Posing and Solving Problems with Barycentric Coordinates

FRANCISCO JAVIER GARCÍA CAPITÁN
I.E.S. Álvarez Cubero, Priego de Córdoba, Spain
e-mail: garciacapitan@gmail.com
web: http://garciacapitan.99on.com/

Abstract. We use barycentric coordinates as a tool both for proposing new problems or solving them. Although barycentric coordinates are not always the most beautiful way to solve a problem, they may be a powerful tool to arrive quickly to a solution of the problem or to the creation of new ones.

Keywords. barycentric coordinates, problem solving, geometric constructions.

Mathematics Subject Classification (2010). 51-04.

1. Introduction

We use the standard notation in triangle geometry. See Yiu [1]. Denote by $a = BC$, $b = CA$ and $c = AB$ the sides of a triangle $ABC$. If we want to visualize all triangles $ABC$ such that $f(a, b, c) = 0$ for some function $f$, we may fix $B$ and $C$ with cartesian coordinates $B = (-\frac{a}{2}, 0)$ and $C = (\frac{a}{2}, 0)$ and let $A = (x, y)$ vary on the plane meeting the conditions

$$b^2 = (x - \frac{a}{2})^2 + y^2, \quad c^2 = (x + \frac{a}{2})^2 + y^2$$

In this way, if we eliminate $b$ and $c$, we get an equation of a curve the form $\varphi(x, y, a) = 0$ as the locus of $A$.

As an example, if we consider the triangles such that $f(a, b, c) = b^2 + c^2 - a^2$, then we easily get $\varphi(x, y, a) = x^2 + y^2 - \frac{a^2}{4}$, and we arrive to the very well known fact that the locus for $A$ is the circle with $BC$ as diameter.

1This article is distributed under the terms of the Creative Commons Attribution License which permits any use, distribution, and reproduction in any medium, provided the original author(s) and the source are credited.
2. Some lines parallel to $BC$

In this section we look for triangles in which some particular lines are parallel to $BC$.

**Problem 2.1.** Find all triangles $ABC$ such that Euler line of $ABC$ is parallel to line $BC$.

**Solution.** The parallel to $BC$ at $G = (1 : 1 : 1)$, containing the infinite point of $BC$, namely $(0 : -1 : 1)$ is the line $2x = y + z$. This line contains the orthocenter $H = (S_BS_C : S_CS_A : S_A S_B)$ if and only if
\[2S_BS_C = S_CS_A + S_A S_B = a^2 S_A \iff a^2 (b^2 + c^2 - a^2) = (c^2 + a^2 - b^2)(a^2 + b^2 - c^2).
\]

Some further calculations using (1) lead to
\[12x^2 + 4y^2 = 3a^2 \iff \frac{4x^2}{a^2} + \frac{4y^2}{3a^2} = 1 \iff \frac{x^2}{\left(\frac{a}{2}\right)^2} + \frac{y^2}{\left(\frac{\sqrt{3}a}{2}\right)^2} = 1.
\]

This shows that the locus of $A$ is an ellipse. The minor axis is the segment $BC$. The endpoints of the major axis are the vertices of the two equilateral triangles erected on the segment $BC$.

**Problem 2.2.** Find all triangles $ABC$ such that the Brocard axis of $ABC$ is parallel to line $BC$.
Solution. In this case, the line joining \((0 : -1 : 1)\) and \(K = (a^2 : b^2 : c^2)\) has equation \((b^2 + c^2)x = a^2(y + z)\). This line goes through the circumcenter \(O = (a^2S_A : b^2S_B : c^2S_C)\) if and only if \((b^2 + c^2)a^2S_A = a^2(b^2S_B + c^2S_C)\), hence

\[
(b^2 + c^2)^2 - (b^2 + c^2)a^2 = b^2(c^2 + a^2 - b^2) + c^2(a^2 + b^2 - c^2)
\]

\[
= (b^2 + c^2)a^2 + b^2(c^2 - b^2) + c^2(b^2 - c^2)
\]

\[
= (b^2 + c^2)a^2 - (b^2 - c^2)^2,
\]

and using (1) we get the equation

\[
16y^4 + (32x^2 - 8a^2)y^2 + 16x^4 - 8a^2x^2 - 3a^4 = 0,
\]

that using \(a = 2m\) is equivalent to

\[
y^4 + (2x^2 - 2m^2)y^2 + x^4 - 2m^2x^2 - 3m^4 = 0.
\]

