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Collinearity of the reflections of the intercepts of a line in the angle bisectors of a triangle

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Abstract. We show that when the intercepts of a line on the sidelines of a triangle are reflected in the respective angle bisectors, the reflections are collinear if and only if the given line either contains the incenter or is tangent to an inscribed parabola.

Keywords. Reflections, dual conics, inscribed parabola, triangle centers.

Mathematics Subject Classification (2010). 51M04, 51M15.

1. Reflections of intercepts of a line in the angle bisectors

In this note we present two animations with a dynamic software each involving the collinearity of the reflections of three points in the angle bisectors of a triangle. In the plane of a triangle ABC, consider a line $\mathscr{L}$ intersecting the sidelines BC, CA, AB at X, Y, Z respectively. Construct the reflections X' of X in the bisector of angle A, and similarly the reflections Y' of Y in the bisector of angle B, and Z' of Z in the bisector of angle C. We show that there are only two ways of choosing the line $\mathscr{L}$ appropriately so that the three reflections X', Y', Z' lie on a line $\mathscr{L}'$.

We work with homogeneous barycentric coordinates with reference to ABC, and refer to [2] for basic terminology and results. If the line $\mathscr{L}$ has equation $ux+vy+wz=0$, then its intercepts on the sidelines are the points

$$X = (0 : w : -v), \quad Y = (-w : 0 : u), \quad Z = (v : -u : 0).$$

The reflections of these points in the respective angle bisectors are

$$\begin{aligned} X' &= ((b-c)(bv+cw) : -b^2v : c^2w), \\ Y' &= (a^2u : (c-a)(cw+au) : -c^2w), \\ Z' &= (-a^2u : b^2v : (a-b)(au+bv)). \end{aligned}$$

These reflections are collinear if and only if

$$\begin{vmatrix} (b-c)(bv+cw) & -b^2v & c^2w \\ a^2u & (c-a)(cw+au) & -c^2w \\ -a^2u & b^2v & (a-b)(au+bv) \end{vmatrix} = 0.$$

Simplifying, we obtain, after cancelling a factor abc, the following two conditions.

(1) $au + bv + cw = 0$,

(2) $(b-c)(b+c-a)vw + (c-a)(c+a-b)wu + (a-b)(a+b-c)uv = 0$.

Condition (1) shows that the line $\mathscr{L}$ contains the incenter $I = (a : b : c)$. Condition (2) means that $\mathscr{L}$ is tangent to the dual conic of the line-conic

(3) $$\frac{(b-c)(b+c-a)}{x} + \frac{(c-a)(c+a-b)}{y} + \frac{(a-b)(a+b-c)}{z} = 0,$$

which clearly contains the sidelines of the triangle and the line at infinity $[1 : 1 : 1]$.

We shall make use of the following basic result in identifying dual conics.

Theorem 1. ([2, §10.6.4]) The dual conic of the line-conic $\frac{p}{x} + \frac{q}{y} + \frac{r}{z} = 0$ is the inscribed conic

$$-p^2x^2 - q^2y^2 - r^2z^2 + 2qryz + 2rpzx + 2pqxy = 0,$$

with perspector $\left(\frac{1}{p} : \frac{1}{q} : \frac{1}{r}\right)$ and center $(q + r : r + p : p + q)$.

2. Reflections of intercepts of lines through the incenter

We change notation and take $\mathscr{L}$ to be a line containing the incenter and a point P with homogeneous barycentric coordinates $(u : v : w)$. Thus, $\mathscr{L}$ has equation

$$(cv - bw)x + (aw - cu)y + (bu - av)z = 0.$$

The three reflections X', Y', Z' are the points

$$\begin{aligned} X' &= (-a(b-c)(cv-bw) : \ -b^2(aw-cu) : \ c^2(bu-av)), \\ Y' &= (a^2(cv-bw) : \ -b(c-a)(aw-cu) : \ -c^2(bu-av)), \\ Z' &= (-a^2(cv-bw) : \ b^2(aw-cu) : \ -c(a-b)(bu-av)). \end{aligned}$$

