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Iterations of sum of powers of digits

Paul Yiu
Department of Mathematical Sciences, Florida Atlantic University,
Boca Raton, FL 33431-0991 USA
e-mail: yiu@fau.edu

Abstract. We show that for $k = 2, 3, \ldots, 6$, iterations of the sum of k-th powers of digits of natural numbers result in a small number of fixed points and limit cycles.

Keywords. Sums of powers of digits, iterations, fixed points, limit cycles.

Mathematics Subject Classification (2010). 97F30.

1. Iteration of sum of powers of digits

A natural number is congruent modulo 9 to the sum of its decimal digits. This simple observation leads to the famous method of "casting out of nines" for the test of divisibility by 9. It also follows that the digit sum operation, repeated indefinitely, will eventually lead to a single digit number, its remainder when divided by 9. This is called the digital root of the number. See, for example, [1].

Let k be a fixed positive integer. Consider the "sum of k-th powers of digits" function

$$s_k(n) := \sum_{j=0}^{\ell} a_j^k \qquad \text{for} \quad n = \sum_{j=0}^{\ell} a_j \cdot 10^{\ell-j}, \quad 0 \le a_j \le 9.$$

Clearly, $s_1(n)$ is the sum of the digits of n. Here are some simple examples:

$$s_2(16) = 37, \quad s_3(153) = 153, \quad s_4(2178) = 6514, \quad s_4(6514) = 2178.$$

Iterations of s_1 beginning with an integer will result in its digital root. In this note we consider iterations of s_k, $k \ge 2$, from the various positive integers. For a positive integer N, consider the sequence

$$\mathcal{S}_k(N): \qquad N,\ s_k(N),\ s_k^2(N),\ \ldots,\ s_n^m(N),\ \ldots,$$

where $s_k^m(N)$ is obtained from N by m applications of s_k. For example, the sequence

$$\mathcal{S}_2(4): \qquad 4,\ 16,\ 37,\ 58,\ 89,\ 145,\ 42,\ 20,\ 4,\ \ldots$$

is periodic with period 8. The main result of this note is that every such $\mathcal{S}_k(N)$ is eventually periodic (Theorem 2), and we determine the fixed points and limit cycles for $k = 2, 3, \ldots, 6$.

Lemma 1. If n has $k+2$ or more digits, then $n > s_k(n)$.

Proof. It is easy to establish by mathematical induction that $(k+2)\cdot 9^k < 10^{k+1}$ for $k \geq 1$. This is clearly true for $k = 1$. Assuming $(k+2)\cdot 9^k < 10^{k+1}$, we have

$$(k+3)\cdot 9^{k+1} = (k+2)\cdot 9^{k+1} + 9^{k+1} < 9\cdot 10^{k+1} + 9^{k+1} < 9\cdot 10^{k+1} + 10^{k+1} = 10^{k+2}.$$

Now, if n has $k+2+\ell$ digits for $\ell \geq 0$, then

$$n \geq 10^{k+1+\ell} = 10^{\ell}(10^{k+1}) > 10^{\ell}\cdot(k+2)\cdot 9^k > (k+2+\ell)\cdot 9^k \geq s_k(n).$$

Theorem 2. For every positive integer N, the sequence $\mathcal{S}_k(N)$ is eventually periodic.

Proof. By Lemma 1, the sequence $\mathcal{S}_k(N)$ is strictly decreasing as long as the terms have $k+2$ or more digits. If n has $k+1$ or fewer digits, then $s_k(n) \leq (k+1)\cdot 9^k < 10^{k+1}$. This means the terms of the sequence $\mathcal{S}_k(N)$ are eventually less than 10^{k+1}. Therefore, there is a sufficiently large integer m such that $s_k^m(N) = s_k^{m+\ell}(N)$ for some $\ell > 0$. It follows that $s_k^{m+h\ell}(N) = s_k^m(N)$ for every $h \geq 0$. The sequence is eventually periodic.

Corollary 3. For a fixed positive integer k, iterations of s_k results in a fixed point or a limit cycle.