We can plot any of these curves. However, it is more interesting to find a ruler and compass constructions of these triangles \(ABC\). If we solve (2) for \(y\) we find

\[
y^2 = m^2 - x^2 \pm 2m\sqrt{m^2 - x^2} = \sqrt{m^2 - x^2} \left(\sqrt{m^2 - x^2} \pm 2m\right).
\]

This gives a simple construction for points \(A = (x, y)\) satisfying (2).

Call \(M\) the midpoint of \(BC\) and take any \(X\) on \(BC\), center of the circle \((BC)\) with \(BC\) as diameter. Let \(NN'\) the diameter of \((BC)\) perpendicular to \(BC\) and \(PP'\) the chord through \(X\) perpendicular to \(BC\). Let \(Q\) be the intersection of \(XP\) and the line through \(N\) parallel to \(N'P\). Let \(Z\) be one of the intersection points of \(BC\) and the circle \((P'Q)\) with \(P'Q\) as diameter. Let \(A\) any of the intersection points of \(XP\) and the circle centered at \(X\) with \(XZ\) as radius. Then the triangle \(ABC\) is a solution of our problem.

Figure 3 shows the locus of \(A\), a closed symmetric curve respect to \(BC\).
3. CONCURRENCES

Problem 3.1. Let $Y_a, Z_a$ be the contact points of the sides of the triangle $ABC$ and the $A$-excircle, and define $Z_b, X_b$ and $X_c, Y_c$ cyclically. The triangle formed by the lines $Y_aZ_a$, $Z_bX_b$ and $X_cY_c$ is perspective with the medial triangle of $ABC$.

![Figure 4](image.png)

From $CY_a : Y_aA = -(s-b) : s$ and $AZ_a : Z_aB = s : -(s-c)$ we have $Y_a = (s-b : 0 : -s)$ and $Z_a = (s-c, -s, 0)$, and we can calculate the equation of the line $Y_aZ_a : sx + (s-c)y + (s-b)z = 0$. Similarly we have the lines $Z_bX_b : (s-c)x + sy + (s-a)z = 0$ and $X_cY_c : (s-b)x + (s-a)y + sz = 0$. These lines intersect at $X = (-a(b+c) : S_C : S_B)$. If $A'B'C'$ is the medial triangle of $ABC$, the line $XA'$ has equation $(b-c)x + a(y-z) = 0$. In the same way we can calculate the lines $Y'B' : (c-a)y + b(z-x) = 0$ and $Z'C' : (a-b)z + c(x-y) = 0$. The lines $XA'$, $YB'$ and $CZ'$ concur at the point $X_9 = (a(s-a) : b(s-b) : c(s-c))$, known as the Mittenpunkt of $ABC$.

4. PROBLEM MATHEMATICAL REFLECTIONS O333

This is problem O333 of magazine Mathematical Reflections:

Problem 4.1. Let $ABC$ be a scalene acute triangle and denote by $O$, $I$, $H$ its circumcenter, incenter, and orthocenter, respectively. Prove that if the circumcircle of triangle $OIH$ passes through one of the vertices of triangle $ABC$ then it also passes through one other vertex.

