A generalization of the Zeeman-Gossard perspector theorem

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Abstract. In this note, we introduce a generalization of the Zeeman-Gossard perspector theorem and a generalization of the Dao’s twelve Euler lines point \(X(4240)\) in the Kimberling’s Encyclopedia of Triangle Centers. We present problems related to the concurrence of four Newton lines in the generalized Zeeman-Gossard perspector configuration.

Keywords. Zeeman-Gossard perspector, Dao’s twelve Euler lines point, triangle geometry, Euclidean geometry.

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The following theorem is well known:

**Theorem 1** ([1], Zeeman-Gossard perspector theorem). Let the Euler line of triangle \(ABC\) meets the sidelines \(BC, CA, AB\) at \(A_0, B_0, C_0\) respectively. The three Euler’s lines of triangles \(AB_0C_0, BC_0A_0\) and \(CA_0B_0\) form triangle \(A_gB_gC_g\). Then triangles \(A_gB_gC_g\) and \(ABC\) are homothetic and congruent, and the homothetic center (called Zeeman-Gossard’s perspector) lies on the Euler line.

You can see details about the Zeeman-Gossard perspector in [2], [3]. We present the following problem:

**Problem 1** ([4], [5], A generalization of the Zeeman-Gossard perspector theorem). Let \(ABC\) be a triangle. Let \(H\) and \(O\) be two points, and let the line \(HO\) meets \(BC, CA, AB\) at \(A_0, B_0, C_0\) respectively. Let \(A_H\) and \(A_O\) be two points such that \(C_0A_H \parallel BH, B_0A_H \parallel CH\) and \(C_0A_O \parallel BO, B_0A_O \parallel CO\). Define \(B_H, B_O, C_H, C_O\) cyclically. Then the triangle formed by the lines \(A_HA_O, B_HB_O, C_HC_O\) and triangle \(ABC\) are homothetic and congruent, and the homothetic center lies on the line \(OH\).
Figure 1. A generalization of Zeeman-Gossard perspector theorem

Special Cases.

- If $OH$ is any line parallel to the Euler line, this problem is the Zeeman-Gossard theorem.
- If $OH$ is any line through the centroid of triangle $ABC$, this problem is the Yiu’s generalization of the Gossard perspector theorem [2].

Simple statement of problem 1:
Let $ABC$ be a triangle, let a line $L$ meets $BC, CA, AB$ at $A_0, B_0, C_0$ respectively. Let $P$ be a point on the line $L$. Let $P_A$ be a point such that $B_0P_A \parallel CP$ and $C_0P_A \parallel BP$. Define $P_B$ and $P_C$ cyclically. Show that the three lines through $PA_0, PB_0, PC_0$ and parallel to $BC, CA, AB$ respectively, form a triangle congruent and homothetic with $ABC$. The homothetic center lies on the line $L$.

Figure 2. Simple statement of problem 1

Problem 2 ([4]). We use the notation of Problem 1. Then the Newton lines of the four quadrilaterals formed by the four lines $AB, AC, A_HA_O, HO$, the four lines
A generalization of the Zeeman-Gossard perspector theorem

BC, BA, B\textsubscript{H}B\textsubscript{O}, HO, the four lines CA, CB, C\textsubscript{H}C\textsubscript{O}, HO and the four lines AB, BC, CA, HO, pass through the homothetic center of triangles ABC and A\textsubscript{g}B\textsubscript{g}C\textsubscript{g}.

![Figure 3. Problem 2: Concurrence of four Newton lines](image)

**Figure 3.** Problem 2: Concurrence of four Newton lines

See [6], [7] for description of point X(4240).

**Problem 3** ([5], A generalization of point X(4240)). Let ABC be a triangle. Let H and O be two points, let the line HO meets BC, CA, AB at A\textsubscript{0}, B\textsubscript{0}, C\textsubscript{0} respectively. Let A\textsubscript{H}', A\textsubscript{O}' be two points such that C\textsubscript{0}A\textsubscript{H}' \parallel CH, B\textsubscript{0}A\textsubscript{H}' \parallel BH and C\textsubscript{0}A\textsubscript{O}' \parallel CO, B\textsubscript{0}A\textsubscript{O}' \parallel BO. Define B\textsubscript{H}', B\textsubscript{O}', C\textsubscript{H}', C\textsubscript{O}' cyclically. Then the four lines A\textsubscript{H}'A\textsubscript{O}', B\textsubscript{H}'B\textsubscript{O}', C\textsubscript{H}'C\textsubscript{O}' and OH are concurrent. If OH is the Euler line, then the point of concurrence is point X(4240).

![Figure 4. Problem 3: A generalization of point X(4240)](image)
Simple statement of problem 3:
Let $ABC$ be a triangle, let a line $L$ meets $BC, CA, AB$ at $A_0, B_0, C_0$ respectively. Let $P$ be a point on the line $L$. The three lines through $A_0, B_0, C_0$ and parallel to $AP, BP, CP$ form triangle $P_AP_BP_C$ respectively. Show that the three lines through $P_A, P_B, P_C$ and parallel to $BC, CA, AB$ respectively are concurrent. The point of concurrence lies on line $L$.

Figure 5. Simple statement of problem 3.

REFERENCES