These are all on the line $\mathscr{L}'$

$$\frac{b+c-a}{a(cv-bw)}x + \frac{c+a-b}{b(aw-cu)}y + \frac{a+b-c}{c(bu-av)}z = 0.$$

The line coordinates of $\mathscr{L}$ clearly shows that it lies on the conic

$$\frac{b+c-a}{x} + \frac{c+a-b}{y} + \frac{a+b-c}{z} = 0,$$

which is the dual conic of the incircle (see [2, §10.6.4, Exercise 2]). This means that the line $\mathscr{L}'$ is tangent to the incircle. The point of tangency is

$$Q = \left(\frac{a^2(cv-bw)^2}{b+c-a} : \frac{b^2(aw-cu)^2}{c+a-b} : \frac{c^2(bu-av)^2}{a+b-c}\right).$$

The point of tangency Q has a simple description in terms of reflections. Let $\mathscr{L}''$ be the parallel of $\mathscr{L}$ through the orthocenter H' of the intouch triangle. The reflections of $\mathscr{L}''$ in the sidelines of the intouch triangle is the point Q (see Figure 1). Here are some examples.

$\mathscr{L}$	IO	IG	IH	IN
Q	$X(11)$	$X(1357)$	$X(1364)$	$X(3025)$

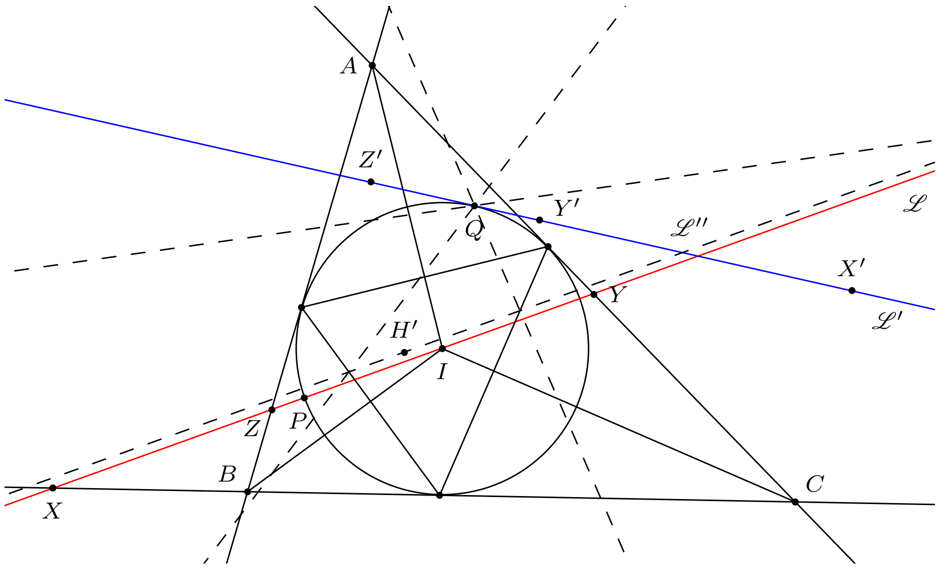

Figure 1.

Remark. The indexing of triangle centers as $X(n)$, apart from the common notation, follows Kimberling's Encyclopedia of Triangle Centers [1]. $X(11)$, for example, is the Feuerbach center, the point of tangency of the incircle with the nine-point circle.

With a dynamic software one animates a point P on the incircle of triangle ABC, construct
(i) the intercepts X, Y, Z of the line IP in the sidelines, and their reflections X', Y', Z' in the respective angle bisectors,
(ii) the parallel of IP through the orthocenter H' of the intouch triangle, and its reflections in the sidelines of the intouch triangle to concur at a point Q on the incircle.
As P traverses the incircle, the reflections X', Y', Z' lie on the moving tangent to the incircle at Q.

