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Computer Discovered Mathematics: Dividing Directed Segments

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Abstract. Dividing directed segments is one of the introductory topics in Geometry. Now is the time of computers. In this paper we use the computer program "Discoverer" and we investigate the remarkable points dividing directed segments whose endpoints are Kimberling points. The theorems in this paper could be used as problems for high school and university students. We recommend to the reader to find the proofs of te theorems.

Keywords. remarkable point, division of directed segment, triangle geometry, computer discovered mathematics, Euclidean geometry, "Discoverer".

Mathematics Subject Classification (2010). 51-04, 68T01, 68T99.

1. Introduction

In this paper we study the points dividing directed segments whose endpoints are Kimberling points.

The enclosed file "Supplementary material" contains a number of theorems related to the topic. These theorems are discovered by the computer program "Discoverer".

We use barycentric coordinates. The reader may find some info about barycentric coordinates in [11],[2],[1],[8],[5],[6]. The labeling of triangle centers follows Kimberling's ETC [7]. For example, X(1) denotes the Incenter, X(2) denotes the Centroid, X(37) is the Grinberg Point, X(75) is the Moses point (Note that in the prototype of the "Discoverer" the Moses point is the X(35)), etc.

The reader may find definitions in [4], Contents, Definitions, and in [10].

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The reference triangle ABC has vertices $A=(1,0,0), B(0,1,0)$ and C(0,0,1). The side lengths of $\triangle ABC$ are denoted by $a=BC, b=CA$ and $c=AB$. A point is an element of $\mathbb{R}^3$, defined up to a proportionality factor, that is,

$$\text{For all } k \in \mathbb{R}^3 - \{0\}: \ P=(u,v,w) \text{ means that } P=(u,v,w)=(ku,kv,kw).$$

Let P be a point with barycentric coordinates (u,v,w). If $u+v+w \neq 0$, we obtain the *normalized* barycentric coordinates by scaling the coordinates to have a unit sum: $P=\left(\frac{u}{u+v+w}, \frac{v}{u+v+w}, \frac{w}{u+v+w}\right)$.

In this paper we denote by p and q positive integers.

We denote by G the Centroid and by I the Incenter of triangle ABC. Their barycentric coordinates are as follows: $G=(1,1,1)$ and $I=(a,b,c)$.

2. Dividing Directed Segments

2.1. **Internal Division.** Point R divides internally the directed segment $\overrightarrow{PQ}$ in ratio $p:q$, if $\overrightarrow{PR}:\overrightarrow{RQ}=p:q$.

If points P and Q are given in normalized barycentric coordinates, the point R which divides internally the directed segment $\overrightarrow{PQ}$ in the ratio $p:q$ has normalized barycentric coordinates

$$R=\frac{q.P+p.Q}{p+q} \tag{1}$$

From (1) we deduce the formula for barycentric coordinates. Suppose that $P=(u,v,w)$ and $Q=(x,y,z)$ are given in (not necessarily normalized) barycentric coordinates. Then $R=(uR,vR,wR)$ has barycentric coordinates as follows:

$$\begin{aligned} uR &= qux+quy+quz+pxu+pxv+pxw, \\ vR &= qvx+qvy+qvz+pyu+pyv+pyw, \\ wR &= qwx+qwy+qwz+pzu+pzv+pzw. \end{aligned} \tag{2}$$

Example 2.1. *The point R dividing the directed segment* $\overrightarrow{GI}$ *in the ratio 2:1 has barycentric coordinates as follows:*

$$R=(7a+b+c, a+7b+c, a+b+7c).$$

Note that this point is not available in the Kimberling's ETC [7].

Special case: Midpoint.

The point dividing internally a segment in the ratio 1:1, is the midpoint of this segment. If $p=1$ and $q=1$, formula (1) reduces to the formula for the midpoint of $P=(u,v,w)$ and $Q=(x,y,z)$ as follows:

$$\begin{aligned} uR &= 2ux+uy+uz+xv+xw, \\ vR &= xv+2vy+vz+uy+yw, \\ wR &= xw+yw+2wz+uz+vz. \end{aligned}$$

Example 2.2. *The midpoint R of the Centroid and the Incenter has barycentric coordinates as follows:*

$$R=(4a+b+c, a+4b+c, a+b+4c).$$

This is point X(551) in Kimberling's ETC [7].