For $k = 2, 3, \ldots, 6$, the number of fixed points and limit cycles is relatively small. Here are the fixed points of s_k, integers n satisfying $s_k(n) = n$:

k	fixed points of s_k
2	1
3	1, 153, 370, 371, 407.
4	1, 1634, 8208, 9474
5	1, 4150, 4151, 54748, 92727, 93084, 194979.
6	1, 548834.

Table 1. Fixed points of s_k

2. ITERATIONS OF SUM OF SQUARES OF DIGITS

For $k = 2$, if a sequence $\mathcal{S}_2(N)$ does not terminate in the fixed point 1, it will eventually enter the cycle (4, 16, 37, 58, 89, 145, 42, 20). This was established by A. Porges in [2]. We outline a proof here by determining the limit cycles in the iterations of the sum of squares of digits. By Lemma 1, we need only examine those sequences beginning with 3-digit numbers. We show that we may simply begin with 2-digit numbers.

First of all, if n is a 3-digit number, $s_2(n) \leq 3\cdot 9^2 = 243$. For $200 \leq n \leq 243$, $s_2(n) \leq 2^2 + 4^2 + 9^2 = 101 < n$. Therefore we may restrict to 3-digit numbers "beginning with 1". Table 2 shows all such numbers n with $s_2(n) \geq 100$. In each case, $s(n) < n$.

n	159	168	169	178	179	188	189	199
	195	186	196	187	197		198	
$s_2(n)$	107	101	118	114	131	129	146	163

Table 2. 3-digit numbers n with $s_2(n) \geq 100$

Therefore it is enough to consider $\mathcal{S}_2(N)$ for $N \leq 99$. Figures 1 and 2 together show that beginning with a number with at most 2 digits, iterations of the sum of squares of digits either converge to 1 or eventually enter the cycle (4, 16, 37, 58, 89, 145, 42, 20).