Proposed by Josef Tkadlec, Charles University, Czech Republic

Solution. We consider an inversion with respect to the circumcircle. The circle $HIO$ inverts on the line $X_{36}X_{186}$ joining $X_{36}$, the inverse of $I$ and $X_{186}$, the inverse
of \( H \). We calculate later that this line has equation

\[
\frac{(b-c)(s-a)S_A}{\cos A - \frac{1}{2}} x + \frac{(c-a)(s-b)S_B}{\cos B - \frac{1}{2}} y + \frac{(a-b)(s-c)S_C}{\cos C - \frac{1}{2}} z = 0,
\]

where we have used the usual notation

\[
S_A = \frac{b^2 + c^2 - a^2}{2}, \quad S_B = \frac{c^2 + a^2 - b^2}{2}, \quad S_C = \frac{a^2 + b^2 - c^2}{2}, \quad s = \frac{a + b + c}{2}.
\]

The line \( X_{36}X_{186} \) goes through \( A \) if and only if

\[
(b-c)S_A \left( \cos B - \frac{1}{2} \right) \left( \cos C - \frac{1}{2} \right) = 0.
\]

Since \( ABC \) is a scalene acute triangle, we must have \( B = 60^\circ \) or \( C = 60^\circ \). Suppose, for example, that \( B = 60^\circ \). Then working backwards, the circle \( HIO \) also goes through \( C \).

To complete the proof we calculate the barycentric coordinates of \( X_{36} \) and \( X_{186} \) and the equation of line \( X_{36}X_{186} \).

**Coordinates of \( X_{36} \).** Let \( I' \) be the inverse of \( I \) with respect to the circumcircle. From \( OI \cdot OI' = R^2 \) and \( OI^2 = R^2 - 2Rr \) (Euler formula), we get

\[
\frac{OI'}{II'} = \frac{OI'}{OI - OI'} = \frac{O1 \cdot OI'}{OI^2 - OI' \cdot OI'} = \frac{R^2}{R^2 - R^2} = \frac{R}{2r} = \frac{abc}{2\Delta} \cdot \frac{2\Delta}{s} = \frac{abc}{2s^2},
\]

where \( S = 2\Delta \) is twice the area of the triangle \( ABC \).

Now the points \( O = (a^2S_A : b^2S_B : c^2S_C) \) and \( I = (a : b : c) \) have sum of its coordinates 2\( s^2 \) and 2\( s \) respectively; therefore \( sO \) and \( sI \) have equal sum (weight) and we can get algebraically \( I' = 2S^2(sO) - abcs(S^2I) = S^2s(2O - abcI) \). Hence the first coordinate of \( I' \) is

\[
a^2(b^2 + c^2 - a^2) - a^2bc = a^2(b^2 + c^2 - a^2 - bc) = 2abc \cdot a \left( \cos A - \frac{1}{2} \right),
\]

and the barycentric coordinates or \( X_{36} \) are

\[
X_{36} = I' = \left( a \left( \cos A - \frac{1}{2} \right) : b \left( \cos B - \frac{1}{2} \right) : c \left( \cos C - \frac{1}{2} \right) \right).
\]

**Coordinates of \( X_{186} \).** Let \( H' \) be the inverse of \( H \). From \( OH \cdot OH' = R^2 \) and the very well known formul \( OH^2 = R^2 - 8R^2 \cos A \cos B \cos C \), we get

\[
\frac{OH'}{HH'} = \frac{R^2}{OH^2 - R^2} = \frac{R^2}{-8R^2 \cos A \cos B \cos C} = \frac{-1}{8 \cos A \cos B \cos C} = \frac{a^2b^2c^2}{8S_A S_B S_C}.
\]

The sum of the coordinates of \( H = (S_B S_C : S_C S_A : S_A S_B) \) is \( S^2 \), therefore \( H' = (8S_A S_B S_C)O - (a^2b^2c^2)(2H) \). The first coordinate of \( H' \) is

\[
8S_A S_B S_C \cdot a^2S_A - a^2b^2c^2 \cdot 2S_B S_C = 2a^2S_B S_C \left( 4S_A^2 - b^2c^2 \right) = 8a^2b^2c^2 S_B S_C \left( \cos A + \frac{1}{2} \right) \left( \cos A - \frac{1}{2} \right),
\]

and the barycentric coordinates of \( H' = X_{186} \) are

\[
X_{186} = \left( \frac{\cos A^2 - \frac{1}{4}}{S_A} : \frac{\cos B^2 - \frac{1}{4}}{S_B} : \frac{\cos C^2 - \frac{1}{4}}{S_C} \right).
\]
Equation of the line $X_{36}X_{186}$. The equation of $X_{36}X_{186}$ has the form $ux + vy + wz = 0$, where the coefficient $u$ is