Special case 2: Complement of Complement.

If P is an arbitrary point, then the point dividing internally the directed segment $\overrightarrow{PG}$ in the ratio $3:1$, is the complement of the complement of P. The complement of the complement R of $P=(u,v,w)$ has barycentric coordinates as follows:

$$R = 2u+v+w, 2v+u+w, 2w+u+v).$$

Example 2.3. *The complement of the complement R of the Incenter has barycentric coordinates as follows:*

$$R = (2a+b+c, 2b+a+c, 2c+a+b).$$

This is point X(1125) in the Kimberling's ETC [7].

2.2. **External Division.** Point R divides externally the directed segment $\overrightarrow{PQ}$ in the ratio $p:q$, $(p \neq q)$, if $\overrightarrow{PR}:\overrightarrow{QR} = p:q$.

If points P and Q are given in normalized barycentric coordinates, then the point R which divides externally the directed segment $\overrightarrow{PQ}$ in the ratio $p:q$ has normalized barycentric coordinates as follows:

$$R = \frac{q.P - p.Q}{q-p}. \tag{3}$$

From (3) we deduce the formula for barycentric coordinates. Suppose that $P = (u,v,w)$ and $Q = (x,y,z)$ are given in (not necessarily normalized) barycentric coordinates. Then $R = (uR, vR, wR)$ has barycentric coordinates as follows:

$$\begin{aligned} uR &= pxu + pxv + pxw - qux - quy - quz, \\ vR &= pyu + pyv + pyw - qvx - qvy - qvz, \\ wR &= pzu + pzv + pzw - qwx - qwy - qwz. \end{aligned} \tag{4}$$

Example 2.4. *The point R dividing externally the directed segment $\overrightarrow{GI}$ in the ratio 1:3 (or equivalently, the point dividing the directed segment $\overrightarrow{IG}$ in the ratio 3:1) has barycentric coordinates as follows:*

$$R = (b+c, c+a, a+b).$$

It is easy to see that this is the Spieker center, point X(10) in the Kimberling's ETC [7].

Special case 1: Reflection.

Point dividing externally the directed segment $\overrightarrow{PQ}$ in ratio 2:1, is the reflection of point P in point Q. The reflection R of $P=(u,v,w)$ in $Q=(x,y,z)$ has barycentric coordinates as follows:

$$\begin{aligned} uR &= ux - uy - uz + 2xv + 2xw, \\ vR &= -xv + vy - vz + 2uy + 2yw, \\ wR &= -xw - yw + wz + 2uz + 2vz. \end{aligned}$$

Example 2.5. *The reflection R of the Centroid in the Incenter has barycentric coordinates as follows:*

$$R = (5a-b-c, -a+5b-c, -a-b+5c)$$

This is point X(3241) in Kimberling's ETC [7].

Special case 2: Complement.

If P is an arbitrary point, then the point dividing externally the directed segment $\overrightarrow{PG}$ in the ratio 3:1, is the complement of P. The complement R of $P = (u, v, w)$ has barycentric coordinates as follows:

$$R = (v + w, w + u, u + v).$$

Example 2.6. *The complement R of the Incenter $I = (a, b, c)$ has barycentric coordinates as follows:*

$$R = (b + c, c + a, a + b).$$

This is the Spieker center, point X(10) in the Kimberling's ETC [7].

Special case 3: Anticomplement.

If P is an arbitrary point, then the point dividing externally the directed segment $\overrightarrow{PG}$ in the ratio 3:2, is the anticomplement of P. The anticomplement R of $P = (u, v, w)$ has barycentric coordinates as follows:

$$R = (v + w - u, w + u - v, u + v - w).$$

Example 2.7. *The anticomplement R of the Incenter $I = (a, b, c)$ has barycentric coordinates as follows:*

$$R = (b + c - a, c + a - b, a + b - c).$$

This is the Nagel point, point X(8) in the Kimberling's ETC [7].