3. Reflections of intercepts of tangents to an inscribed parabola

Now suppose the line $\mathscr{L}$ satisfies condition (2), so that it is a tangent to the inscribed conic dual to the line-conic given by (3). Since the line-conic contains the sidelines and the line at infinity (with line-coordinates $[1:1:1]$), the inscribed conic is a parabola with infinite point the perspector of the line-conic (3), namely,

$$X(522) = ((b-c)(b+c-a) : \ (c-a)(c+a-b) : \ (a-b)(a+b-c)).$$

The focus of the parabola is the isogonal conjugate of $X(522)$, which is the point $X(109)$ on the circumcircle. The directrix is the line (through the orthocenter H)

containing the reflections of $X(109)$ in the sidelines. This is precisely the line IH. From (3), a typical line tangent to the parabola has equation

$$\frac{b+c-a}{b+c-a+t}x+\frac{c+a-b}{c+a-b+t}y+\frac{a+b-c}{a+b-c+t}z=0.$$

The line $\mathscr{L}'$ containing the reflections X', Y', Z' is

$$\begin{aligned}&((b+c-a)(c+a-b)(a+b-c)-(a^2+b^2+c^2-2ca-2ab)t)x\\+&((b+c-a)(c+a-b)(a+b-c)-(a^2+b^2+c^2-2ab-2bc)t)y\\+&((b+c-a)(c+a-b)(a+b-c)-(a^2+b^2+c^2-2bc-2ca)t)z=0.\end{aligned}$$

This line has infinite point $X(513)=(a(b-c):\ b(c-a):c(a-b))$. From this, the line $\mathscr{L}'$ is perpendicular to OI, independent of the choice of t (see Figure 2).

Again, one animates a point P on the line IH, and construct
(i) the perpendicular bisector $\mathscr{L}$ of the segment $PX(109)$,
(ii) the intercepts X, Y, Z of the line $\mathscr{L}$ in the sidelines, and their reflections X', Y', Z' in the respective angle bisectors.
As P traverses the line IH, these reflections X', Y', Z' lie on a moving line $\mathscr{L}'$ perpendicular to the line OI (joining the circumcenter and incenter of triangle ABC).

In particular, if $\mathscr{L}$ is the perpendicular bisector of $IX(109)$, then $\mathscr{L}'$ is the perpendicular to OI at I.

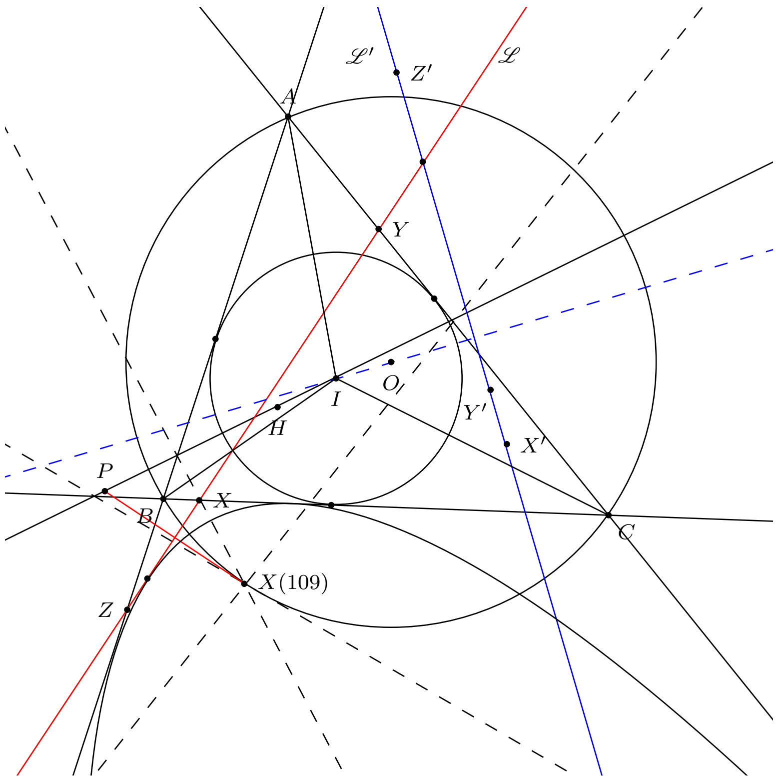

FIGURE 2.

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http://faculty.evansville.edu/ck6/encyclopedia/ETC.html.

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