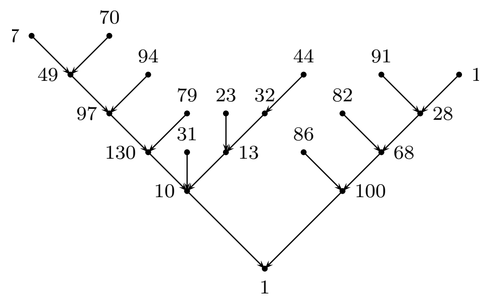

FIGURE 1. Iterations of sum of squares of digits converging to 1

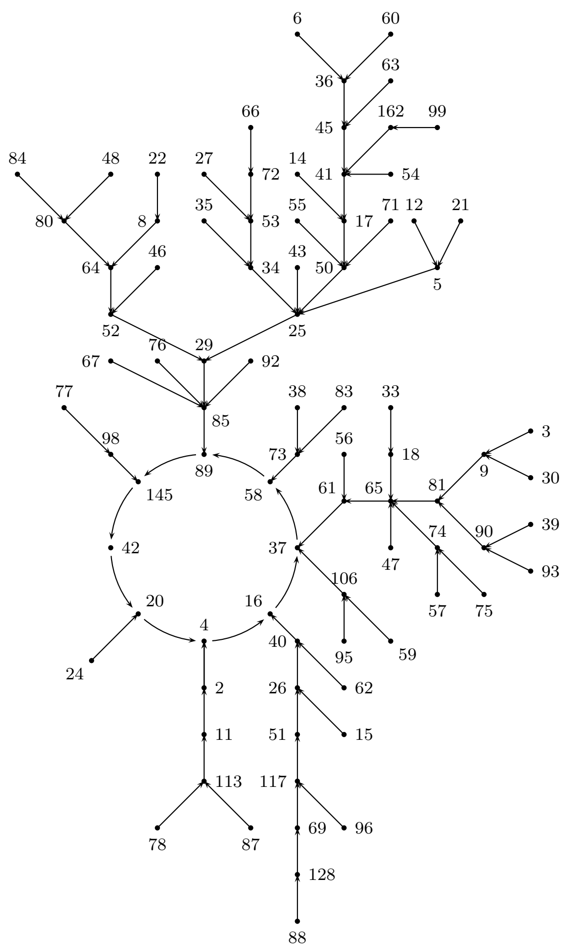

FIGURE 2. Iterations of sum of squares of digits to limit cycle

3. Iterations of sum of cubes of digits

The function s_3 has a number of fixed points other than 1. These are 153, 370, 371, and 407 (see Table 1). Sequences $\mathcal{S}_3(N)$ not converging to one of these fixed points will eventually enter a limit cycle. To enumerate these limit cycles, by Lemma 1, we need only consider N with no more than 4 digits. One can further show that $n > s_3(n)$ for 4-digit numbers $N \geq 2000$. Therefore, we need only consider $\mathcal{S}_3(N)$ for $N \leq 1999$. Note that $s_3(1999) = 2188 > 1999$. The number of sequences converging to the various fixed points are given in Table 3 below.

Fixed points	1	153	370	371	407	Total
sequences	34	666	343	588	78	1709

Table 3. Number of sequences $\mathcal{S}_3(N)$ with fixed points for $N \leq 1999$

Each of the remaining $1999 - 1709 = 290$ sequences enters one of the limit cycles below. Table 4 gives the number of sequences entering the cycles at various points.

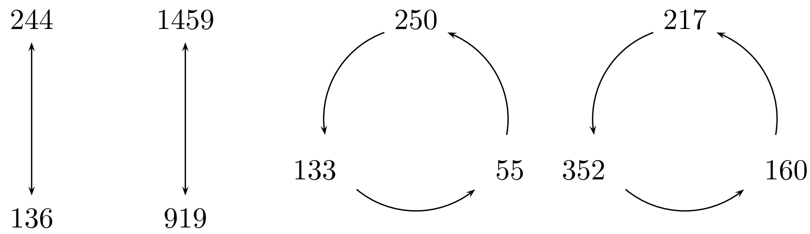

Figure 3. Limit cycles of iterations of sum of cubes of digits

Cycle	136	244	919	1459	55	250	133	160	217	353	Total
sequences	3	18	15	54	16	3	91	24	18	48	290

Table 4. Number of sequences $\mathcal{S}_3(N)$ entering limit cycles

4. Iterations of sum of 4-th powers of digits

To enumerate the limit cycles, we need only check $\mathcal{S}_4(N)$ for $N \leq 19999$. Table 5 shows the numbers of sequences converging to the four fixed points. Table 6 shows that the remaining sequences eventually enter one of two limit cycles, one of length 2 and another of length 7.

Fixed point	1	1634	8208	9474	Total
sequences	17	60	1246	60	1383

Table 5. Number of sequences $\mathcal{S}_4(N)$ with fixed points for $N \leq 19999$

Cycle	2178	6514	1138	4179	9219	13139	6725	4338	4514	Total
sequences	744	150	44	1775	48	15749	12	72	22	18616

Table 6. Number of sequences $\mathcal{S}_4(N)$ entering limit cycles

5. Iterations of s_5 and s_6

We record the fixed points and limit cycles for iterations of s_5 and s_6 in Figures 4 and 5 respectively. For s_5, it is enough to restrict to $\mathcal{S}_5(N)$ for $N < 2 \cdot 10^5$.

Cycle length	1	2	4	6	10	12	22	28
cycles	7	2	1	1	2	1	1	1

Table 7. Numbers of fixed points and limit cycles for s_5.

