
X(80)	Reflection of the Incenter in the Feuerbach Point	X(6246)
X(83)	Isogonal Conjugate of the Brocard Midpoint	X(6249)
X(84)	Perspector of Triangle ABC and the Hexyl Triangle	X(6245)

TABLE 2

An extension of Table 2 is given in the enclosed files.

If we apply theorem 3.1 to the first row of Table 2, we obtain: The Symmedian Point of the Euler triangle of the Orthocenter is the midpoint of the Orthocenter and the Symmedian Point. This is point X(5480).

Theorem 7.2. *The Perspector of the Euler Triangle of the Orthocenter and the*

(1) *Medial Triangle is the X(5) Nine-Point Center.*
(2) *Antimedial Triangle is the X(3091).*
(3) *Inner Grebe Triangle is the X(6202).*
(4) *Outer Grebe Triangle is the X(6201).*
(5) *Johnson Triangle is the X(381) Center of the Orthocentroidal Circle.*
(6) *Inner Yff Triangle is the X(1479) Center of the Outer Johnson-Yff Circle.*
(7) *Outer Yff Triangle is the X(1478) Center of the Inner Johnson-Yff Circle.*
(8) *Half-Median Triangle is the X(3) Circumcenter.*

8. SPECIAL CASE: P = NINE-POINT CENTER

The barycentric coordinates of the Euler triangle of the Nine-Point Center are as follows (See Theorem 1.1):

$$Ea = (5a^2b^2 + 5a^2c^2 - 3b^4 + 6b^2c^2 - 3c^4 - 2a^4, b^2(c^2 + a^2) - (c^2 - a^2)^2,$$

$$c^2(a^2+b^2)-(a^2-b^2)^2).$$

$$Eb = (a^2(b^2+c^2)-(b^2-c^2)^2, 5a^2b^2+6a^2c^2-2b^4+5b^2c^2-3c^4-3a^4,$$
$$c^2(a^2+b^2)-(a^2-b^2)^2).$$

$$Ec = (a^2(b^2+c^2)-(b^2-c^2)^2, b^2(c^2+a^2)-(c^2-a^2)^2,$$
$$6a^2b^2+5a^2c^2-3b^4+5b^2c^2-2c^4-3a^4).$$

Note that the enclosed files contain results about identification of remarkable points of the Euler triangle of the Nine-Point Center.

Theorem 8.1. *The Perspector of the Euler Triangle of the Nine-Point Center and the*

1. *Medial Triangle is the X(3628).*
2. *Antimedial Triangle is the X(1656).*
3. *Johnson Triangle is the X(5) Nine-Point Center.*
4. *Inner Yff Triangle is the X(496) Harmonic Conjugate of the Johnson Midpoint.*
5. *Outer Yff Triangle is the X(495) Johnson Midpoint.*
6. *Half-Median Triangle is the X(140) Nine-Point Center of the Medial Triangle.*

9. Special case: P = Symmedian Point

The barycentric coordinates of the Euler triangle of the Symmedian Point are as follows (See Theorem 1.1):

$Ea = (2a^2+b^2+c^2, b^2, c^2)$, $Eb = (a^2, a^2+2b^2+c^2, c^2)$, $Ec = (a^2, b^2, a^2+b^2+2c^2)$.

Note that the enclosed files contain results about identification of remarkable points of the Euler triangle of the Symmedian Point.

Theorem 9.1. *The Perspector of the Euler Triangle of the Symmedian Point and the*

1. *Medial Triangle is the X(3589).*
2. *Antimedial Triangle is the X(3618).*
3. *Inner Yff Triangle is the X(613).*
4. *Outer Yff Triangle is the X(611).*
5. *Half-Median Triangle is the X(141).*

Supplementary Material

The enclosed file `http://www.journal-1.eu/2015/01/zips/Euler_Tringles.zip` contains the files quoted in this paper.

Acknowledgement

The authors are grateful to Professor René Grothmann for his wonderful computer program *C.a.R.* `http://car.rene-grothmann.de/doc_en/index.html` and to Professor Troy Henderson for his wonderful computer program *MetaPost Previewer* `http://www.tlhiv.org/mppreview/`.

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