Special case 4: Anticomplement of Anticomplement.

The point dividing externally the directed segment $\overrightarrow{PG}$ in the ratio 3:4, is the anticomplement of the anticomplement of P. The anticomplement of the anticomplement R of $P = (u, v, w)$ has barycentric coordinates as follows:

$$R = (3u - v - w, 3v - w - u, 3w - u - v.)$$

Example 2.8. *The anticomplement of the anticomplement R of the Incenter has barycentric coordinates as follows:*

$$R = (3a - b - c, 3b - c - a, 3c - a - b.)$$

This is the Nagel Point of the Antimedial Triangle, point X(145) in the Kimberling's ETC [7].

Special case 5: Complement of Complement of Complement.

The point R dividing externally the directed segment $\overrightarrow{PG}$ in the ratio 9:1, is the complement of the complement of the complement R of P. The barycentric coordinates of R are as follows:

$$R = (2u + 3v + 3w, 2v + 3u + 3w, 2w + 3u + 3v).$$

Example 2.9. *The barycentric coordinates of the complement of the complement of the complement R of the Incenter are as follows:*

$$R = 2a + 3b + 3c, 2b + 3a + 3c, 2c + 3a + 3b).$$

This is point X(3634) in the Kimberling's ETC [7].

Special case 6: Anticomplement of Anticomplement of Anticomplement.

The point R dividing externally the directed segment $\overrightarrow{PG}$ in the ratio 9:8, is the anticomplement of the anticomplement of the anticomplement of P. The barycentric coordinates of R are as follows:

$$R = 5u - 3v - 3w, 5v - 3u - 3w, 5w - 3u - 3v).$$

Example 2.10. *The barycentric coordinates of the anticomplement of the anticomplement of the anticomplement R of the Incenter are as follows:*

$$R = (5a - 3b - 3c, 5b - 3a - 3c, 5c - 3a - 3b).$$

This is point X(3621) in the Kimberling's ETC [7].

Note that dividing internally (resp. externally) of the directed segment $\overrightarrow{PQ}$ in the ratio $p : q$ is the same as dividing internally (resp. externally) of the directed segment $\overrightarrow{QP}$ in the ratio $q : p$.

3. INTERNAL DIVISION

3.1. **Kimberling Points.** //

Many of the points dividing directed segments are available in the Kimberling's ETC [7].

In this section we consider points available in [7].

Below we give an example related to point X(3845), [7]. Additional examples are given in the enclosed Supplementary material.

The computer program "Discoverer" discovers many new theorem about points available in the [7]. The following theorems 3.1, 3.2 and 3.3 extend the article X(3845) of [7]:

Theorem 3.1. *Point Dividing Internally the Directed Segment from the Centroid to the Orthocenter in the Ratio of 3:1 is the Point X(3845).*

By using this theorem we may easily calculate the barycentric coordinates of point X(3845). The first barycentric coordinate of X(3845) is as follows:

$$10c^2b^2 - 5c^4 + 4a^4 - 5b^4 + c^2a^2 + a^2b^2.$$

The barycentric coordinates of X(3845) are also given in [7].

Theorem 3.2. *Point X(3845) is the*

1. *Centroid of the Euler Triangle of the Euler Triangle.*
2. *Centroid of the Johnson Triangle of the Euler Triangle.*
3. *Circumcenter of the Fourth Brocard Triangle of the Euler Triangle.*
4. *Orthocenter of the Euler Triangle of the Center of the Orthocentroidal Circle.*
5. *Orthocenter of the Triangle of the Circumcenters of the Pedal Corner Triangles of the Center of the Orthocentroidal Circle.*
6. *de Longchamps Point of the Triangle of the Centroids of the Triangulation Triangles of the Nine-Point Center.*
7. *Steiner Point of the Triangle of the Nine-Point Centers of the Triangulation Triangles of the Centroid.*