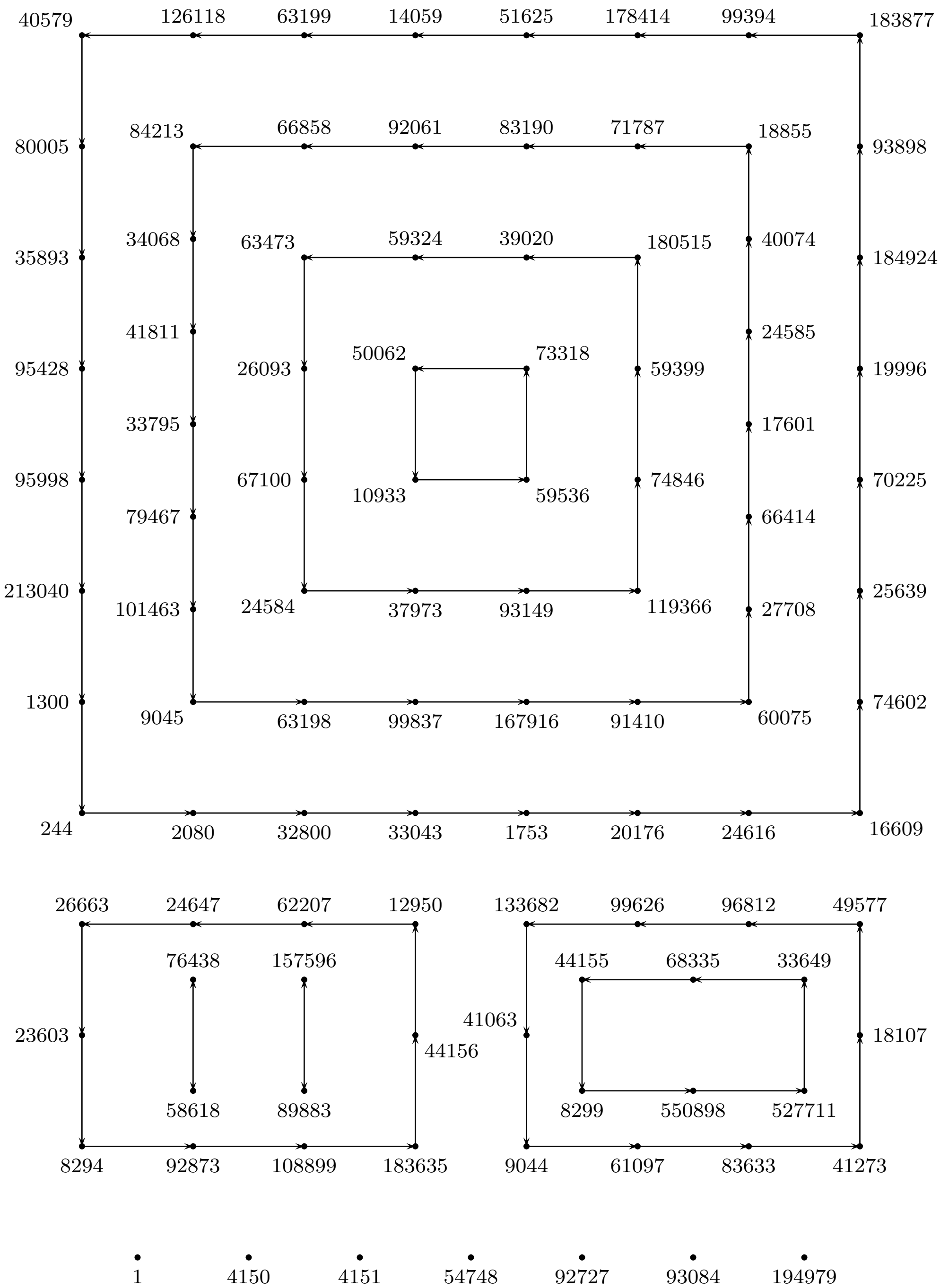

Figure 4. Fixed points and limit cycles for iterations of s_5

The result for s_6 is simpler. For $N < 2130000$, there are two fixed points, and one limit cycle each of length 2, 3, 4, 10, 30.