$$
\begin{vmatrix}
\frac{b}{\cos^2 B - \frac{1}{2}} & \frac{c}{\cos^2 C - \frac{1}{2}} \\
\frac{1}{2} & \frac{1}{2} \\
\end{vmatrix} = \frac{bS_B}{\cos B + \frac{1}{2}} \frac{cS_C}{\cos C + \frac{1}{2}}.
$$

because

$$
\begin{vmatrix}
\frac{bS_B}{\cos B + \frac{1}{2}} & \frac{cS_C}{\cos C + \frac{1}{2}} \\
\frac{1}{2} & \frac{1}{2} \\
\end{vmatrix} = \frac{1}{abc} \begin{vmatrix}
\frac{bS_B}{abc} & \frac{cS_C}{abc} \\
\frac{1}{2} & \frac{1}{2} \\
\end{vmatrix} = \frac{1}{2} \left( \frac{bS_B - cS_C}{S_B + \frac{abc}{2}} \right).
$$

5. Right triangle at the incenter

**Problem 5.1.** Let $ABC$ be a triangle and $X$ the contact point of the $C$–excircle and $BC$, and $Y$ the intersection of $CA$ and the line parallel to $AX$ through $B$. Then we have $\angle YIC = 90^\circ$.

![Figure 5](https://talentomatematico.files.wordpress.com/2014/01/geometria-moderna.pdf)

Since $CY : YA = CB : BX = a : s - a$, we have $Y = (a : 0 : s - a)$. The incenter is $I = (a : b : c)$, with sum of coordinates (weight) $a + b + c = 2s$. The infinite point of line $IY$ is $(2a - a - b : 2(s - a) - c) = (a : -b : b - a)$, the same as the infinite point of the line $bx + ay = 0$, which is the external angle bisector of angle $C$. Therefore $IY$ and $IC$ are perpendicular.

6. The same homothety center

**Problem 6.1.** Let $ABC$ be a triangle and $P, Q$ two points on line segment $BC$. Let $(P_B), (P_C), (Q_B)$ and $(Q_C)$ be the incircles of the triangles $ABP$, $APC$, $ABQ$ and $AQC$. Then the external center of homothety of the circles $(P_B)$ and $(Q_C)$ is the same as the external center of homothety of the circles $P_C$ and $Q_B$.

https://talentomatematico.files.wordpress.com/2014/01/geometria-moderna.pdf
Solution. We first use barycentric coordinates to prove some lemmas:

**Lemma 6.1.** If $X$ lies on $BC$ and $BX : XC = t : 1 - t$, then the incenter $X_B$ of triangle $ABX$ has homogeneous barycentric coordinates $X_B = (at : AX + (1 - t)c : ct)$.

**Proof.** If $X' = AX_B \cap BC$, we have $X = (0 : 1 - t : t)$ and, by the bisector theorem, $BX' : X'X = AB : AX = c : AX$ we get $X' = (0 : AX + (1 - t)c : tc)$. On the other side, since $X_B$ lies on the angle bisector $BI : cx - az = 0$, we find the intersection point $X_B = (at : AX + (1 - t)c : ct)$.

Symmetrically, the incenter $X_C$ of triangle $AXC$ has homogeneous barycentric coordinates $X_C = ((1 - t)a : (1 - t)b : bt + AX)$.