(8) *Center of the Orthocentroidal Circle of the Euler Triangle.*
(9) *Retrocenter of the Honsberger Triangle of the Medial Triangle.*
(10) *Midpoint of the Center of the Orthocentroidal Circle and the Orthocenter.*
(11) *Midpoint of the Circumcenter and the Reflection of the Centroid in the Orthocenter.*
(12) *Reflection of the Nine-Point Center in the Center of the Orthocentroidal Circle.*
(13) *Reflection of the Midpoint of the Centroid and the Circumcenter in the Nine-Point Center.*
(14) *Reflection of the Circumcenter in the Midpoint of the Centroid and the Nine-Point Center.*
(15) *Reflection of the Center of the Orthocentroidal Circle in the Midpoint of the Nine-Point Center and the Orthocenter.*

Theorem 3.3. *The point X(3845) lies on the*

(1) *Orthocentroidal Circle of the Euler Triangle of the Center of the Orthocentroidal Circle.*
(2) *Circumcircle of the Triangle of the Nine-Point Centers of the Triangulation Triangles of the Centroid.*
(3) *Orthocentroidal Circle of the Euler Triangle of the Euler Triangle.*
(4) *Parry Circle of the Euler Triangle of the Euler Triangle.*
(5) *Parry Circle of the Johnson Triangle of the Euler Triangle.*
(6) *Brocard Circle of the Fourth Brocard Triangle of the Euler Triangle.*
(7) *Lester Circle of the Fourth Brocard Triangle of the Euler Triangle.*

3.2. **New Remarkable Points.** //

Many of the points dividing directed segments are not available in the Kimberling's ETC [7]. In this section we consider points not available in [7]. We could consider these points as new remarkable points of the triangle. The reader may find additional examples in the enclosed Supplementary material.

Table 1 gives examples of internal divisions. In Table 1 point R divides internally the directed segment $\overrightarrow{PQ}$ in the ratio $2:1$. The points in the last column of the table are not available in the Kimberling's ETC [7].

We produce the table by using the following theorem:

Theorem 3.4. *Given points $P = (u, v, w)$ and $Q = (x, y, z)$. Then point R dividing internally the directed segment $\overrightarrow{PQ}$ in the ratio $2:1$ has barycentric coordinates as follows:*

$$uR = 3ux + uy + uz + 2xv + 2xw,$$

$$vR = xv + 3vy + vz + 2uy + 2yw,$$

$$wR = xw + yw + 3wz + 2uz + 2vz.$$

	P	Q	First barycentric coordinate of R
1	X(1)	X(2)	$5a+2b+2c$
2	X(1)	X(6)	$a(3a^2+b^2+c^2+2ab+2ac)$
3	X(1)	X(7)	$a^3-3ab^2+6abc-3ac^2+4a^2c+4a^2b-2b^3+2b^2c+2bc^2-2c^3$
4	X(2)	X(1)	$7a+b+c$
5	X(2)	X(6)	$7a^2+b^2+c^2$
6	X(2)	X(7)	$5a^2-7b^2+14bc-7c^2+2ac+2ab$
7	X(2)	X(8)	$7b+7c-5a$
8	X(3)	X(1)	$a(2b^3-3ab^2-2cb^2+4cba-2a^2b-2bc^2-3ac^2+3a^3-2a^2c+2c^3)$
9	X(3)	X(8)	$2b^4-4ab^3-a^2b^2-4b^2c^2+4ab^2c+4ac^2b+4a^3b-8bca^2-a^4-a^2c^2+2c^4-4ac^3+4a^3c$
10	X(4)	X(6)	$6a^2c^2b^2-3a^2c^4-a^6-3a^2b^4+b^4c^2+b^2c^4+5b^2a^4-b^6-c^6+5c^2a^4$

TABLE 1

The "Discoverer" discovers new theorems about new remarkable points.

An example: Denote by D the Point Dividing Internally the Directed Segment from the Incenter to the Centroid in the Ratio of 2:1.