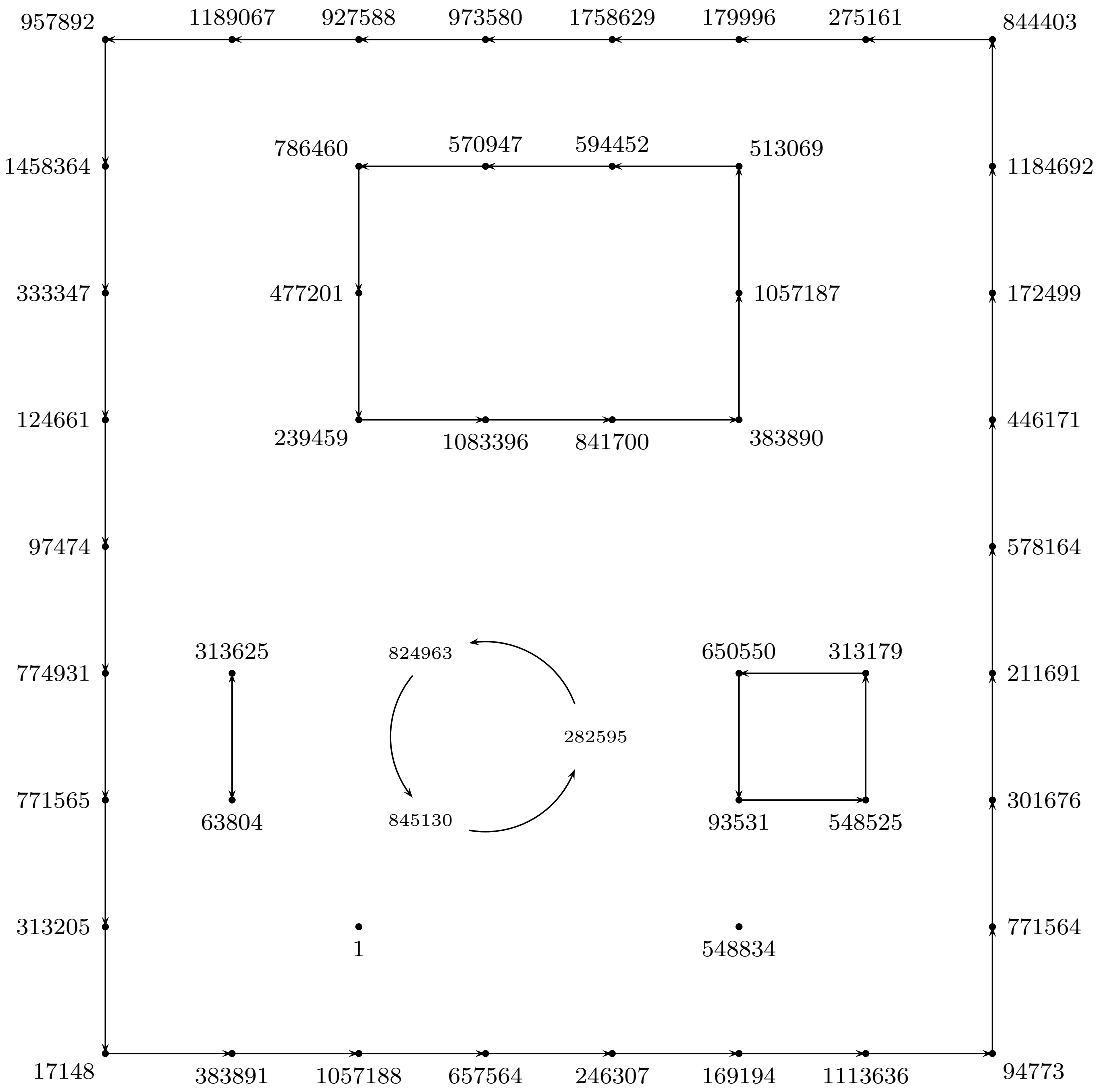

FIGURE 5. Fixed points and limit cycles for iterations of s_6

REFERENCES

[1] H. E. Dudeney, *Amusements in Mathematics*, 1917; Dover reprint, 1970.
[2] A. Porges, *A set of eight numbers*, American Mathematical Monthly, 52 (1945) 379–382.