**Lemma 6.2.** If $X$ lies on $BC$ and $BX : XC = t : 1 - t$, then we have $AX^2 = tb^2 + (1 - t)c^2 - t(1 - t)a^2$.

**Proof.** From Stewart theorem for cevian $AX$,

\[
 a \cdot (AX^2 + BX \cdot XC) = b^2 \cdot BX + c^2 \cdot XC
\]

\[
 \Rightarrow a \cdot (AX^2 + ta \cdot (1 - t)a) = b^2 \cdot ta + c^2 \cdot (1 - t)a
\]

\[
 \Rightarrow AX^2 = tb^2 + (1 - t)c^2 - t(1 - t)a^2.
\]

**Lemma 6.3.** If $X$ lies on $BC$ and $BX : XC = t : 1 - t$, then

\[
 AX^2 - (tb - (1 - t)c)^2 = (1 - t)t(a + b + c)(b + c - a).
\]
Proof. We sum the corresponding sides of the identities
\[(tb - (1 - t)c)^2 = t^2b^2 + (1 - t)^2c^2 - 2t(1 - t)bc\]
\[(1 - t)(a + b + c)(b + c - a) = (1 - t)(b^2 + c^2 + 2bc - a^2)\]
and we get
\[(tb - (1 - t)c)^2 + (1 - t)(a + b + c)(b + c - a) = tb^2 + (1 - t)c^2 - t(1 - t)a^2\]
\[= AP^2.\]

Lemma 6.4. Let \(ABC\) be a triangle and \(P = (u : v : w)\) a point, \(P'\) the feet of the perpendicular to \(BC\) through \(P\), and \(P''\) the reflection of \(P'\) on \(P\). Then \(P''\) has coordinates
\[P'' = (2a^2u : a^2v - uS_C : a^2w - uS_B).\]

Proof. We calculate first the coordinates of \(P'\), by finding the line through \(P\) and \(P\) and \((-a^2 : S_C : S_B)\), the infinite point of a perpendicular to \(BC\), then finding its intersection with line \(BC\) : \(x = 0\). To do that, we calculate the determinant
\[
\begin{vmatrix}
0 & y & z \\
u & v & w \\
-a^2 & S_C & S_B
\end{vmatrix}
= 0 \Rightarrow (S_Bu + a^2w)y = (S_Cu + a^2v)z,
\]
giving the point \(P' = (0 : S_Cu + a^2v : S_Bu + a^2w)\). The sum of coordinates of \(P'\) is \(a^2(u + v + w)\), hence the coordinates of \(P''\) follow from the algebraic relation
\[P'' = 2P - P' = 2(a^2u, a^2v, a^2w) - (0, S_Cu + a^2v, S_Bu + a^2w)\]
\[= (2a^2u, a^2v - S_Cu, a^2w - S_Bu).\]
Now to solve our problem we take \(P = (0 : 1 - p : p)\) and \(Q = (0 : 1 - q : q)\). Then by Lemma 1, \(P_B = (ap : AP + c(1-p) : cp)\) and \(Q_C = (a(1-q) : b(1-q) : AQ + bq)\). If \(P_B'\) and \(Q_C'\) are the orthogonal projections of \(P\) and \(P'\) on \(BC\) and \(P_B''Q_C''\) are their reflection on \(P_B, Q_C,\) respectively, obtenemos:
\[P_B'' = (2pa^2 : aAP + (1 - p)ac - pS_C, 2p(s - a)(s - c))\],
\[Q_C'' = (2(1-q)a^2 : 2(1-q)(s - a)(s - b) : aAQ + qab - (1 - q)S_B)\].
The line \(P_B''Q_C''\) intersects \(BC\) at the point
\[K_{PQ} = (0 : -(1 - q)(AP - pb + (1 - p)c) : p(AQ + qb - (1 - q)c)).\]
Symmetrically, if we consider the points \(P_C\) and \(Q_B\) we get the that the corresponding line \(P_B''Q_B''\) intersects \(BC\) at the point
\[K_{QP} = (0 : -(1 - p)(AQ - qb + (1 - q)c) : q(AP + pb - (1 - p)c)).\]
Since points of the form \((0 : m : n)\) and \((0 : m' : n')\) are the same if and only if \(mn' = m'n\), to prove \(K_{PQ} = K_{QP}\) we only need that

\[ q(1 - q) \left( AP^2 - (pb - (1 - p)c)^2 \right) = p(1 - p) \left( AQ^2 - (qb - (1 - q)c)^2 \right), \]

and this follows immediately from Lemma 4.

**References**