Theorem 3.5. *Point D is the:*

(1) *Centroid of the Triangle of the Centroids of the Triangulation Triangles of the Incenter.*
(2) *Center of the Fuhrmann Circle of the Triangle of the Centroids of the Triangulation Triangles of the Center of the Orthocentroidal Circle.*
(3) *Harmonic Conjugate of the Midpoint of the Centroid and the Nagel Point with respect to the Centroid and the Incenter.*

Theorem 3.6. *Point D lies on the:*

(1) *Orthocentroidal Circle of the Triangle of the Centroids of the Triangulation Triangles of the Incenter.*
(2) *Parry Circle of the Triangle of the Centroids of the Triangulation Triangles of the Incenter.*

Theorem 3.7. *Point D lies on the:*

(1) *Nagel Line*
(2) *Line through the Inverse of the Incenter in the Circumcircle and the Midpoint of the Centroid and the Gergonne Point*

4. EXTERNAL DIVISION

4.1. **Kimberling Points.** The enclosed file "Supplementary material" contains a number of theorems about remarkable points which divide externally directed segments whose endpoints are Kimberling points and which are available in [7]. These theorems extend the corresponding article in [7].

4.2. **New Remarkable Points.** In Table 2 point R divides externally the directed segment $\overrightarrow{PQ}$ in the ratio $3:1$. The points in the last column of the Table 2 are not available in the Kimberling's ETC [7].

We use the following theorem:

Theorem 4.1. *Given points $P=(u,v,w)$ and $Q=(x,y,z)$. The point R dividing externally the directed segment $\overrightarrow{PQ}$ in the ratio $3:1$ has barycentric coordinates*

$$uR = 2ux + 3xv + 3xw - uy - uz,$$
$$vR = 2vy + 3yw + 3yu - vz - vx,$$
$$wR = 2wz + 3zu + 3zv - wx - wy.$$

	P	**Q**	**First barycentric coordinate of R**
1	X(1)	X(4)	$3b^4-b^3a-6c^2b^2+a^2b^2+cb^2a+ba^3+c^2ba-2cba^2+3c^4-4a^4+ca^3+c^2a^2-c^3a$
2	X(1)	X(5)	$3b^4-2ab^3-a^2b^2-6b^2c^2+2cab^2+2ba^3+2bac^2-4cba^2+3c^4-a^2c^2-2a^4+2ca^3-2ac^3$
3	X(1)	X(7)	$4a^3-2ab^2+4abc-2ac^2+a^2c+a^2b-3b^3+3b^2c+3bc^2-3c^3$
4	X(2)	X(1)	$8a-b-c$
5	X(2)	X(7)	$5a^2-4b^2+8bc-bc^2-ac-ab$
6	X(3)	X(1)	$a(3b^3-2ab^2-3cb^2+6cba-3a^2b-3bc^2-2ac^2+2a^3-3a^2c+3c^3)$

TABLE 2

In Table 3 point R divides externally the directed segment $\overrightarrow{GQ}$ in the ratio $p:q$. The points in the last column of the table are not available in the Kimberling's ETC [7].

	P	**Q**	**p : q**	**First barycentric coordinate of R**
1	X(2)	X(4)	$3:2$	$14c^2b^2-7c^4+11a^4-7b^4-4c^2a^2-4a^2b^2$
2	X(2)	X(6)	$3:2$	$7a^2-2b^2-2c^2$
3	X(2)	X(7)	$3:2$	$11a^2-7b^2+14bc-7c^2-4ac-4ab$
4	X(2)	X(8)	$3:2$	$11a-7b-7c$
5	X(2)	X(1)	$4:1$	$11a-b-c$
6	X(2)	X(3)	$4:1$	$10a^2b^2+10a^2c^2-11a^4-2b^2c^2+b^4+c^4$
7	X(2)	X(4)	$4:1$	$22c^2b^2-11c^4+13a^4-11b^4-2c^2a^2-2a^2b^2$
8	X(2)	X(7)	$4:1$	$13a^2-11b^2+22bc-11c^2-2ac-2ab$
9	X(2)	X(8)	$4:1$	$13a-11b-11c$
10	X(2)	X(7)	$4:3$	$5a^2-3b^2+6bc-3c^2-2ac-2ab$

TABLE 3

The enclosed file "Supplementary material" contains a number of theorems about new remarkable points which divide externally directed segments whose endpoints are Kimberling points and which are not available in [7].

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